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Fundamental frequency for large amplitude vibrations of uniform Timoshenko beams with central point concentrated mass using coupled displacement field method

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Abstract

Complex structures used in many fields of engineering are made up of simple structural elements like beams, plates, etc. These structural elements, sometimes carry concentrated point masses at discrete points, and when subjected to severe dynamic environment tend to vibrate with large amplitudes. Both the continuum and the finite-element solutions are available in the open literature to tackle this coupled nonlinear problem, without concentrated point masses with particular emphasis on the fundamental linear and nonlinear frequencies. However, for short beams and moderately thick plates, one has to consider the effects of shear deformation and rotary inertia to evaluate their fundamental linear and nonlinear frequencies. A commonly used method for obtaining the same is the energy method, or a finite-element analogue of the same. In this paper the authors used a coupled displacement field method where in the number of undetermined coefficients '2*n*' existing in the classical energy method are reduced to '*n*', which significantly simplifies the procedure to obtain the analytical solution. The large amplitude free vibration behaviour of the most commonly used uniform shear flexible hinged–hinged and clamped–clamped beams with central point concentrated masses is studied here. This study reveals some interesting aspects concerned with the problem considered. The numerical results in terms of the linear frequency parameter and the ratios of nonlinear to linear radian frequencies for the uniform with a central point concentrated mass are given in the digital form.

1. Introduction

The importance of large amplitude free vibrations of commonly used structural elements like beams, circular and rectangular plates, etc. is well recognized now because of the works of many researchers starting from Woinowsky–Krieger [1]. Refs. [2–4] are some of the pioneering works, including both the continuum and the finite-element methods, on the large amplitude free vibrations of beams. Secondary effects like shear deformation and rotary inertia on the large amplitude free vibrations of beams are considered in Ref. [5],

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Nomenclature

A	area of cross section
a	central lateral displacement
a_m	maximum amplitude
E	Young's modulus
G	shear modulus
I	area moment of inertia
k	shear correction factor
L	length of the beam
m	mass of the beam per unit length
M	concentrated mass
$p_{(x)}$	static load per unit length of the beam
q	amplitude ratio
q_m	maximum amplitude ratio
r	radius of gyration
T	kinetic energy
$\bar{u}$	axial displacement at a point on the beam
U	strain energy
$\bar{w}$	transverse displacement at a point on the beam
W	work done by the lateral load
W_1	work done by the tension developed in the beam
x	axial coordinate of the beam
z	lateral coordinate of the beam
α_1, α_2	as given in equations
β	slenderness ratio
$\bar{\beta}$	shear rotation
ε_x	axial strain
γ_{xz}	shear strain
v	Poisson ratio
ψ	bending rotation
ω_L	linear radian frequency
ω_{NL}	nonlinear radian frequency
θ	total rotation
$(\ )'$	differentiation with respect to x
$(^\bullet)$	differentiation with respect to time

wherein a simplified finite-element formulations are used. Ref. [6] presents exhaustively, many investigations on this topic available in the open literature till recently.

The recent work, reported in Refs. [7,8] on the development of coupled displacement field finite elements for Timoshenko beam prompted the authors to develop a continuum analogue of the same using the energy method, for the large amplitude free vibrations of Timoshenko beams considering the shear deformation and rotary inertia. A successful preliminary attempt of the authors in this direction can be seen in Ref. [9].

To solve the large amplitude free vibration problem of shear flexible structural elements like short beams, which are commonly used, 'n' admissible functions satisfying the essential boundary conditions, for the lateral displacement field 'w' and another 'n' admissible functions for the total rotation θ, which are compatible with the admissible functions for 'w' are necessary to be used in the classical energy method. The Lagrangian is minimized with respect to the '$2n$' unknown coefficients, corresponding to the '$2n$' admissible functions, to obtain a system of coupled nonlinear differential equations in time, the solution of which is quite involved if not impossible.

For some practically used beam configurations, like uniform hinged–hinged and clamped–clamped beams, the large amplitude free vibration problem is generally studied with single term exact or nearly exact trigonometric functions [10,11], where the main emphasis is to obtain the nonlinear fundamental frequency. Even for this situation, one gets two nonlinear coupled differential equations in time, in terms of the two undetermined coefficients corresponding to the two admissible functions. The solution of which, if the beams are short, is not amenable to obtain simple and accurate closed-form solutions. On the other hand, if one uses the coupled displacement field method [9], where in the lateral displacement 'w' and total rotation 'θ' are coupled by a coupling equation, only one undetermined coefficient remains and the use of the energy method gives only one nonlinear ordinary differential equation in time. The coupling equation is obtained from one of the differential equation governing the static equilibrium of the beam and is independent of the boundary conditions. This final nonlinear temporal equation is of Duffing type with cubic nonlinearity and can be solved either numerically or with a reasonable simplifying approximation, as in the case of the harmonic balance method (HBM) [12]. The closed-form solution obtained through the use of HBM is simple and elegant which gives quick and accurate solutions to the large amplitude free vibration problem, with reference to the nonlinear fundamental frequency of the uniform Timoshenko beams as shown in Ref. [9].

In the present paper, the authors applied the coupled displacement field method for the large amplitude free vibrations of uniform shear flexible hinged–hinged and clamped–clamped beams with a point concentrated mass at the centre of the beam. The effect of the point concentrated mass is studied in terms of the linear natural frequency and the ratios of nonlinear to linear radian frequencies, treating the point concentrated mass and the slenderness ratios as parameters. The present study reveals some interesting aspects about the effect of the symmetric point concentrated mass on the large amplitude free vibrations of uniform shear flexible symmetric beam configurations, like the uniform hinged–hinged and clamped–clamped beams.

2. Coupling equation

The kinematics of the Timoshenko beam theory (Fig. 1) can be written as

$$\overline{u}(x,z) = z\theta(x) = z(-\psi + \overline{\beta}), \tag{1}$$

$$\overline{w}(x,z) = w(x), \tag{2}$$

where $\overline{u}$ and $\overline{w}$ are the axial and transverse displacements at a generic point of the beam, z is the distance of the generic point from the neutral axis, w is the transverse displacement and θ is the total rotation anywhere on the beam axis, ψ is the bending rotation (Euler–Bernoulli theory) and $\overline{\beta}$ is the shear rotation or shear strain

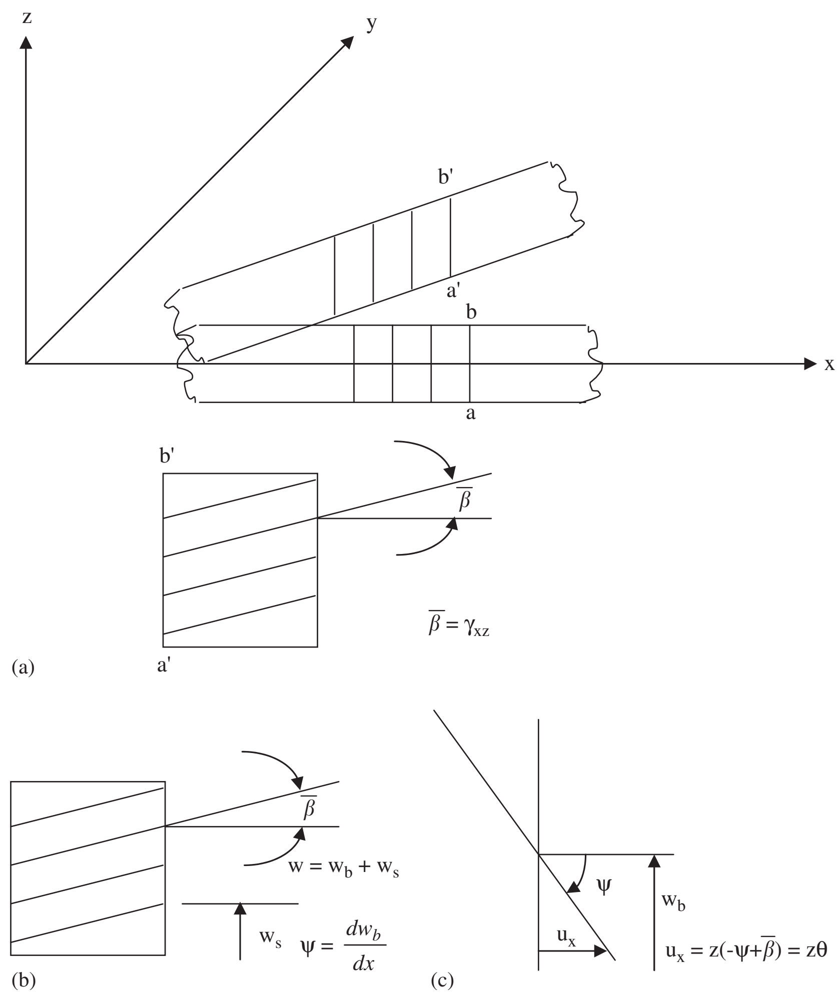

Fig. 1. Displacement fields for Timoshenko Beam: a) Shear deflection of a beam; b) Shear and c) bending (Euler–Bernoulli).

and x, z are the independent spatial variables. The axial and shear strains consistent with coordinate system (Fig. 1) are given by

$$\varepsilon_x = z\frac{\mathrm{d}\theta}{\mathrm{d}x}, \tag{3}$$

$$\gamma_{xz} = \frac{\mathrm{d}w}{\mathrm{d}x} + \theta. \tag{4a}$$

The expression for γ_{xz} can also be obtained from Ref. [13] as

$$\gamma_{xz} = \frac{\mathrm{d}\overline{w}}{\mathrm{d}x} + \frac{\mathrm{d}\overline{u}}{\mathrm{d}z} = \frac{\mathrm{d}w}{\mathrm{d}x} + \theta. \tag{4b}$$

It is to be noted here that Eqs. (4a) and (4b) are exactly the same confirming that the terminology used is consistent with the coordinate system taken.

Now, the expressions for the strain energy 'U' and the work done W by the externally applied loads are given by

$$U = \frac{EI}{2}\int_0^L \left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^2 \mathrm{d}x + \frac{kGA}{2}\int_0^L \left(\frac{\mathrm{d}w}{\mathrm{d}x} + \theta\right)^2 \mathrm{d}x, \tag{5}$$

$$W = \int_0^L p(x)w(x)\,\mathrm{d}x, \tag{6}$$

where EI is the flexural rigidity, GA is the shear rigidity, k is the shear coefficient (taken as 5/6, valid for beams of rectangular cross-section. For further discussion k, Timoshenko and Gere [14] may be referred, in the present study, $p(x)$ is the static lateral load per unit length acting on the beam, E is the Young's modulus, G is the shear modulus, I is the area moment of inertia, A is the area of cross-section, x is the axial coordinate and L is the length of the beam.

Applying the principle of minimization of total potential energy, as

$$\delta(U\text{-}W) = 0 \tag{7}$$

the following equilibrium equations can be obtained [15]:

$$kGA\left(\frac{\mathrm{d}^2w}{\mathrm{d}x^2} + \frac{\mathrm{d}\theta}{\mathrm{d}x}\right) + p = 0, \tag{8}$$

$$EI\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} - kGA\left(\frac{\mathrm{d}w}{\mathrm{d}x} + \theta\right) = 0. \tag{9}$$

Eqs. (8) and (9) are coupled equations and can be solved for obtaining the solution for the static analysis of the shear deformable beams.

A close observation of Eq. (8) shows that it is dependent on the load term 'p' and Eq. (9) is independent of the load terms 'p'. Hence, Eq. (9) is used to couple the total rotation θ and the transverse displacement w, so that the two undetermined coefficients problem (for single admissible functions) becomes a single undetermined coefficient problem and the resulting large amplitude free vibration problem becomes much simpler to solve.

3. Coupled displacement field method

The coupled displacement field method is explained in detail in this section with reference to a short, uniform hinged–hinged beam with a central point concentrated mass (Fig. 2a). To start with an admissible function for θ which satisfies all the applicable essential/natural boundary conditions and symmetric condition is assumed in the beam domain, and the coupled lateral displacement w distribution using Eq. (9), is evaluated. θ distribution along the length of the beam is assumed as

$$\theta = a\cos\frac{\pi x}{L}, \tag{10}$$

where 'a' is the nondimensional maximum central transverse amplitude of the beam, which is also the nondimensional maximum transverse amplitude a_m in this case.

Eq. (9) can be rewritten as

$$\frac{\mathrm{d}w}{\mathrm{d}x} = -\theta + \gamma\theta'', \tag{11}$$

where

$$\gamma = \frac{EI}{kGA}.$$

Substituting the function θ in Eq. (11), we obtain the coupled displacement field for 'w', after integration as

$$w = \lambda\sin\frac{\pi x}{L}, \tag{12}$$

where

$$\lambda = -\frac{L}{\pi}\left\{1 + \left(\frac{\pi}{L}\right)^2\frac{EI}{kGA}\right\}a. \tag{13}$$

It may be noted here that because of the coupled displacement field 'w', the transverse displacement distribution, contains the same undetermined coefficient 'a' as the θ distribution and satisfies all the applicable essential boundary and symmetric conditions

$$w(0) = w(L) = \left.\frac{\mathrm{d}w}{\mathrm{d}x}\right|_{x=L/2}. \tag{14}$$

4. Large amplitude free vibrations

Large amplitude vibrations can be studied, once the coupled displacement field for the lateral displacement 'w', for an assumed 'θ' distribution is evaluated using the principle of conservation of total energy at any instant of time, neglecting damping, which states that

$$U + T + W_1 = \text{Constant}, \tag{15}$$

where U is the strain energy, T is the kinetic energy, W_1 is the work done by the tension developed in the beam because of large amplitudes (deformations).

The expressions for U, T and W_1 are given by

$$U = \frac{EI}{2}\int_0^L\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^2\mathrm{d}x + \frac{kGA}{2}\int_0^L\left(\frac{\mathrm{d}w}{\mathrm{d}x} + \theta\right)^2\mathrm{d}x, \tag{16}$$

$$T = \frac{\rho A}{2}\int_0^L \dot{w}^2\,\mathrm{d}x + \frac{\rho I}{2}\int_0^L \dot{\theta}^2\,\mathrm{d}x + \frac{1}{2}M\dot{w}^2|_{x=L/2}, \tag{17}$$

$$W_1 = \frac{T_a}{2}\int_0^L \frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2\mathrm{d}x, \tag{18}$$

where 'ρ' is the mass density, T_a is the tension developed in the beam because of large deformations, and $(\dot{\ })$ denotes differentiation with respect to time.

The expression for T_a is obtained following Woinowsky–Krieger [1] as

$$T_a = \frac{EI}{2Lr^2}\int_0^L \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \mathrm{d}x, \tag{19}$$

where 'r' is the radius of gyration and T_a is evaluated in terms of the amplitude parameter $q\left(=\frac{a}{r}\right)$.

Substituting the expressions for θ and w, obtained from the coupled displacement field, the expressions for U, T and W_1 are given by

$$U = \frac{EI\pi^2 a^2}{4L}\left[1+\frac{\pi^2 EI}{L^2 kGA}\right], \tag{20}$$

$$T = \dot{a}^2\left(\frac{L}{\pi}\right)^2\left\{\frac{\rho AL}{4}\left[1+\frac{\pi^2 EI}{L^2 kGA}\right]^2 + \frac{\rho AL}{4}\frac{I}{A}\left(\frac{\pi}{L}\right)^2 + \frac{M}{2}\left[1+\frac{\pi^2 EI}{L^2 kGA}\right]^2\right\}, \tag{21}$$

$$W_1 = \frac{EI}{32r^2}\left(\frac{\pi}{L}\right)^2 L a^4\left[1+\frac{\pi^2 EI}{L^2 kGA}\right]^2. \tag{22}$$

Substituting the expressions for U, T and W_1 in Eq. (15) and simplifying, noting that $I = Ar^2$, we get the following energy balance equation:

$$\dot{q}^2 + \alpha_1 q^2 + \alpha_2 q^4 = \text{constant} \tag{23}$$

with the following expressions for α_1 and α_2

$$\alpha_1 = \frac{EI\left(\frac{\pi}{L}\right)^4\left[1+\frac{\pi^2 E}{kG\beta^2}\right]}{m\left\{\left[1+\frac{\pi^2 E}{kG\beta^2}\right]\left[1+\frac{2M}{mL}\right]+\frac{\pi^2}{\beta^2}\right\}}, \tag{24}$$

$$\alpha_2 = \frac{EI\left(\frac{\pi}{L}\right)^4\left[1+\frac{\pi^2 E}{kG\beta^2}\right]^2}{8m\left\{\left[1+\frac{\pi^2 E}{kG\beta^2}\right]^2\left[1+\frac{2M}{mL}\right]+\frac{\pi^2}{\beta^2}\right\}}. \tag{25}$$

In Eqs. (24) and (25) E and G in the square brackets can be eliminated by the standard relation,

$$G = \frac{E}{2(1+\upsilon)}, \tag{26}$$

where v is the Poisson ratio (taken as 0.3 in the present study) and $\beta = L/r$, the slenderness ratio of the beam. The Harmonic balance method discussed in the next section is used to solve Eq. (23) to obtain closed-form expressions for the ratio of nonlinear to linear radian frequencies in terms of q and β.

5. Harmonic balance method

The direct numerical integration method (DNI) [10], proposed by the first author, can be used to solve Eq. (23) to the desired degree of accuracy. However, for an elegant closed-form solution, one can advantageously use the HBM which is briefly discussed in this section.

Differentiating Eq. (23), we obtain

$$\ddot{q} + \alpha_1 q + 2\alpha_2 q^3 = 0. \tag{27}$$

This is the famous Duffing equation and is solved by assuming

$$q = q_m \sin \omega_{\mathrm{NL}} t, \tag{28}$$

where ω_{NL} is the nonlinear radian frequency and q_m is the maximum amplitude ratio a_m/r.

Substituting Eq. (28) in Eq. (27), we obtain

$$-\omega_{\mathrm{NL}}^2 \sin \omega_{\mathrm{NL}} t + \alpha_1 \sin \omega_{\mathrm{NL}} t + 2\alpha_2 q_m^2 \sin^3 \omega_{\mathrm{NL}} t = 0, \tag{29}$$

where $\sin^3 \omega_{\mathrm{NL}}\, t$ can be written as

$$\sin^3 \omega_{\mathrm{NL}} t = \tfrac{3}{4} \sin \omega_{\mathrm{NL}} t - \tfrac{1}{4} \sin 3\omega_{\mathrm{NL}} t. \tag{30}$$

From Eqs. (30) and (29), dropping the term corresponding to the third harmonic of ω_{NL}, Eq. (29) can be written as

$$\omega_{\mathrm{NL}}^2 = \alpha_1 + \tfrac{3}{2}\alpha_2 q_m^2. \tag{31}$$

From Eq. (31), if $q_m = 0$, which corresponds to the case of linear free vibrations, the linear radian frequency, $\omega_L = \omega_{\mathrm{NL}}$ and

$$\omega_L^2 = \alpha_1 \tag{32}$$

and hence, the ratio of the nonlinear to the linear radian frequency is given by

$$\left(\frac{\omega_{\mathrm{NL}}}{\omega_L}\right)^2 = 1 + \frac{3}{2}\left(\frac{\alpha_2}{\alpha_1}\right) q_m^2. \tag{33}$$

From Eqs. (24), (25) and (31), we obtain, after simplification,

$$\left(\frac{\omega_{\mathrm{NL}}}{\omega_L}\right)^2 = 1 + \frac{3}{16}\left[1 + \frac{\pi^2 E}{kG\beta^2}\right]\left(\frac{a_m}{r}\right)^2. \tag{34}$$

Eq. (34) is an elegant form to calculate the ratios of $\omega_{\mathrm{NL}}^2/\omega^2{}_L$ for various values of the maximum amplitude and slenderness ratios of the beam and can be written, expressing G in terms of E, as

$$\left(\frac{\omega_{\mathrm{NL}}}{\omega_L}\right)^2 = 1 + \frac{3}{16}\left[1 + \frac{2\pi^2(1+\upsilon)}{k\beta^2}\right]\left(\frac{a_m}{r}\right)^2. \tag{35}$$

For very large β, i.e., for slender beams, where shear deformation can be neglected, Eq. (35) becomes

$$\left(\frac{\omega_{\mathrm{NL}}}{\omega_L}\right)^2 = 1 + \frac{3}{16}\left(\frac{a_m}{r}\right)^2 \tag{36}$$

which is a standard result [10].

6. Clamped–clamped beam

In this section, we consider the large amplitude free vibrations of a uniform, short clamped–clamped beam (Fig. 2b) using the coupled displacement field method. As the proposed method is explained in detail for the case of the hinged–hinged beam, the same is briefly discussed below.

In the case of the clamped–clamped beam, the admissible function for the total rotation is taken, as

$$\theta = a \sin \frac{2\pi x}{L}, \tag{37}$$

where 'a' has the same definition as given for the case of the hinged–hinged beam.

And the coupled displacement field 'w' is obtained, from Eq. (9), as

$$w = a\frac{L}{2\pi}\left\{1 + \left(\frac{2\pi}{L}\right)^2 \frac{EI}{kGA}\right\}\left[\cos\frac{2\pi x}{L} - 1\right]. \tag{38}$$

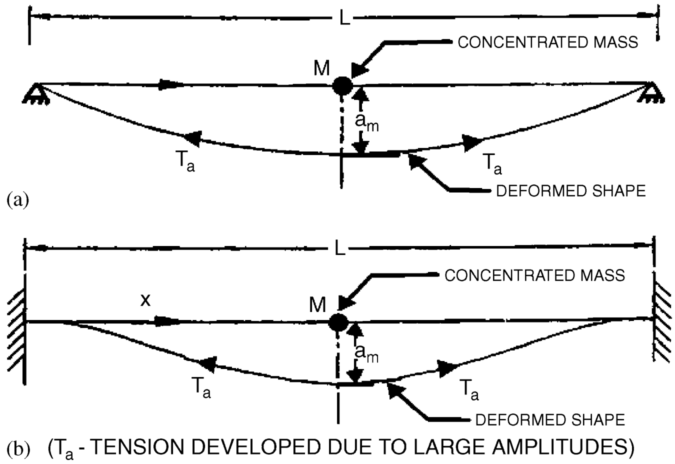

Fig. 2. Uniform beams with axially immovable ends with a central point concentrated mass: a) Hinged–hinged beam; b) Clamped–Clamped beam.

It is to be noted here that these two displacement fields satisfy the essential boundary conditions (four in number) of the clamped–clamped beam at both the ends, and both the θ and w expressions contain only one undetermined coefficient 'a'.

The expressions for U, T and W_1 are the same as given in Eqs. (16)–(18) in terms of 'w' and θ and their derivatives. After substituting the coupled displacement field admissible functions for θ and w, we get the expressions for U, T and W_1 after integrating, as

$$U = \frac{EIL}{4} a^2 \left(\frac{2\pi}{L}\right)^2 \left[1 + \left(\frac{2\pi}{L}\right)^2 \frac{EI}{kGA}\right], \tag{39}$$

$$T = \frac{3mL}{4} \dot{a}^2 \left(\frac{L}{\pi}\right)^2 \left\{\left[1 + \left(\frac{2\pi}{L}\right)^2 \frac{EI}{kGA}\right]\left[\frac{1}{4} + \frac{2M}{3mL}\right] + \frac{I}{3A}\left(\frac{\pi}{L}\right)^2\right\}, \tag{40}$$

$$W_1 = \frac{EI}{32r^2}\left(\frac{\pi}{L}\right)^2 \left[1 + \left(\frac{2\pi}{L}\right)^2 \frac{EI}{kGA}\right]. \tag{41}$$

For the clamped–clamped beam, from Eq. (15), we obtain the same Eq. (23) and the corresponding expressions for α_1 and α_2 are

$$\alpha_1 = \omega_L^2 = \frac{4EI\left(\frac{\pi}{L}\right)^4\left[1 + \left(\frac{2\pi}{L}\right)^2 \frac{EI}{kGA}\right]}{3m\left\{\left[1 + \left(\frac{2\pi}{L}\right)^2 \frac{EI}{kGA}\right]^2\left[\frac{1}{4} + \frac{2M}{3mL}\right] + \frac{I}{3A}\left(\frac{\pi}{L}\right)^2\right\}}, \tag{42}$$

$$\alpha_2 = \frac{EI\left(\frac{\pi}{L}\right)^4\left[1 + \left(\frac{2\pi}{L}\right)^2 \frac{EI}{kGA}\right]^2}{24mr^2\left\{\left[1 + \left(\frac{2\pi}{L}\right)^2 \frac{EI}{kGA}\right]^2\left[\frac{1}{4} + \frac{2M}{3mL}\right] + \frac{I}{3A}\left(\frac{\pi}{L}\right)^2\right\}}. \tag{43}$$

Following the procedure given for the hinged–hinged beam, the ratio of the nonlinear to linear radian frequency, after simplification of Eq. (33) with the corresponding values of α_1 and α_2 for the clamped–clamped beam and expressing G in terms of E, is given by

$$\left(\frac{\omega_{\mathrm{NL}}}{\omega_L}\right)^2 = 1 + \frac{3}{64}\left[1 + \frac{8\pi^2(1+\upsilon)}{k\beta^2}\right]\left(\frac{a_m}{r}\right)^2. \tag{44}$$

For very large slenderness ratio β, Eq. (44) becomes

$$\left(\frac{\omega_{\mathrm{NL}}}{\omega_L}\right)^2 = 1 + \frac{3}{64}\left(\frac{a_m}{r}\right)^2, \tag{45}$$

which is a standard result for a slender, uniform clamped–clamped beam with ends immovable axially [10].

7. Numerical results and discussion

Using the coupled displacement field method proposed in this paper, the large amplitude free vibration behaviour of uniform short beams with a central point concentrated mass, wherein the effects of shear deformation and rotary inertia are considered. Two commonly encountered beams, namely, hinged–hinged and clamped–clamped beams are studied in the present paper. As the point concentrated mass is located at the centre of the beam, which is a symmetric configuration, only the translational inertia of the point concentrated mass is taken into account in the present study. The rotary inertia of the point concentrated mass is zero as this mass is located at the symmetric (central) point of the beam where the slope of the mode shape corresponding to the fundamental frequency of the vibrating beam is zero. Widely used simple trigonometric admissible functions, which represent the fundamental mode of the vibrating beam are considered here. The two admissible functions contain same undetermined coefficient because of the coupled displacement field method developed here in. The numerical results are presented in the digital form to facilitate the other researchers in this field to compare their solutions directly for the present problem considered.

In the present study elegant closed-form expressions are developed for $\omega_{\mathrm{NL}}/\omega_L$ and for ω_L. It is very interesting to note that expressions for $\omega_{\mathrm{NL}}/\omega_L$ are independent of the point concentrated mass (mass parameter) and depends only on the slenderness ratio and the amplitude ratio. However, ω_L, the linear fundamental radian frequency, is dependent on the mass parameter and slenderness ratio, and consequently ω_{NL}, the nonlinear fundamental radian frequency is also dependent on the mass parameter and slenderness ratios. Further, for a given slenderness ratio both the values of ω_L and ω_{NL} are increased by a same multiplying factor, which is dependent on the amplitude ratio and hence the ratios $\omega_{\mathrm{NL}}/\omega_L$ are independent of the mass parameter and are dependent on the slenderness and amplitude ratios.

Since the ratios of $\omega_{\mathrm{NL}}/\omega_L$ are independent of the mass ratio, the values of ω_L for hinged–hinged are given first in Table 1 and for clamped–clamped beam are given in Table 2 for various mass and slenderness ratios. In Table 3 the ratios of $\omega_{\mathrm{NL}}/\omega_L$ are given for the hinged–hinged beam and the same are give in Table 4 for clamped–clamped beam for different amplitude and slenderness ratios. Comparison of the present results with those available in the open literature are made wherever possible, to show the efficacy of the present proposed method and it is observed that the agreement is good.

Table 1
$(\lambda_f)^{1/2}$ Values for a uniform shear flexible hinged–hinged beam

M/mL	β					Slender beam $\beta \to \infty$
	10	25	50	100	500	
0	8.3913	9.5667	9.7906	9.8496	9.8688	9.8696
	(8.3875)[a]	(9.5666)	(9.7903)	(9.8494)	(9.8686)	(9.8696)[b]
0.25	6.9746	7.8297	7.9991	8.0435	8.0579	8.0585
0.5	6.0161	6.7887	6.9296	6.9664	6.9784	6.9787
0.75	5.3961	6.0764	6.1992	6.2313	6.2417	6.2421
1	4.9353	5.5496	4.9098	5.6885	5.6978	5.6982
2	3.8373	4.3028	4.3852	4.4066	4.4135	4.4138
3	3.2484	3.6380	3.7066	3.7244	3.7361	3.7361
4	2.2484	3.2091	3.2691	3.2846	3.2897	3.2949
5	2.5952	2.9032	2.9571	2.9711	2.9756	2.9804

[a]Values given in the parentheses are taken from Ref. [15] for $\beta = 10$–500.
[b]Values given in the parentheses are taken from Ref. [16] for $\beta \to \infty$.

Table 2
$(\lambda_f)^{1/2}$ Values for a uniform shear flexible clamped–clamped beam

M/mL	β					Slender beam $\beta \to \infty$
	10	25	50	100	500	
0	14.6902	20.3931	22.0992	22.6161	22.7914	22.7969
	(15.0532)[a]	(20.6809)	(22.1991)	(22.6385)	(22.7860)	(22.7969)[b]
0.25	8.7703	11.9982	12.8366	13.0764	13.1561	13.1595
0.5	6.8053	9.3029	9.9463	10.1298	10.1907	10.1933
0.75	5.7559	7.8656	8.4073	8.5615	8.6127	8.6149
1	5.0783	6.9385	7.4151	7.5507	7.5957	7.5976
2	3.6973	5.0504	5.3960	5.4941	5.5267	5.5281
3	3.6976	4.1652	4.4498	4.5306	4.5575	4.5586
4	3.0499	3.6256	3.8732	3.9434	3.9668	3.9677
5	2.6549	3.2529	3.4749	3.5379	3.5588	3.5596

[a]Values given in the parentheses are taken from Ref. [15] for $\beta = 10$–500.
[b]Values given in the parentheses are taken from Ref. [17] for $\beta \to \infty$.

Table 3
ω_{NL}/ω_L Values for a uniform shear flexible hinged–hinged beam

a_m/r	β										
	25		50		100		500		$\to \infty$		
	Present study	DNI [10]	Present study	DNI [10]	Present study	DNI [10]	Present study	DNI [10]	Present study	DNI [10]	FEM [18]
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.2	1.0039	1.0039	1.0038	1.0038	1.0038	1.0038	1.0037	1.0037	1.0037	1.0037	1.0037
0.4	1.0156	1.0156	1.0151	1.0151	1.0149	1.0149	1.0149	1.0149	1.0149	1.0149	1.0148
0.6	1.0348	1.0347	1.0336	1.0335	1.0333	1.0332	1.0332	1.0331	1.0332	1.0331	1.0331
0.8	1.0611	1.0608	1.0590	1.0588	1.0585	1.0582	1.0583	1.0581	1.0583	1.0580	1.0581
1.0	1.0940	1.0933	1.0905	1.0902	1.0900	1.8940	1.0897	1.0892	1.0897	1.0892	1.0892
2.0	1.3368	1.3313	1.3264	1.3212	1.3237	1.3186	1.3229	1.3178	1.3229	1.3178	1.3178
3.0	1.6645	1.6501	1.6457	1.6318	1.6409	1.6272	1.6394	1.6257	1.6394	1.6257	1.6257
4.0	2.0366	2.0116	2.0092	1.9850	2.0023	1.9783	2.0001	1.9761	2.0000	1.9760	1.9761
5.0	2.4328	2.3968	2.3969	2.3619	2.3879	2.3531	2.3850	2.3506	2.3848	2.3501	2.3502

Table 1 gives the values of the fundamental frequency parameter, defined as

$$\lambda_f = \left[\frac{m\omega_L^2 L^4}{EI}\right]^{1/2},$$

for a uniform shear flexible hinged–hinged beam for several values of the central mass ratios defined as $\left[\overline{M} = M/mL\right]$. It can be seen from this table that the frequency parameter λ_f is significantly affected in the presence of $\overline{M}$ even for the value of $\overline{M}$ is as low as 0.25 irrespective of the values of the slenderness ratios, defined as $(\beta = L/r)$, of the beam considered. However, for higher values of $\overline{M}$ the decrease in the value of λ_f is not as significant as in the initial lower values of $\overline{M}$. The values of λ_f, for $\overline{M} = 0$ for $\beta = 25$–500 and for $\beta \to \infty$ for this beam configuration are matching very well with the values given in Refs. [15,16], respectively.

For the case of clamped–clamped uniform, the trend of the variation of λ_f, for any given β, with $\overline{M}$ is similar to that of the hinged–hinged beam, as can be seen in Table 2. In this case it may be noted that for the initial small values of $\overline{M}$ the reduction in the frequency parameter is much higher for this beam configuration compared to the hinged–hinged beam. But for higher values of $\overline{M}$ the trend is the same as seen for the

Table 4
ω_{NL}/ω_L Values for a uniform shear flexible clamped–clamped beam

a_m/r	β										
	25		50		100		500		$\to\infty$		
	Present study	DNI [10]	Present study	DNI [10]	Present study	DNI [10]	Present study	DNI [10]	Present study	DNI [10]	FEM [18]
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.2	1.0011	1.0011	1.0010	1.0010	1.0009	1.0009	1.0009	1.0009	1.0009	1.0009	1.0009
0.4	1.0045	1.0045	1.0039	1.0039	1.0038	1.0038	1.0037	1.0037	1.0037	1.0037	1.0036
0.6	1.0100	1.0100	1.0088	1.0088	1.0085	1.0085	1.0084	1.0084	1.0084	1.0084	1.0080
0.8	1.0178	1.0178	1.0156	1.0156	1.0151	1.0151	1.0149	1.0149	1.0149	1.0149	1.0142
1.0	1.0277	1.0276	1.0243	1.0242	1.0235	1.0234	1.0232	1.0231	1.0232	1.0231	1.0221
2.0	1.1065	1.1058	1.0940	1.0933	1.0908	1.0902	1.0898	1.0892	1.0897	1.0892	1.0854
3.0	1.2268	1.2239	1.2011	1.1987	1.1946	1.1924	1.1925	1.1903	1.1924	1.1902	1.1825
4.0	1.3776	1.3711	1.3368	1.3313	1.3264	1.3212	1.3230	1.3179	1.3229	1.3178	1.3055
5.0	1.5501	1.5389	1.4932	1.4836	1.4786	1.4694	1.4739	1.4649	1.4737	1.4647	1.4474

hinged–hinged beam. Again, for this beam configuration also, the values of λ_f, for $\overline{M}=0$ for $\beta = 25$–500 and for $\beta\to\infty$ are matching very well with the values given in Refs. [15,17], respectively.

Tables 3 and 4 give the ratios of ω_{NL}/ω_L, for the hinged–hinged and clamped–clamped shear flexible uniform beams, obtained in the present work for several values of amplitude and slenderness ratios. As has already been mentioned that these values are independent of $\overline{M}$. The same results from Ref. [18] are included in these tables for the sake of comparison, and the agreement of the values of ω_{NL}/ω_L is good, for various values of the amplitude and slenderness ratios, q_m and β, respectively.

8. Conclusions

Coupled displacement field method is applied in this paper, to study the large amplitude free vibration behaviour of uniform shear flexible hinged–hinged and clamped–clamped beams with a centrally placed point concentrated mass. In general the method used in this paper in conjunction with the classical energy method reduces the number of undetermined coefficients involved in the admissible function for the lateral displacement and the total rotation from '2n' to 'n' and hence reduces the computational effort to solve the coupled nonlinear temporal equations, for the studies made on the fundamental frequency, which is an important design parameter. When suitable single term admissible functions for the total rotation and the lateral displacement are used the number of nonlinear temporal equations are reduced from two to one in the present method. This equation can effectively be solved using the harmonic balance method to obtain simple and accurate closed-form solutions for obtaining the ratios of nonlinear to linear radian frequencies as a function of the slenderness, mass and maximum amplitude ratios.

From a careful study of the numerical results, obtained from the present study, given in the digital form, for both the hinged–hinged and clamped–clamped beams with a central point concentrated mass, the following major conclusions can be arrived at.

1. The ratios of the nonlinear to linear radian frequencies are independent of the mass ratio, which is an unanticipated interesting phenomenon.
2. The effect of the point concentrated mass is to reduce the linear frequency parameter and this reduction is very significant up to the values of the point concentrated mass equal to the beam mass.
3. This effect is more severe in the case of the clamped-clamped beam compared to the hinged–hinged beam.
4. Comparision of the present results, wherever possible, for the first mode of vibration, with those available in the open literature shows that the coupled displacement field method gives accurate values for the linear fundamental frequency parameter and the ratios of nonlinear to linear radian frequencies.

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