Available online at www.sciencedirect.com
ScienceDirect
Journal of Sound and Vibration 298 (2006) 439–445
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Short Communication

Apparently the first closed-from solution of inhomogeneous elastically restrained vibrating beams

Issac Elishakoff*, Demetris Pentaras

Department of Mechanical Engineering, Florida Atlantic University, Boca Raton, FL 33431-0991, USA

Received 26 January 2006; received in revised form 10 May 2006; accepted 19 May 2006
Available online 28 July 2006

Abstract

In this paper we consider free vibration of inhomogeneous Bernoulli–Euler beam which is clamped at one end and elastically restrained at the other. The closed-form solution is obtained for the beam of constant material density and constant cross-section but of modulus of elasticity, which varies in a polynomial manner. The semi-inverse method is utilized; namely, the fundamental mode of vibration is postulated as a polynomial too. It turns out that such a formulation leads to infinite number of solutions; one can obtain an unique solution by introducing an additional requirement inherent in vibration tailoring: namely, designing the system that possesses the pre-specified natural frequency. It is shown that if in addition to the fundamental mode shape the natural frequency is also specified, the unique solution is derived.

1. Introduction

The free vibration of uniform and non-uniform beams attracted many investigators since Bernoulli and Euler derived the governing differential equation in the 18th century. The beams with end springs have been dealt with by many investigators. In the Handbook by Karnovsky and Lebed [1] both analytical and numerical are summarized. Due to the numerous papers it is virtually impossible to do justice to the accumulated literature. Therefore, only representative works will be cited. The studies by Liu and Chen [2], Hibbeler [3], Maurizi et al. [4], Laura and Gutierez [5], Lee and Kuo [6] and Lizarev [7] should be mentioned. The above papers dealt with direct problems, i.e. the ones in which the flexural rigidity and the inertial coefficient are specified, and one needs to determine the natural frequencies and mode shapes. Inverse vibration problems were attacked by Barcilon [8,9], Lowe [10] and other investigators. In these problems one deals with construction of the Euler–Bernoulli beam from the spectral data. In these problems circumstances the natural question arises on how can one get the spectra that serve as inputs for the construction problem.

Elishakoff and Candan [11] dealt with a semi-inverse problem with a more modest objective than that repeated in Refs. [8–10]. Elishakoff and Candan [11] dealt with a situation when only the fundamental mode shape is specified, in the form of a simple polynomial; one assumes the inertial coefficient as given and seeks the polynomial flexural rigidity that is compatible with the mode shape and attendant postulated flexural

*Corresponding author. Tel.: +1 561 297 2729; fax: +1 561 297 2825.
E-mail addresses: elishaki@fau.edu (I. Elishakoff), dpentara@fau.edu (D. Pentaras).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2006.05.028

rigidity. For the inhomogeneous beams with ideal boundary conditions, including pinned, clamped or free ends, Elishakoff and Candan [11] constructed Bernoulli–Euler beams that correspond to the provided information. It turned out that there are an infinite number of beams that posses the specified natural mode. To have a unique solution it is thus necessary to impose some additional requirements.

In this paper the results of Ref. [11] are generalized to include an inhomogeneous beam with an end spring. According to Einstein, the theories and methods ought to be as simple as possible, but not simpler; here the simplest possible solution is presented for this problem. The results show the usefulness of such a formulation. It turns out that by postulating the mode shape and setting a natural frequency at the preselected level, one obtains an unique solution.

2. Basic equations

The governing differential equation for the inhomogeneous beam reads

$$\frac{\partial^2}{\partial x^2}\left[D(x)\frac{\partial^2 w}{\partial x^2}\right] + R(x)\frac{\partial^2 w}{\partial t^2} = 0, \tag{1}$$

where $w(x,t)$ is the transverse displacement, $D(x) = E(x)I(x)$ is the flexural rigidity, E the modulus of elasticity, $I(x)$ the moment of inertia, $R(x) = \rho(x)A(x)$ inertial coefficient, $\rho(x)$ the mass density, $A(x)$ the cross-sectional area, x the axial coordinate, t the time. We set $R(x) = \text{const}$, namely that $\rho(x) = \text{const} = \rho_0$, $A(x) = \text{const} = A_0$, $I(x) = \text{const} = I_0$. The only function that varies along the beam's axis is the modulus of elasticity, as a result of which the flexural rigidity is a function of x.

We introduce a non-dimensional axial coordinate

$$\xi = x/L, \tag{2}$$

where L is the length of the beam. The inertial coefficient $R(x)$ is considered to be a constant

$$R(x) = \rho A = \text{const}, \tag{3}$$

so that Eq. (1) reduces to

$$\frac{d^2}{d\xi^2}\left[D(\xi)\frac{d^2 W}{d\xi^2}\right] - \rho A L^4 \omega^2 W = 0, \tag{4}$$

where ω is the sought natural frequency, $W(\xi)$ the mode shape. Eq. (4) is obtained from Eq. (1) by the substitution

$$w(x,t) = W(x)\sin\omega t. \tag{5}$$

We study a beam that has a rotational spring at the left end and is clamped at the right end, so that the boundary conditions are

$$W(\xi) = 0 \qquad \text{at } \xi = 0, \tag{6}$$

$$kL\frac{dW(\xi)}{d\xi} = D(\xi)\frac{d^2 W(\xi)}{d\xi^2} \quad \text{at } \xi = 0, \tag{7}$$

$$W(\xi) = 0 \quad \text{at } \xi = 1, \tag{8}$$

$$\frac{dW(\xi)}{d\xi} = 0 \quad \text{at } \xi = 1, \tag{9}$$

where k is the spring stiffness.

The simplest function that can be postulated for the mode shape is a fourth-order polynomial

$$W(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 + a_4\xi^4. \tag{10}$$

The enforcement of condition (6) yields

$$a_0 = 0. \tag{11}$$

Prior to satisfaction of the boundary condition in Eq. (7) we need a specific form for the flexural rigidity $D(\xi)$. We are solving a semi-inverse problem where the flexural rigidity is determined in such a manner that the function in Eq. (10) represents an exact mode shape. The function for $D(\xi)$ that is compatible with Eq. (10) is a fourth-order polynomial

$$D(\xi) = b_0 + b_1\xi + b_2\xi^2 + b_3\xi^3 + b_4\xi^4. \tag{12}$$

Eqs. (10) and (12) are substituted into Eq. (7) to result in

$$kLa_1 = 2b_0a_2. \tag{13}$$

The boundary conditions in Eqs. (8) and (9) lead to

$$a_2 = -2a_3 - 3a_4, \tag{14}$$

$$a_1 = a_3 + 2a_4, \tag{15}$$

and

$$a_2 = \frac{2b_0}{4b_0 + kL}a_4. \tag{16}$$

In view of Eq. (13) a_1 becomes

$$a_1 = \frac{2b_0}{4b_0 + kL}a_4. \tag{17}$$

Substituting Eq. (17) into Eq. (15) leads to

$$a_3 = -2\frac{(3b_0 + kL)}{4b_0 + kL}a_4. \tag{18}$$

Thus, the mode shape becomes

$$W(\xi) = a_4\left(\frac{2b_0}{4b_0 + kL}\xi + \frac{kL}{4b_0 + kL}\xi^2 - \frac{(6b_0 + 2kL)}{4b_0 + kL}\xi^3 + \xi^4\right), \tag{19}$$

where, a_4 is an arbitrary constant. We fix it at unity. The expression for the mode shape is

$$W(\xi) = \frac{2b_0}{4b_0 + kL}\xi + \frac{kL}{4b_0 + kL}\xi^2 - \frac{6b_0 + 2kL}{4b_0 + kL}\xi^3 + \xi^4. \tag{20}$$

The result of substitution of Eqs. (12) and (20) into Eq. (4) is

$$\sum_{j=0}^{4} C_j\xi^j = 0, \tag{21}$$

where

$$C_0 = 24b_0kL - 24b_1kL + 4b_2kL + 96b_0^2 - 72b_1b_0, \tag{22}$$

$$C_1 = 72b_2kL + 72b_1kL - 2\rho A\omega^2L^4b_0 - 216b_2b_0 + 288b_1b_0 + 12b_3kL, \tag{23}$$

$$C_2 = 144b_2kL - 144b_3kL + 576b_0b_2 - 432b_3b_0 + 24b_4kL - \rho AL^5\omega^2k, \tag{24}$$

$$C_3 = 960b_3b_0 - 720b_4b_0 + 240b_3kL + 2\rho AL^5\omega^2k + 6\rho A\omega^2L^4b_0 - 240b_4kL, \tag{25}$$

$$C_4 = 360b_4kL - 4\rho AL^4\omega^2b_0 + 1440b_4b_0 - \rho A\omega^2kL^5. \tag{26}$$

Eq. (26) results in

$$\omega^2 = 360b_4/\rho AL^4. \tag{27}$$

It should be noted that although Eq. (26) depends on the spring stiffness coefficient kL, the expression for ω^2 is independent of it. Remarkably, formula (27) coincides with its counterpart that is valid for the beam with classical boundary conditions, as reported in Ref. [11] (see also the monograph [12]). Whereas the relation between ω^2 and b_4 remains unchanged, the rest of the coefficients in $D(\xi)$ depend upon the stiffness of the end spring. By substituting Eq. (27) into Eq. (25) we get

$$b_3 = \frac{-2(3b_0 + kL)}{4b_0 + kL} b_4, \tag{28}$$

and Eqs. (23) and (24) yield

$$b_2 = \frac{k^2L^2 - 8b_0kL - 54b_0^2}{3(4b_0 + kL)^2} b_4, \tag{29}$$

$$b_1 = \frac{2\left(k^3L^3 - 9b_0^2kL^5 + 87b_0^2kL + 159b_0^3\right)}{3(4b_0 + kL)^3} b_4. \tag{30}$$

From Eq. (22) we get

$$b_4 = \frac{18b_0(4b_0 + kL)^4}{5256kLb_0^3 + 1706k^2L^2b_0^2 + 232b_0k^3L^3 + 5724b_0^4 + 11k^4L^4}. \tag{31}$$

The flexural rigidity $D(\xi)$ is obtained by substituting Eqs. (28)–(31) into Eq. (12). Note that when $k = 1$, we get $b_3 = 1.5b_4$ as in Ref. [12]for the pinned–clamped inhomogeneous beam. When k tends to infinity, b_3 approaches value $-2b_4$ as is the case for a clamped–clamped inhomogeneous beam [12].

It is seen that when the end spring stiffness k is fixed, we get an infinite number beams since the coefficient b_0 is arbitrary. For example, for $kL = 1$ and $b_0 = 1$, we have

$$\begin{aligned} D(\xi) &= 1 + \frac{15780}{12929}\xi - \frac{9150}{12929}\xi^2 - \frac{18000}{12929}\xi^3 + \frac{11250}{12929}\xi^4 \\ &\approx 1 + 1.2205\xi - 0.70771\xi^2 - 1.3922\xi^3 + 0.87014\xi^4. \end{aligned} \tag{32}$$

For $kL = 10$ and $b_0 = 2$ we get

$$\begin{aligned} D(\xi) &= 2 + \frac{26856}{12281}\xi - \frac{7452}{12281}\xi^2 - \frac{46656}{12281}\xi^3 + \frac{26244}{12281}\xi^4 \\ &\approx 2 + 2.1868\xi - 0.60679\xi^2 - 3.799\xi^3 + 2.137\xi^4. \end{aligned} \tag{33}$$

For $kL = 100$ and $b_0 = 3$

$$\begin{aligned} D(\xi) &= 3 + \frac{7222900131543161}{2251799813685248}\xi - \frac{7365930465590695}{9007199254740992}\xi^2 \\ &\quad - \frac{296257640492991}{35184372088832}\xi^3 \frac{4870584328104953}{1125899906842624}\xi^4 \\ &\approx 3 + 3.2076\xi - 0.81778\xi^2 - 8.4201\xi^3 + 4.3259\xi^4. \end{aligned} \tag{34}$$

These are shown in the Figs. 1–3. It may appear at the first glance that the variation shown in Fig. 1 is symmetric with respect to the middle cross-section $\xi = 0.5$. It is however not a symmetric function with respect to $\xi = 0.5$, because the presence of the end spring introduces a lack of symmetry. For $k = 0$, the figures is symmetric because of symmetric pinned–clamped boundary conditions [12]. At low values of kL the mode shape may appear symmetric, but the deviation from symmetry is more apparent for a beam with end springs of larger stiffness.

In Fig. 2, kL is set at ten, whereas in Fig. 3, $kL = 100$. In these figures the lack of symmetry is apparent.

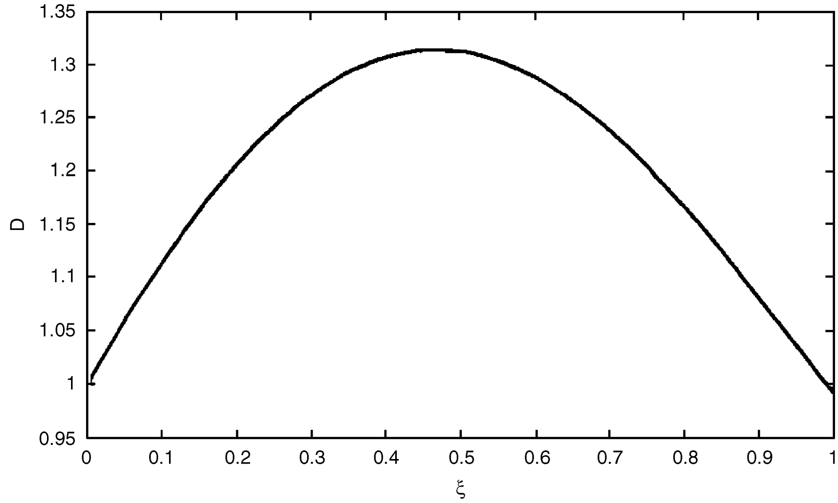

Fig. 1. Variation of $D(\xi)$ for $kL = 1$ and $b_0 = 1$, $\xi \in [0;1]$.

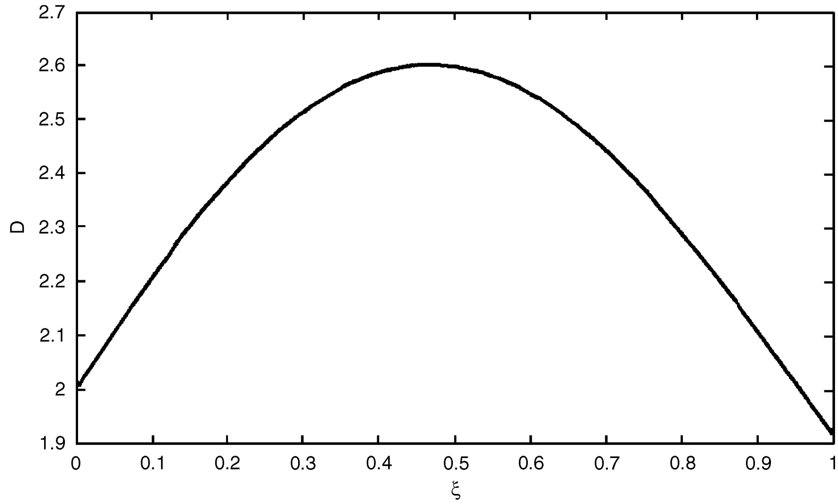

Fig. 2. Variation of $D(\xi)$ for $kL = 10$ and $b_0 = 2$, $\xi \in [0;1]$.

The presence of an infinite number of solutions is a *favorable* consequence of the present formulation of the semi-inverse problem. It allows a designer to introduce an additional requirement. For example, if the design requires the beam to possess a specified natural frequency Ω, then from Eq. (27) we get the expression for b_4 by setting $\omega = \Omega$:

$$b_4 = \rho A L^4 \Omega^2 / 360. \tag{35}$$

Then, by equating the left-hand sides of Eq. (31) and (35), we get

$$\frac{18b_0(4b_0 + kL)}{5256kLb_0^3 + 1706k^2L^2b_0^2 + 232b_0k^3L^3 + 5724b_0^4 + 11k^4L^4} = \frac{\rho A L^4 \Omega^2}{360}, \tag{36}$$

which is a cubic equation in b_0. Solving Eq. (36) for a specified kL and Ω and substitution the resulting b_0 into Eqs. (28)–(30) yields the coefficients b_1, b_2 and b_3, resulting in an unique beam.

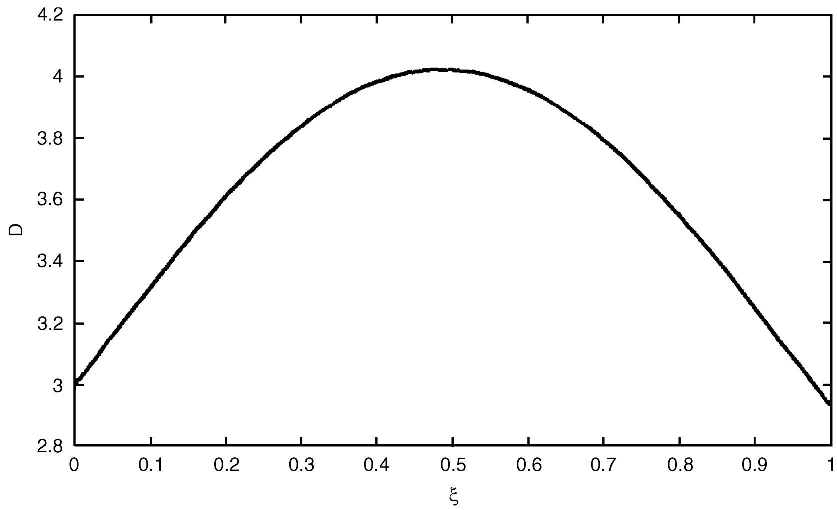

Fig. 3. Variation of $D(\xi)$ for $kL = 100$ and $b_0 = 3$, $\xi \in [0;1]$.

3. Discussion

It is instructive to contrast the above closed-form solution for a class of polynomially inhomogeneous beams with that valid for an uniform, homogeneous beam. In the latter case the mode shape is of the form [13]

$$W(x) = B_1 \sin(\alpha x) + B_2 \cos h(\alpha x) + B_3 \sin(\alpha x) + B_4 \cos(\alpha x), \tag{37}$$

where

$$\alpha^4 = \rho A \omega^2 / EI. \tag{38}$$

Satisfaction of the boundary conditions in Eqs. (6–9) leads to the following transcendental equation:

$$\frac{kL}{EI}\left(1 + \frac{1}{\cos\,\mathrm{h}(\alpha L)\cos(\alpha L)}\right) = \tan(\alpha L) - \tan\,h(\alpha L). \tag{39}$$

Note that when k vanishes, the above equation reduces to

$$\tan(\alpha L) = \tan\,h(\alpha L), \tag{40}$$

which is the frequency equation for the pinned-clamped beam. For kL approaching infinity

$$\cos(\alpha L)\cos\,h(\alpha L) + 1 = 0, \tag{41}$$

recovering the frequency equation of the clamped–clamped beam. For general kL, the fundamental natural frequencies are, for example, for $kL/EI = 0.01$, $\alpha L \approx 0.527$; for $kL/EI = 0.1$, $\alpha L \approx 0.6759$; for $kL/EI = 1$, $\alpha L \approx 1.305$; for $kL/EI = 10$, $\alpha L \approx 1.793$; for $kL/EI = 100$, $\alpha L \approx 1.877$. Finally, for $kL/EI \to \infty$, $\alpha L \approx 1.885$. The results are obtained from a numerical solution of the characteristic equation given in Eq. (39).

It is worth noting that while the solution of the uniform, homogeneous beam necessitates *numerical* tackling of the *transcendental* equation, the semi-inverse method in the present setting furnishes the solution in a closed, polynomial form. It is anticipated that there could be closed-form solutions available for non-polynomial variation of the mode shape and/or the flexural rigidity functions.

References

[1] A. Karnovsky, O.I. Lebed, *Formulas for Structural Dynamics: Tables, Graphs, and Solution*, Mc Graw-Hill, New York, 2000, pp. 159–194.

[2] W.H. Liu, K.S. Chen, Effects of lateral support on the fundamental natural frequencies and bucklin coefficients, *Journal of Sound and Vibration* 129 (1) (1989) 155–160.

[3] R.C. Hibbeler, Free vibration of a beam supported by unsymmetrical spring-hinges, *Journal of Applied Mechanics* (1975) 501–502.
[4] M.J. Maurizi, R.E. Rossi, J.A. Reyes, Comments on A note of generally restrained beams, *Journal of Sound and Vibration* 147 (1) (1991) 167–171.
[5] P.A.A. Laura, R.H. Gutierez, Vibration of an elastically restrained cantilever beam of varying cross-section with tip mass of finite length, *Journal of Sound and Vibration* 108 (1986) 123–131.
[6] S.Y. Lee, Y.H. Kuo, Exact solution for the analysis of generally elastically restrained non-uniform beams, *Journal of Sound and Vibration* 59 (1992) 205–212.
[7] A.D. Lizarev, Free vibration of beams with elastically clamped ends, *Izvestiya VUZOV, Stoirlstvo I Arkhitektura* (10) (1959) (in Russian)
[8] V. Barcilon, Inverse problem of a vibration beam, *Journal of Applied Mechanics and Physics* 27 (1976) 147–158.
[9] V. Barcilon, On the multiplicity of solution of the inverse problem for vibrating beam, *SIAM Journal of Applied Mechanics* 37 (1979) 605–613.
[10] B. Lowe, On the construction of an Euler–Bernoulli beam from spectral data, *Journal of Sound and Vibration* 163 (1993) 165–171.
[11] I. Elishakoff, S. Candan, Apparently first closed-form solution for vibration inhomogeneous beams, *International Journal of Solids and Structures* 38 (2201) 3411–3441.
[12] I. Elishakoll, *Eigenvalues of Inhomogeneous Structures: Unusual Closed-Form Solutions*, CRC Press, Boca Raton, 2005.
[13] S.S. Rao, *Mechanical Vibration*, Pearson Education, Inc., Upper Saddle River, NY, 2004.