Available online at www.sciencedirect.com

ScienceDirect

Journal of Sound and Vibration 298 (2006) 432–438

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

Short Communication

On superaccurate finite elements and their duals for eigenvalue computation

E. Dokumaci*

Department of Mechanical Engineering, Dokuz Eylul University, Izmir, Turkey

Received 8 October 2004; received in revised form 25 April 2006; accepted 25 April 2006
Available online 18 July 2006

Abstract

In eigenvalue problems of elastic media in free vibration, a discrete model is usually derived by using the finite element method and the accuracy of the elements is of concern in practical applications. The present note reviews the accuracy of certain classes of finite element models in two-point boundary eigenvalue problems and points to some special elements that exist within these classes.

1. Introduction

In eigenvalue problems of elastic media in free vibration, a discrete model is usually derived by using the finite element method and the accuracy of the elements is of concern in practical applications. However, the accuracy of the approximate eigenvalues can be analyzed in explicit analytical forms only in few simple problems that can be modeled by a uniform mesh of identical elements. Such a problem is the determination of the eigenvalues of uniform bars in elementary axial or torsional vibration, or of a vibrating string. The basic two-degree-of-freedom (2dof) two-node finite element model for this problem is now a classical one and can be found in textbooks on the finite element method. The stiffness and mass matrices of this element are, in non-dimensional form,

$$\mathbf{K} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \text{ and } \mathbf{M} = \frac{1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \tag{1}$$

respectively. This model is based on a linear displacement field. It is well known from the variational ramifications of the finite element method that such complete and conforming elements give upper bounds to the exact eigenvalues over the whole of the approximate eigenvalue spectrum and, for the eigenvalue problem under consideration, for an element based on a complete polynomial displacement field of order m, the approximate upper bounds to the exact eigenvalues are accurate to $O(n^{-2m})$ over a uniform mesh of n elements of equal length. Thus, for the above basic element, the accuracy of the upper bounds is $O(n^{-2})$.

*Tel.: +232 388 7863; fax: +232 388 7864.
E-mail address: dokumaci@deu.edu.tr.

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2006.04.044

Recently, Fried and Chavez [1] noted that, lower bounds to the eigenvalues of a vibrating string can be predicted to $O(n^{-4})$ by using the element mass matrix

$$\mathbf{M} = \frac{1}{12}\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}, \tag{2}$$

in place of the mass matrix of Eq. (1) (here, the notation is slightly different than that of Ref. [1] due to use of non-dimensional element mass and stiffness matrices and slight changes in the nomenclature). The authors suggest an ingenious formulation of Eq. (2) as the $\delta = 0.5$ case of the element mass matrix

$$\mathbf{M}(\delta) = \frac{1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \frac{\delta}{6}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{3}$$

The case of $\delta = 0$ gives the mass matrix of Eq. (1), and the $\delta = 1$ element, on the other hand, corresponds to the Rayleigh or Duncan model of mass lumping, which is known to yield, when used with the element stiffness matrix **K,** lower bounds to the exact eigenvalues to the accuracy $O(n^{-2})$ [2].

The $O(n^{-4})$ accuracy of $\delta = 0.5$ element is interesting in that, it gives the one order higher accuracy of the quadratic polynomial displacement element at the matrix size of the basic element. For this reason, it is said to be a 'superaccurate' element [1]. Incidentally, it may be also of interest to note that, this element was first proposed, from different considerations, by Murty [3] in relation to axial vibration of uniform bars, which is governed by the same eigenvalue problem as that of the vibrating string.

For this type eigenvalue problem, the spectral convergence criteria for any element having 2dof (called a first-order element) have been given in Ref. [4]. It follows from this study that, that any first-order element having the stiffness matrix **K**, Eq. (1), will converge to the exact eigenvalues over a uniform mesh, if the element mass matrix is of the form

$$\mathbf{M}(\gamma) = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{1}{\gamma}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \tag{4}$$

where $\gamma > 0$. It is clear that, the $\delta = 0$, 0.5 and 1 elements of Ref. [1] correspond, respectively, to the above $\gamma = 12$, 6 and 4 elements. It is proved in Ref. [4] that, the $\gamma > \pi^2$ elements give upper bounds and the $\gamma \leqslant 6$ elements give lower bounds to the accuracy $O(n^{-2})$, and that the accuracy of the $\gamma = 6$ element is $O(n^{-4})$. A first-order element of accuracy $O(n^{-6})$ does not exist, because first-order elements have only a single tuning parameter, that is, γ.

Also considered in Ref. [1] is an element having 3dof. This is based on the finite element model having a quadratic displacement field with an internal node at the center of the element. In non-dimensional form, the mass and stiffness matrices of this element are, respectively,

$$\mathbf{M} = \frac{1}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}, \tag{5}$$

and

$$\mathbf{K} = \frac{1}{3}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}. \tag{6}$$

The element displacement vector associated with these matrices is $\{u(0)\, u(0.5)\, u(1)\}$, where $u(x)$ denotes a non-dimensional displacement and $0 \leqslant x \leqslant 1$ is the element domain. In Ref. [1], the foregoing element mass matrix is generalized, in the spirit of Eq. (3), as

$$\mathbf{M}(\beta) = \frac{1}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} + \frac{\beta}{30}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}, \tag{7}$$

so that the $\beta = 0$ gives Eq. (5) and $\beta = 1$ corresponds to a lumped element mass matrix. The authors show that the $\beta = 2/3$ element gives lower bounds to the smallest eigenvalue to the accuracy of $O(n^{-6})$, whereas the corresponding accuracy of $\beta = 0$ or 1 elements is $O(n^{-4})$. The proof is numerical and limited to the smallest eigenvalue.

Convergence criteria for any element having 3dof, called a second-order element, have been given in Ref. [4] in analytical form, over the whole of the approximate eigenvalue spectrum. The present note will establish the place of the above β elements within possible second-order elements, and point out to further special elements that exist within this class.

2. Second-order elements

Without loss of generality, the dynamic stiffness matrix, $\mathbf{D} = \mathbf{K} - \lambda\mathbf{M}$, of a finite element of the type of problem under consideration can be condensed to a 2×2 matrix by eliminating the dof corresponding to the internal dofs. Here, λ is a non-dimensional eigenvalue parameter referred to the element length and, for a uniform mesh of n elements of equal length (which is always the case throughout the present analysis), the non-dimensional eigenvalue parameter referred to the string length is given by $\sigma^2 = \lambda n^2$. The condensed element dynamic stiffness matrix is independent of the location and type of the internal dofs and can be expressed as

$$\mathbf{Z} = \frac{1}{2}\begin{bmatrix} G(\lambda) + H(\lambda) & G(\lambda) - H(\lambda) \\ G(\lambda) - H(\lambda) & G(\lambda) + H(\lambda) \end{bmatrix}. \tag{8}$$

The approximate eigenvalues are determined by the roots of the equation $R(\lambda_r) = Q(\lambda)$ where

$$R(\lambda_r) = \tan^2\left(\frac{\sqrt{\lambda_r}}{2}\right), \quad Q(\lambda) = -\frac{G(\lambda)}{H(\lambda)}, \tag{9}$$

and $\lambda_r = (\pi r/n)^2$, $r = 1, 2, \ldots$ for a fixed–fixed string (for systems allowing free–free boundary condition, $r = 0, 1, 2, \ldots$ and for the fixed–free condition, $r = 0.5, 1.5, \ldots$). $Q(\lambda)$ denotes any function which increases asymptotically from zero to infinity in the λ interval $(0, \gamma)$, where γ is the smallest positive pole of $G(\lambda)$ or the smallest root of $H(\lambda)$, whichever is the smaller. This formulation is also valid for the first-order elements, but in this case condensation is not necessary and Eq. (9) represents the actual element dynamic-stiffness matrix of the admissible first-order elements, which are given by $G(\lambda) = \gamma\lambda/4$ and $H(\lambda) = \lambda - \gamma$.

For the β elements under consideration, it can be shown that, $H(\lambda) = \lambda - 12$ and that

$$Q(\lambda) = -\frac{\lambda(2 + 3\beta)\lambda - 120}{[(4 + \beta)\lambda - 40](\lambda - 12)}. \tag{10}$$

This function and $R(\lambda_r)$ are shown Fig. 1 for $\beta = 0$, that is, for the quadratic polynomial finite element model for which $\gamma = 40/(4 + \beta) = 10$. It can be shown that, for a given number of elements, half of the approximate eigenvalues, λ, lie in the interval $(0, \gamma)$ and the other half in (h, g), where h denotes the root of $H(\lambda)$, $h = 12$, and g denotes the non-zero root of $G(\lambda)$, $g = 120/(2 + 3\beta) = 60$. It can be shown that [4], as the number of elements is increased, the approximate eigenvalues in the interval (h, g) pass to the interval $(0, \gamma)$, whilst those already in the interval $(0, \gamma)$ are pushed towards the origin, $\lambda = 0$, and are given by the power series

$$\lambda = \lambda_r\left(1 + \frac{c_4}{4!}\sigma_r^2 n^{-2} + \frac{c_6}{6!}\sigma_r^4 n^{-4} + \frac{c_8}{8!}\sigma_r^6 n^{-6} + \frac{c_{10}}{10!}\sigma_r^8 n^{-8} + \cdots\right), \tag{11}$$

ultimately, where $\sigma_r^2 = n^2\lambda_r$, which is in fact the exact value of the rth eigenvalue σ^2. The coefficients c_4 and c_6 can be computed using the general equations, Eqs. (22) and (23), respectively, of Ref. [4]. For the β elements, it is found that

$$c_4 = 0, \quad c_6 = 1 - 3\beta/2. \tag{12}$$

Thus, it follows that, unless $\beta = 2/3$, the β elements yield eigenvalues with accuracy $O(n^{-4})$, in which case, these eigenvalues occur (ultimately) as upper bounds if $\beta < 2/3$ and lower bounds if $\beta > 2/3$. Then, obviously,

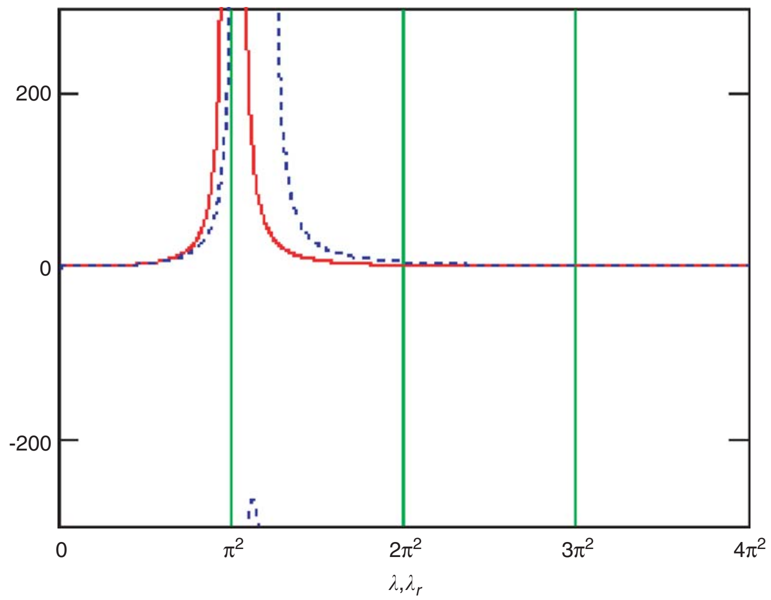

Fig. 1. Characteristics of the second-order $\beta = 0$ matrix displacement model: – – – $Q(\lambda)$; —— $R(\lambda_r)$.

the $\beta = 2/3$ element is accurate to $O(n^{-6})$. This proves that, the $O(n^{-6})$ accuracy applies ultimately not only for the smallest eigenvalue, but for the whole of the approximate eigenvalue spectrum (ultimately, all eigenvalues appear in the interval $(0, \gamma)$.

Whether the $\beta = 2/3$ element ultimately gives upper or lower bounds to the eigenvalues depends on the sign of c_8. Since the expression for c_8 was not given in Ref. [4], it is presented here:

$$c_8 = -2 + 10c_4 \frac{84B''(0) - 84G'''(0) - 7B'(0) - 3.5c_4G''(0)}{G'(0)} - c_6 \frac{56G''(0) - 28B'(0)}{G'(0)} + 56 \frac{60B'''(0) - 15B''(0) + B'(0) - 30G''''(0)}{G'(0)}, \tag{13}$$

where $B(\lambda) = G(\lambda) - H(\lambda)$ and prime denotes differentiation with respect to λ. Upon using the functions G and H for the β elements, it can be shown that $c_8 < 0$ for $\beta = 2/3$. This proves that the $\beta = 2/3$ element is a lower bound element ultimately, as confirmed by Fig. 2, which is the counterpart of Fig. 1 for the $\beta = 2/3$ element.

The foregoing convergence characteristics strictly apply only to the approximate eigenvalues in the interval $(0, \gamma)$ ultimately, however, since the approximate eigenvalues are pushed into this interval as the number of elements in the finite element model increases to infinity, they may be considered as the intrinsic spectral properties of the elements. For a finite number of elements, the largest half of the approximate eigenvalues will lie in the interval (h, g). These are called superfluous eigenvalues, as they do not represent a convergent set of eigenvalues as those in the interval $(0, \gamma)$. Still, the superfluous eigenvalues constitute rough approximations to the corresponding exact eigenvalues in the interval $(\pi^2, 4\pi^2)$. They will occur as upper bounds for the $\beta = 0$ element, and as mixed bounds (i.e., upper bounds for some of the eigenvalues and lower bounds for the other eigenvalues) for the $\beta = 2/3$ element.

A sufficient condition for the approximate eigenvalues to make their initial appearance in the interval $(0, \gamma)$ as upper (lower) bounds is $\gamma > \pi^2$ ($\gamma < \pi^2$). Noting that $\gamma = 40/(4+\beta)$ for the second-order β elements under consideration, $\beta > 40/\pi^2 - 4 = 0.053$ ($\beta < 0.053$) elements will give lower (upper) bounds initially. The initial and ultimate enforcement of lower (upper) bounds in the interval $(0, \gamma)$ ensures lower (upper) bounds throughout the approximate eigenvalue spectrum in this interval.

The above-considered β elements constitute a subclass of admissible second-order elements, which are given by the functions [4]

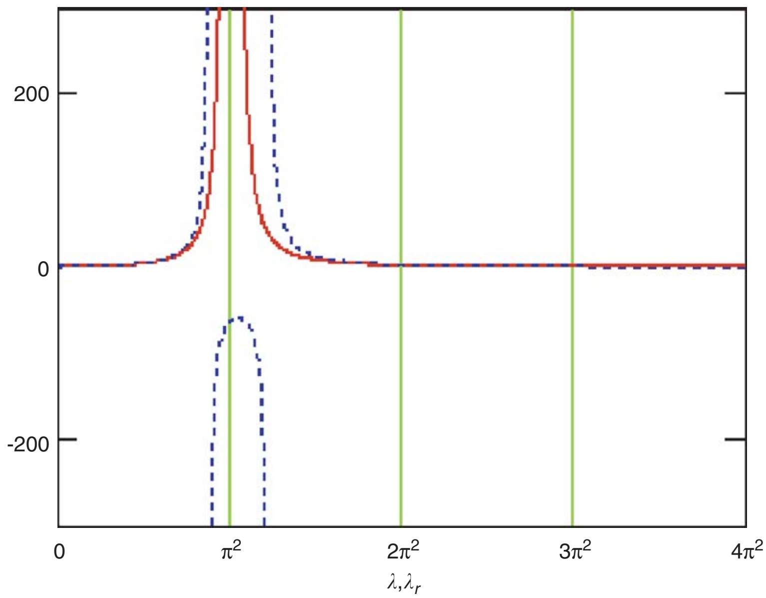

Fig. 2. Characteristics of the second-order $\beta = 2/3$ ($h = 12$)superaccurate element: – – – $Q(\lambda)$; —— $R(\lambda_r)$.

$$G(\lambda) = \frac{\gamma h \lambda(\lambda - g)}{4g(\lambda - \gamma)}, \quad H(\lambda) = \lambda - h, \tag{14}$$

where $\gamma, g, h > 0$. For these elements, the condition for $c_4 = 0$ is

$$\gamma = \frac{6gh}{hg + 6h - 6g}. \tag{15}$$

When this condition is satisfied, the condition for $c_6 = 0$ can be expressed as[1]

$$\gamma = \frac{120(h - 6)}{120 - 17h}. \tag{16}$$

Feasible convergent second-order elements are given by $g, h > \pi^2$ and minimum$(g, h) > \gamma$, and a continuous space of superaccurate elements of accuracy $O(n^{-6})$ exists in the domain of intersection of Eqs. (15) and (16) for $h > \pi^2$. In particular, it transpires that, for $h = 12$, $g(12) = 30$ and $\gamma = 60/7$. Therefore, if h is taken equal to 12, then the only $O(n^{-6})$ element is the $\beta = 2/3$ element. These superaccurate elements tend to improve slightly as h decreases. In Fig. 3 the counterpart of Fig. 2 for the $h = 10$ element, for which $g(10) = 80/3$ and $\gamma = 48/5$ is shown. This is also a lower bound element.

Whether there is any 'hyperaccurate' second-order element of accuracy $O(n^{-8})$ can be examined similarly by searching the condition for $c_8 = 0$ when $c_4 = c_6 = 0$. This condition, which can be derived by using Eqs. (15) and (16) in Eq. (13), turns out to be

$$162h^2 - 3120h + 15120 = 0, \tag{17}$$

which obviously has no real solution.

3. Dual elements

In general, given the condensed dynamic stiffness-matrix, Eq. (9), the parent stiffness and mass matrices can be found by the solution of an inverse problem. For the class of second-order elements that are based on the

[1]Eq. (16) supersedes Eq. (58) of Ref. [4].

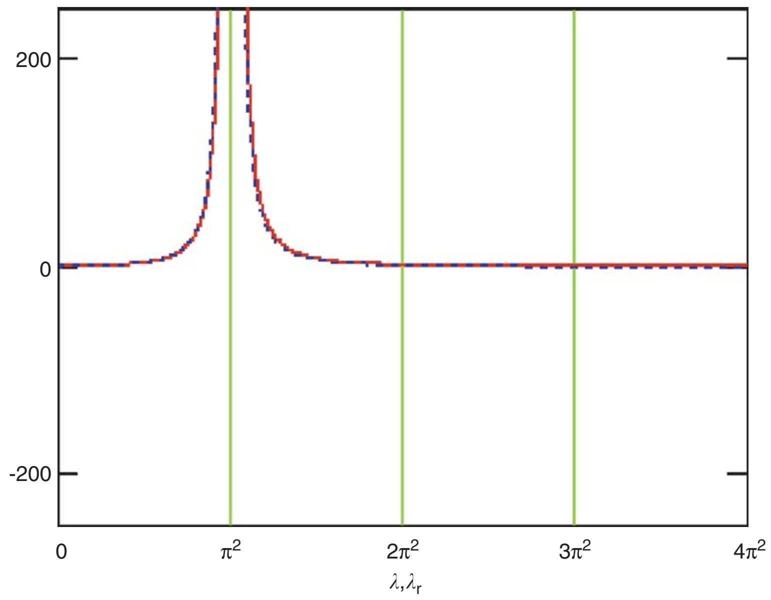

Fig. 3. Characteristics of the second-order $h = 10$ superaccurate element: – – – $Q(\lambda)$; —— $R(\lambda_r)$.

stiffness matrix of Eq. (6), the general form of the uncondensed dynamic-stiffness matrix can be expressed as

$$\mathbf{D} = \frac{1}{3}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} - \frac{\lambda}{30}\begin{bmatrix} m_0 & m_1 & m_2 \\ m_1 & m_3 & m_1 \\ m_2 & m_1 & m_0 \end{bmatrix}. \tag{18}$$

The form of the mass matrix is dictated by the symmetry of the element (the internal node being at the center of the element) and the requirement that the function $H(\lambda)$ is to be linear in λ (see, the second of Eq. (14)). Upon condensing Eq. (18) to the form of Eq. (9), it follows that,

$$m_0 + 2m_1 + m_2 = 15 - 80/\gamma, \tag{19}$$

$$m_3(m_0 + m_2) - 2m_1^2 = 2400/g, \tag{20}$$

$$m_2 - m_0 = 60/h, \quad m_3 = 160/\gamma. \tag{21}$$

The uncondensed mass matrix corresponding to the condensed dynamic-stiffness parameters g, h and γ are found from the solutions of this set of equations. Hence, it transpires that, the mass matrices for a given set of these parameters exist in pairs. For the elements highlighted in the foregoing section, the pairs of the mass matrices are given below.

The quadratic matrix displacement model of accuracy $O(n^{-4})$ ($\beta = 0$):

$$\mathbf{M} = \frac{1}{30}\begin{bmatrix} 14 \mp 10 & -8 \pm 10 & 9 \mp 10 \\ -8 \pm 10 & 16 & -8 \pm 10 \\ 9 \mp 10 & -8 \pm 10 & 14 \mp 10 \end{bmatrix}. \tag{22}$$

The superaccurate element of accuracy $O(n^{-6})$ ($\beta = 2/3$):

$$\mathbf{M} = \frac{1}{90}\begin{bmatrix} 44 \mp 30 & -28 \pm 30 & 29 \mp 30 \\ -28 \pm 30 & 56 & -28 \pm 30 \\ 29 \mp 30 & -28 \pm 30 & 44 \mp 30 \end{bmatrix}. \tag{23}$$

The superaccurate element of accuracy $O(n^{-6})$ $(h = 10)$

$$\mathbf{M} = \frac{1}{90}\begin{bmatrix} 44 \mp 12\sqrt{5} & -25 \pm 12\sqrt{5} & 26 \mp 12\sqrt{5} \\ -25 \pm 12\sqrt{5} & 50 & -25 \pm 12\sqrt{5} \\ 26 \mp 12\sqrt{5} & -25 \pm 12\sqrt{5} & 44 \mp 12\sqrt{5} \end{bmatrix}. \tag{24}$$

4. Conclusion

The theory of second-order two-point boundary finite elements is revisited to prove that the $\beta = 2/3$ element of Ref. [1] is an ultimately true lower bound element and its $O(n^{-6})$ accuracy applies ultimately over the whole of the approximate eigenvalue spectrum. The analysis also reveals that:

There is a multitude of ultimately convergent superaccurate second-order elements of accuracy $O(n^{-6})$.

The second-order elements exist in pairs when transformed to the form of a finite element matrix displacement model with a central internal node. It is believed that, the existence of dual elements is pointed out here for the first time.

There are no second-order element of accuracy $O(n^{-8})$.

Acknowledgment

The author wishes to thank the referee for the comment on the second-order $O(n^{-8})$ element, which motivated the finding of a symbolic differentiation bug.

References

[1] I. Fried, M. Chavez, Superaccurate finite element eigenvalue computation, *Journal of Sound and Vibration* 275 (2004) 415–422.

[2] W.J. Duncan, A critical examination of the representation of massive and elastic bodies by systems of rigid masses elastically connected, *Quarterly Journal of Mechanics and Applied Mathematics* 5 (1952) 97–108.

[3] A.V. Krishna Murty, A lumped inertia force method for vibration problems, *The Aeronautical Quarterly* 17 (1966) 127–140.

[4] E. Dokumaci, A critical examination of discrete models in vibration problems of continuous systems, *Journal of Sound and Vibration* 53 (1977) 153–164.