Available online at www.sciencedirect.com

ScienceDirect

Journal of Sound and Vibration 298 (2006) 385–392

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

Analytic 3-D *p*-element for vibration analyses of plates, Part 2: Laminated plates

B. Zhu[a], C.T.O. Leung[a], A.Y.T. Leung[a,*], Cheng Xiao[a,b], Si Yuan[b]

[a]*Department of Building and Construction, City University of Hong Kong, China*
[b]*Department of Civil Engineering, Tsinghua University, China*

Received 23 June 2004; received in revised form 11 May 2005; accepted 29 May 2006
Available online 1 August 2006

Abstract

An analytic 3-D *p*-version arbitrary quadrilateral plate element is applied to solve the free vibration of laminated plates. The computational accuracy is considerably improved due to the additional hierarchical shape functions and analytic integration. The study on a single layer orthotropic plate with simply supported boundary conditions shows the fast convergence and great accuracy of the present element. Then the analyses of rectangular plates with simply supported, cantilevered and fully clamped boundary conditions are carried out. Because of the great fitness to all kinds of geometry shapes, the present element is easily applied to various polygonal plates.

1. Introduction

Two plate theories are most widely used for laminated plates. One is the classical plate theory (CPT) based on Kirchhoff's assumptions, which restricts the model to thin plates; the other is the first-order shear deformation theory (FSDT) based on Reissner-Mindlin's moderately thick plate theory, which introduces the through-thickness shear and inertia effects and leads to considerable accurate results for both thin and moderately thick laminated plates [1]. Some high-order plate theories such as the three-order shear deformation theory, with the elimination of the shear correct factor, have also received much interest [2–5].

To obtain more accurate solutions, completely 3-D analyses are superior. Through analytical and semi-analytical methods, bending, vibration and buckling problems were introduced [6,7]. They are restricted to plates with rectangular shapes only and could not be easily extended to the application of finite elements. A trapezoidal *p*-element is first put forward by Leung et al. [8] for the free vibrating plane problems. With the exact analytical integration, the accuracy of natural frequencies is greatly improved. Then this kind of *p*-element is applied to the vibration of membranes, thick plates and laminated plates and with excellent results [9–11].

Part 1 has considered the free vibration of isotropic plates; in this part, laminated plates will be of interest. The vibration analyses of rectangular, triangular and polygonal laminated plates with different laminate sequences, boundary conditions are carried out to show the performance of the proposed quadrilateral plate element.

*Corresponding author. Tel.: +852 2788 7600; fax: +852 2788 9643.
E-mail address: Andrew.leung@cityu.edu.hk (A.Y.T. Leung).

0022-460X/$ - see front matter © 2006 Elsevier Ltd. All rights reserved.
doi:10.1016/j.jsv.2006.05.031

Nomenclature

E_{11}, E_{22}, E_{33}	Young's modulus
G_{12}, G_{23}, G_{13}	shear modulus
ν_{12}, ν_{23}, ν_{13}	Poisson's ratio
β	fibre angle
h, t	thickness of the laminate
ρ	mass per unit volume
D_0	flexural rigidity ($= E_{22}h^3/12(1-\nu_{12}\nu_{21})$)
ω	natural frequency
λ	non-dimensional frequency parameters
σ_i, ε_i	stress and strain components
Q_{ij}	reduced stiffness in the material axes of the layer
$\bar{Q}_{ij}$	reduced stiffness in the global coordinate
p_x, p_y, p_z	number of additional hierarchical terms in three co-ordinates
$\mathbf{J}$	Jacobian matrix
$\mathbf{u}$	vector of u, v and w
$\mathbf{K}^e$	stiffness matrix

2. Formulation

2.1. Laminated plate theory

A symmetrically laminated plate with the co-ordinate system at the mid-plane of the laminate is shown in Fig. 1. Each layer of the laminate is of equal thickness for convenience. The fibre direction is indicated by an angle β, which is the positive rotation angle of the principal material axes from the arbitrary xy-axes. The modulus of elasticity for a layer parallel to fibres is E_{11}, perpendicular to fibres is E_{22} and perpendicular to xy-plane is E_{33}. Without loss of generality, in this paper, the x-axes are taken parallel to one side of the geometry of the plates.

For the 3-D analyses of the laminated plates, the stiffness matrix relates to the fibre direction. The constitutive equations in the local coordinate for the kth layer can be expressed as

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_6 \\ \sigma_4 \\ \sigma_5 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 & 0 \\ & Q_{22} & Q_{23} & 0 & 0 & 0 \\ & & Q_{33} & 0 & 0 & 0 \\ & & & Q_{66} & 0 & 0 \\ & \text{sym} & & & Q_{44} & 0 \\ & & & & & Q_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_6 \\ \varepsilon_4 \\ \varepsilon_5 \end{Bmatrix}, \tag{1}$$

where σ_i, ε_i and Q_{ij} are the stress, strain and reduced stiffness coefficients in the material axes of the layer. To transform the above equations from the local coordinate system to the global coordinate, the laminate

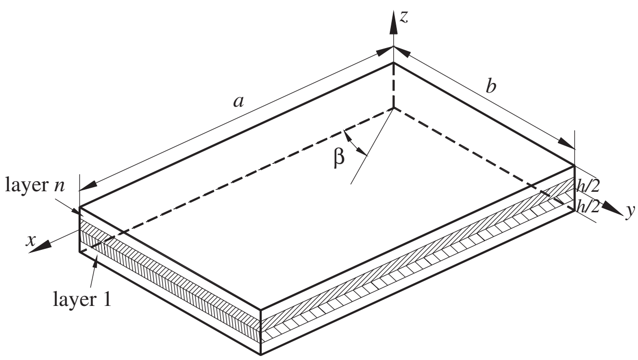

Fig. 1. The configuration of the laminated composite plate.

constitutive equation can be expressed as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix}^{(k)} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{13} & \bar{Q}_{16} & 0 & 0 \\ & \bar{Q}_{22} & \bar{Q}_{23} & \bar{Q}_{26} & 0 & 0 \\ & & \bar{Q}_{33} & \bar{Q}_{36} & 0 & 0 \\ & & & \bar{Q}_{66} & 0 & 0 \\ & \text{sym} & & & \bar{Q}_{44} & \bar{Q}_{45} \\ & & & & & \bar{Q}_{55} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix}, \tag{2}$$

where $\bar{Q} = T^{-1}QT^{-\mathrm{T}}$, here

$$T = \begin{bmatrix} \cos^2\beta & \sin^2\beta & 0 & 2\cos\beta\sin\beta & 0 & 0 \\ \sin^2\beta & \cos^2\beta & 0 & -2\cos\beta\sin\beta & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\cos\beta\sin\beta & \cos\beta\sin\beta & 0 & \cos^2\beta-\sin^2\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\beta & -\sin\beta \\ 0 & 0 & 0 & 0 & \sin\beta & \cos\beta \end{bmatrix}. \tag{3}$$

2.2. Element formulation

The co-ordinate system used to define a 3-D quadrilateral uniform plate element is the same as that mentioned in Part 1. The only difference is the formulation of the stiffness matrix. For the kth layer, let z_k, z_{k+1} denote values of the z-coordinate at the bottom and the top, respectively. Then the element stiffness matrix is given by

$$\mathbf{K}^e = \int_V \mathbf{B}^{\mathrm{T}}\bar{\mathbf{Q}}B\,\mathrm{d}V = \sum_{k=1}^{n}\int_{\zeta_k}^{\zeta_{k+1}}\int_{-1}^{1}\int_{-1}^{1}\mathbf{B}^{\mathrm{T}}\bar{\mathbf{Q}}^k\mathbf{B}\cdot|\mathbf{J}|\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta, \tag{4}$$

where $\zeta_k = \frac{2}{h}(z_k - z_1) - 1$, $\zeta_{k+1} = \frac{2}{h}(z_{k+1} - z_1) - 1$ and n is the total number of layers.

3. Numerical results and discussion

In this section, thin, moderately thick and thick laminate plates are considered with different stacking sequences, geometry parameters and boundary conditions. To simplify the computation and presentation, the same numbers of the additional hierarchical terms in the three dimensions are taken, that is, $p_x = p_y = p_z = p$. Unless stated otherwise, the number of hierarchical terms p is set to be 3, and the orientation of x-axes is parallel to any side of the polygons.

3.1. Vibration of a single layer orthotropic plate

A simply supported square plate is used to perform the convergence study and demonstrate the accuracy of the present p-element. The plate is made of aragonite, the material constant of which is given as

$$Q = \begin{bmatrix} 160 & 37.3 & 1.72 & 0 & 0 & 0 \\ & 86.87 & 15.72 & 0 & 0 & 0 \\ & & 84.81 & 0 & 0 & 0 \\ & & & 42.06 & 0 & 0 \\ & \text{sym} & & & 25.58 & 0 \\ & & & & & 42.68 \end{bmatrix}.$$

The simply supported boundary condition here is defined as follows:

$$x = 0, \quad a : u = w = 0,$$
$$y = 0, \quad b : v = w = 0.$$

The first five non-dimensional frequencies ($\lambda = \omega h\sqrt{\rho/Q_{11}}$) of the plate are computed. As shown in Fig. 2, three meshing schemes are used: one rectangular element (mesh I), two quadrilateral elements (mesh II) and four quadrilateral (mesh III) elements. With the increasing number of hierarchical terms p, the results are shown in Table 1 along with the available 3-D finite element solutions [12]. The fast convergence is observed with the increasing numbers of hierarchical terms and elements. Furthermore, the results are in excellent agreement with the existing FEM solutions.

3.2. Vibration of laminated square plates

Firstly, for different materials, E/E and G/E with material properties are shown in Table 2, the first six fundamental frequency parameters $\lambda = \omega a^2\sqrt{\rho h/D_0}(D_0 = E_{11}h^3/12(1 - v_{12}v_{21}))$ are presented in Table 3 for single- and five-layer angle-ply cantilever plates. Figs. 3 and 4 gives the coordinate systems of the cantilever plate. The material properties are taken from Ref. [13]. It should be noted that the 3-D elasticity analysis needs the complete set of mechanical properties, the transverse Poisson ratio v_{23}^p is obtained from Ref. [14]. With $h/a = 0.05$, the results are compared with the CPT solutions of Narita et al. [15]. The frequencies for cross-ply E/E and G/E laminated plates are presented in Table 4. The lamination sequences are 0°, 0°/90°/0° and

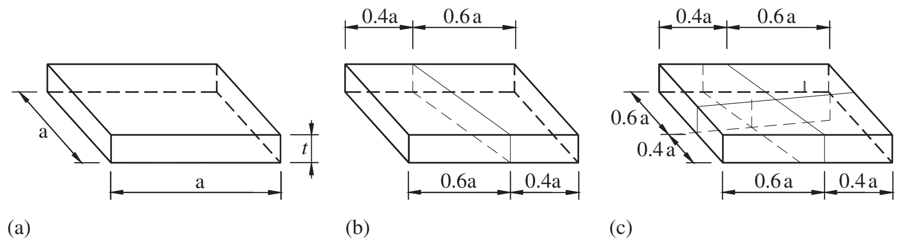

Fig. 2. Meshes of square plates: (a) mesh I; (b) mesh II; and (c) mesh III.

Table 1
Non-dimensional frequencies ($\lambda = \omega h\sqrt{\rho/Q_{11}}$) of a simply supported orthotropic square plate ($a/h = 2$)

Method		Mode sequence				
		1	2	3	4	5
3-D FEM solution [12]		0.7295	0.8054	0.8054	1.0823	1.2144
Mesh I	$p = 1$	0.7874	0.8107	0.8107	1.2317	1.6049
	$p = 2$	0.7383	0.8107	0.8107	1.0865	1.2830
	$p = 3$	0.7294	0.8054	0.8054	1.0827	1.2515
	$p = 4$	0.7293	0.8054	0.8054	1.0823	1.2175
Mesh II	$p = 1$	0.7853	0.8086	0.8107	1.1522	1.3314
	$p = 2$	0.7363	0.8055	0.8106	1.0857	1.2314
	$p = 3$	0.7295	0.8054	0.8054	1.0826	1.2148
	$p = 4$	0.7293	0.8054	0.8054	1.0823	1.2136
Mesh III	$p = 1$	0.7693	0.8084	0.8084	1.0903	1.3106
	$p = 2$	0.7302	0.8054	0.8054	1.0826	1.2265
	$p = 3$	0.7294	0.8054	0.8054	1.0823	1.2145
	$p = 4$	0.7293	0.8054	0.8054	1.0823	1.2136

0°/90°/0°/90°/0°, respectively. It is obvious that when the plate is very thin, the difference of the solution between 3-D theory and CPT is little. The results in Tables 3 and 4 also prove it.

To test the performance of the present element in solving problems with fully clamped boundary conditions, a square laminated plate with different lamination and thickness ratios are taken into accounted. Ye [16] gave the first frequency parameters for this problem. Comparison with the results of ANSYS and Ye [16] is carried out in Table 5. It can be seen that good agreement are achieved for various lamination.

3.3. Vibration of triangular plates

The advantage of the finite element method over the analytical and semi-analytical methods is that the former can be easily applied to various geometries and boundary conditions, while it is rather difficult for the

Table 2
Material properties for E/E and G/E

Material	E_{11}	E_{22} ($=E_{33}$)	G_{12} ($=G_{13}$)	G_{23}	ν_{12} ($=\nu_{13}$)	ν^p_{23}
E/E	60.7	24.8	11.99	8.48	0.23	0.462
G/E	138	8.96	7.1	2.68	0.30	0.675

Table 3
Fundamental frequency parameters ($\lambda = \omega a^2\sqrt{\rho h/D_0}$) for angle-ply square cantilever plates

Mode	1	2	3	4	5	6
(E/E) single-layer (30°)						
Ref. [15]	2.954	7.164	18.14	20.01	26.45	41.67
Present	2.957	7.045	17.83	19.63	25.69	36.24
(E/E) five-layer (30°/−30°/30°/−30°/30°)						
Ref. [15]	3.019	7.398	18.32	20.65	26.95	44.04
Present	3.022	7.283	18.00	20.25	26.17	38.61
(G/E) single-layer (30°)						
Ref. [15]	2.095	4.605	9.999	13.56	18.53	20.96
Present	2.073	4.468	9.625	12.72	17.54	19.79
(G/E) five-layer (30°/−30°/30°/−30°/30°)						
Ref. [15]	2.545	5.952	12.63	16.42	21.55	26.72
Present	2.506	5.674	11.77	15.42	19.70	24.73

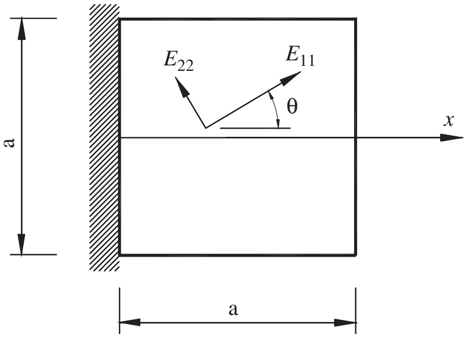

Fig. 3. Cantilever plate with co-ordinate system.

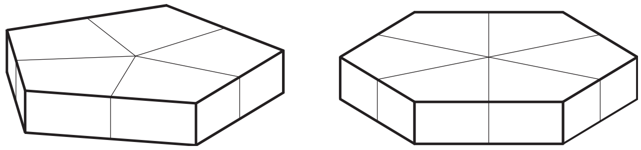

Fig. 4. Meshes of regular polygonal plates.

Table 4
Fundamental frequency parameters ($\lambda = \omega a^2 \sqrt{\rho h/D_0}$) of cross-ply laminated cantilever plates

	Lamination		Mode					
			1	2	3	4	5	6
E/E	0°	Ref. [15]	3.507	6.897	18.41	22.07	27.37	42.14
		Present	3.512	6.801	18.07	21.68	26.64	37.28
	0°/90°/0°	Ref. [15]	3.468	6.874	18.67	21.84	27.19	42.15
		Present	3.473	6.771	18.33	21.40	26.40	35.93
	0°/90°/0°/90°/0°	Ref. [15]	3.284	6.764	19.65	20.94	26.32	42.17
		Present	3.293	6.667	19.34	20.59	25.62	35.59
G/E	0°	Ref. [15]	3.514	4.740	9.116	18.35	22.02	23.61
		Present	3.480	4.627	8.827	17.67	20.46	21.85
	0°/90°/0°	Ref. [15]	3.453	4.707	10.01	21.57	21.64	23.33
		Present	3.401	4.563	9.680	19.46	20.54	21.25
	0°/90°/0°/90°/0°	Ref. [15]	3.155	4.505	13.33	19.77	21.55	28.80
		Present	3.122	4.383	12.96	18.26	19.81	21.52

latter. In this section, for the thickness ratios $h/a = 0.2$ and 0.5 with a is the length of the side, equilateral triangular plates with different lamination are analysed and the first five non-dimensional frequency parameters $\lambda = \omega b^2/\pi^2 \sqrt{\rho h/D_0}$ are shown in Table 6.

3.4. Vibration of polygonal plates.

With five and six present elements as shown in Fig. 3, respectively, the pentagonal and hexagonal plates with fully clamped boundary conditions are analysed. The orientation of x-axes is parallel to any side of the polygons. The circum-radius of the polygons is R. The first five non-dimensional frequency parameters ($\lambda = (4\omega R^2/\pi^2)\sqrt{\rho h/D_0}$) are calculated and shown in Table 7, with different thickness ratios h/R. Other boundary conditions can also be easily dealt with.

4. Conclusions

Following the development and application of an analytic 3-D p-version arbitrary quadrilateral plate element for the free vibration of isotropic plates in Part 1, the element is extended to the vibration analyses of laminated plates in this part. With the additional hierarchical shape functions and analytically integrated element matrices, the computation accuracy is considerably improved. The convergence rate of the element is very fast, which can be observed in the first example. The element is applied for the free vibration of laminated triangular, rectangular and polygonal plates with various boundary conditions.

Table 5
Fundamental frequency parameters ($\lambda = \omega a\sqrt{\rho/E_{22}}$) of a fully clamped square plate with various thickness ratios and laminate sequences

h/a	Lamination	Method	Mode sequence				
			1	2	3	4	5
0.05	0°	Ref. [16]	0.990				
		ANSYS	1.019	1.368	2.042	2.404	2.649
		Present	1.021	1.369	2.080	2.409	2.654
	0°/90°	Ref. [16]	0.772				
		ANSYS	0.780	1.555	1.555	2.156	2.727
		Present	0.782	1.556	1.556	2.160	2.771
	0°/90°/0°	Ref. [16]	0.991				
		ANSYS	1.020	1.440	2.231	2.365	2.650
		Present	0.927	1.488	2.098	2.480	2.499
	[0°/90°]$_2$	Ref. [16]	0.932				
		ANSYS	0.989	1.941	1.941	2.635	3.356
		Present	0.982	1.916	1.916	2.602	3.339
0.1	0°	Ref. [16]	1.570				
		ANSYS	1.680	2.328	3.481	3.483	3.935
		Present	1.684	2.318	3.491	3.502	3.933
	0°/90°	Ref. [16]	1.327				
		ANSYS	1.418	2.653	2.653	3.583	4.339
		Present	1.416	2.639	2.639	3.562	4.348
	0°/90°/0°	Ref. [16]	1.568				
		ANSYS	1.683	2.452	3.418	3.765	3.948
		Present	1.612	2.547	3.259	3.893	4.045
	[0°/90°]$_2$	Ref. [16]	1.504				
		ANSYS	1.717	3.101	3.101	4.106	4.940
		Present	1.680	3.008	3.008	3.982	4.803

$E_{11}/E_{22} = 10$, $E_{33} = E_{22}$, $G_{12} = G_{13} = 0.6E_{22}$, $G_{23} = 0.5E_{22}$, $\nu_{12} = \nu_{13} = \nu_{23} = 0.25$.

Table 6
First five non-dimensional frequency parameters ($\lambda = \omega b^2/\pi^2\sqrt{\rho h/D_0}$) of a fully clamped equilateral triangular plate for different lamination and thickness ratios

	h/a	Mode sequence				
		1	2	3	4	5
0°/90°/0°	0.2	8.07	11.89	13.03	16.14	17.37
	0.5	3.57	5.23	5.60	6.31	6.62
0°/90°/0°/90°	0.2	8.21	12.25	13.04	16.54	17.58
	0.5	3.57	5.32	5.57	6.44	6.64
0°/90°/0°	0.2	7.97	12.22	12.26	16.57	16.92
	0.5	3.57	5.19	5.20	5.77	5.79
0°/90°/0°/90°	0.2	8.20	12.61	12.65	16.94	17.49
	0.5	3.57	5.42	5.43	6.64	7.20

$E_{11}/E_{22} = 40$, $E_{33} = E_{22}$, $G_{12} = G_{13} = 0.6E_{22}$, $G_{23} = 0.5E_{22}$, $\nu_{12} = \nu_{13} = 0.25$, $\nu_{23}^{p} = 0.646$.

Table 7
First five non-dimensional frequency parameters ($\lambda = (4\omega R^2/\pi^2)\sqrt{\rho h/D_0}$) of three-ply polygonal (0°/90°/0°) laminate plate

	h/a	Mode sequence				
		1	2	3	4	5
Pentagon	0.1	2.41	3.42	4.84	4.96	5.93
	0.2	1.60	2.37	2.94	3.39	3.68
	0.5	0.80	1.19	1.36	1.68	1.73
Hexagon	0.1	2.27	3.46	4.45	4.90	5.65
	0.2	1.42	2.35	2.75	3.27	3.53
	0.5	0.77	1.18	1.28	1.60	1.67

$E11/E22 = 40$, $E33 = E22$, $G12 = G13 = 0.6E22$, $G23 = 0.5E22$, $v12 = v13 = 0.25$, $v_{23}^{p} = 0.646$, $D_0 = E_{22}h^3/12(1 - v_{12}v_{21})$.

Acknowledgement

The research is supported by the Hong Kong Research Grant Council No. CityU 1012/04E.

References

[1] J.N. Reddy, *Mechanics of Laminated Composite Plates: Theory and Analysis*, CRC Press, Boca Raton FL, 1996.
[2] J.N. Reddy, A simple higher-order theory for laminated composite plates, *Journal of Applied Mechanics* 45 (1984) 745–752.
[3] J.N. Reddy, C.F. Liu, A higher-order shear deformation theory of laminated elastic shells, *International Journal of Engineering Science* 23 (1985) 319–330.
[4] A.Y.T. Leung, J. Niu, C.W. Lim, K. Song, A new unconstrained third-order plate theory for Navier solutions of symmetrically laminated plates, *Computers and Structures* 81 (2003) 2539–2548.
[5] C.W. Lim, K.M. Liew, A higher-order theory for vibration of shear deformable cylindrical shallow shells, *International Journal of Mechanical Sciences* 37 (1995) 277–295.
[6] S. Srinivas, C.V. Joga Rao, A.K. Rao, An exact analysis for vibration of simply supported homogeneous and laminated thick rectangular plates, *Journal of Sound and Vibration* 12 (1970) 187–199.
[7] S. Srinivas, A.K. Rao, Bending, vibration and buckling of simply supported thick orthotropic rectangular plates and laminates, *International Journal of Solids and Structures* 6 (1970) 1463–1481.
[8] A.Y.T. Leung, B. Zhu, J. Zheng, H. Yang, Analytical trapezoidal Fourier *p*-element for vibration plane problems, *Journal of Sound and Vibration* 271 (2004) 67–81.
[9] A.Y.T. Leung, B. Zhu, J. Zheng, H. Yang, A trapezoidal Fourier *p*-element for membrane vibrations, *Thin-walled Structures* 41 (2003) 479–491.
[10] A.Y.T. Leung, B. Zhu, Transverse vibration of thick polygonal plates using analytical integrated trapezoidal Fourier *p*-element, *Computers and Structures* 82 (2004) 109–119.
[11] A.Y.T. Leung, C. Xiao, B. Zhu, S. Yuan, Free vibration of laminated composite plates subjected to in-plane stresses using trapezoidal p-element, *Composite Structures* 68 (2005) 167–175.
[12] R.C. Batra, L.F. Qian, L.M. Chen, Natural frequencies of thick square plates made of orthotropic, trigonal, monoclinic, hexagonal and triclinic materials, *Journal of Sound and Vibration* 270 (2004) 1074–1086.
[13] C.C. Chao, Comparison of natural frequencies of laminates by 3-D theory, part I: rectangular plates, *Journal of Sound and Vibration* 230 (2000) 985–1007.
[14] T.P. Philippidis, The transverse poisson's ratio in fiber reinforced laminae by means of a hybrid experimental approach, *Journal of Composite Materials* 28 (1994) 252–261.
[15] Y. Narita, A.W. Leissa, Frequencies and mode shapes of cantilevered laminated composite plates, *Journal of Sound and Vibration* 154 (1992) 161–172.
[16] J.Q. Ye, A three-dimensional free vibration of cross-ply laminated rectangular plates with clamped edges, *Computer Methods in Applied Mechanics and Engineering* 140 (1997) 383–392.