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Short Communication

Solutions of the Duffing-harmonic oscillator by an iteration procedure

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Abstract

A modified iteration procedure is applied to the Duffing-harmonic oscillator. With the procedure, the excellent approximate frequencies and the corresponding periodic solutions can be easily obtained.

1. Introduction

Consider a nonlinear oscillator modeled by the equation

$$\ddot{x} + g(x) = 0, \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{1}$$

where $g(x)$ is a nonlinear function of x and has the property:

$$g(-x) = -g(x).$$

If $g(x)$ does not have for small x a dominant term proportional to x, then Eq. (1) is said to be a "truly nonlinear oscillator" (TNO) [1]. One example of such equations is the Duffing-harmonic oscillator described by the equation [2]

$$\ddot{x} + \frac{x^3}{1+x^2} = 0, \quad x(0) = A, \quad \dot{x}(0) = 0. \tag{2}$$

Recently, Lim and Wu [3] proposed a modified iteration procedure for Eq. (1). Mickens [1] generalized this procedure for the following equation:

$$\ddot{x} + g(x) = \varepsilon f(x, \dot{x}), \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{3}$$

where

$$f(-x, -\dot{x}) = -f(x, \dot{x}).$$

But they did not give the details as how to carry out the iteration scheme to deal with Eq. (2). It has been shown that all the curves in the phase-space corresponding Eq. (2) are closed, and all motions for arbitrary

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initial conditions give periodic solutions [2]. Lim and Wu [4] obtained analytical approximate solutions to Eq. (2) by combining the linearization of the governing equation with the method of harmonic balance. The main purpose of this communication is to use an iteration procedure to determine accurate approximations to the periodic solutions of Eq. (2).

2. Solution method

To begin, let the angular frequency of Eq. (1) be ω, which is unknown to be further determined. Then Eq. (1) can be rewritten as [1,3,5–7]

$$\ddot{x} + \omega^2 x = \omega^2 x - g(x) =: G(x), \quad x(0) = A, \quad \dot{x}(0) = 0. \tag{4}$$

The linearized equation of Eq. (1) is

$$\ddot{x} + \omega^2 x = 0, \quad x(0) = A, \quad \dot{x}(0) = 0. \tag{5}$$

Comparing Eq. (1) with Eq. (5), we see that even though $g(x)$ is not "small", the function $G(x) = \omega^2 x - g(x)$ is "small". Therefore, the left-hand side of Eq. (4) is linear and the term $G(x)$ on the right-hand side is a "small" function. This is the reason that we prefer Eq. (4) to Eq. (1).

The iteration scheme is [5]

$$\ddot{x}_{k+1} + \omega^2 x_{k+1} = G(x_k), \quad x_k(0) = A, \quad \dot{x}_k(0) = 0, \quad k = 0, 1, 2, \ldots, \tag{6}$$

where the input or starting function is

$$x_0(t) = A \cos\theta = A \cos\omega t. \tag{7}$$

Usually, x_1 can easily be obtained from Eq. (6). Timoshenko et al. [8] have applied this technique to the Duffing equation, but they only gave the first iteration result. When $k \geqslant 1$, we have

$$G(x_k) = G[x_{k-1} + (x_k - x_{k-1})] \approx G(x_{k-1}) + G_x(x_{k-1})(x_k - x_{k-1}), \tag{8}$$

where

$$G_x(x) = \frac{\mathrm{d}G}{\mathrm{d}x}. \tag{9}$$

Therefore, Eq. (6) can be rewritten as [1,3]

$$\ddot{x}_{k+1} + \omega^2 x_{k+1} = G(x_{k-1}) + G_x(x_{k-1})(x_k - x_{k-1}),$$

$$x_k(0) = A, \quad \dot{x}_k(0) = 0, \quad k = 0, 1, 2, \ldots, \tag{10}$$

where $x_{-1}(t) = x_0(t)$ [1,3]. Instead of Eq. (8) we may also have

$$G(x_k) = G[x_0 + (x_k - x_0)] \approx G(x_0) + G_x(x_0)(x_k - x_0). \tag{11}$$

Now Eq. (6) can be written as

$$\begin{aligned} &\ddot{x}_{k+1} + \omega^2 x_{k+1} = G(x_0) + G_x(x_0)(x_k - x_0), \\ &x_k(0) = A, \quad \dot{x}_k(0) = 0, \quad k = 0, 1, 2, \ldots. \end{aligned} \tag{12}$$

In what follows, we will use formula (12) to solve Eq. (2). In this case, formula (12) becomes

$$\begin{aligned} &\ddot{x}_{k+1} + \omega^2 x_{k+1} = \omega^2 x_k - \frac{x_0^3}{1 + x_0^2} - \frac{3x_0^2 + x_0^4}{(1 + x_0^2)^2}(x_k - x_0), \\ &x_k(0) = A, \quad \dot{x}_k(0) = 0, \quad k = 0, 1, 2, \ldots \ . \end{aligned} \tag{13}$$

Using Eq. (7), we have the following Fourier series expansions:

$$\frac{x_0^3}{1 + x_0^2} = a_1 \cos\theta + a_3 \cos 3\theta + a_5 \cos 5\theta + \cdots, \tag{14}$$

$$\frac{3x_0^2 + x_0^4}{(1 + x_0^2)^2} = \frac{b_0}{2} + b_2 \cos 2\theta + b_4 \cos 4\theta + b_6 \cos 6\theta + \cdots, \tag{15}$$

where [4]

$$\begin{aligned}
a_1 &= A - \frac{2}{A} + \frac{2}{A(1 + A^2)^{1/2}},\\
a_3 &= \frac{8}{A^3} + \frac{2}{A} - \frac{8}{A^3(1 + A^2)^{1/2}} - \frac{6}{A(1 + A^2)^{1/2}},\\
b_0 &= 2 + \frac{2}{(1 + A^2)^{1/2}} - \frac{2(2 + A^2)}{(1 + A^2)^{3/2}},\\
b_2 &= \frac{4}{A^2} - \frac{2}{(1 + A^2)^{1/2}}\left(1 + \frac{2}{A^2}\right) + \frac{2A^2}{(1 + A^2)^{3/2}},\\
b_4 &= \frac{16}{A^4}\left[-3 - \frac{1}{2}A^2 + \frac{8 + 8A^2 + A^4}{8(1 + A^2)^{1/2}}\right.\\
&\qquad \left. + \frac{16 + 24A^2 + 6A^4 - A^6}{8(1 + A^2)^{3/2}}\right],\\
b_6 &= \frac{16}{A^4}\left[\frac{20}{A^2} + 12 + \frac{3}{4}A^2 + \frac{48 + 22A^2 + 3A^4}{8(1 + A^2)^{1/2}}\right.\\
&\qquad \left. + \frac{4}{A^2(1 + A^2)^{1/2}} - \frac{192 + 416A^2 + 280A^4 + 60A^6 + 3A^8}{8A^2(1 + A^2)^{3/2}}\right]
\end{aligned} \tag{16a−f}$$

and

$$\begin{aligned}
a_5 &= \frac{2}{\pi}\int_0^{\pi} \frac{A^3 \cos^3\theta \cos 5\theta}{1 + A^2\cos^2\theta}\, d\theta = -\frac{2}{A} - \frac{24}{A^3} - \frac{32}{A^5} + \frac{10}{A(1 + A^2)^{1/2}}\\
&\quad + \frac{40}{A^3(1 + A^2)^{1/2}} + \frac{32}{A^5(1 + A^2)^{1/2}}.
\end{aligned} \tag{16g}$$

Substituting Eq. (14) into Eq. (13) and letting $k = 0$ gives

$$\begin{aligned}
&\ddot{x}_1 + \omega^2 x_1 = (\omega^2 A - a_1)\cos\theta - a_3 \cos 3\theta - a_5 \cos 5\theta,\\
&x_1(0) = A, \quad \dot{x}_1(0) = 0.
\end{aligned} \tag{17}$$

The requirement of no secular terms in $x_1(t)$ implies that

$$\omega = \omega_1 = \sqrt{\frac{a_1}{A}}. \tag{18}$$

This equation is identical to Eq. (13) in Ref. [4]. The corresponding approximate periodic solution $x_1(t)$ becomes

$$x_1(t) = A \cos \omega t + c_3(\cos \omega t - \cos 3\omega t) + c_5(\cos \omega t - \cos 5\omega t), \tag{19}$$

where ω is given by Eq. (18) and

$$c_3 = -\frac{a_3}{8\omega_1^2} = -\frac{a_3 A}{8a_1}, \tag{20a}$$

$$c_5 = -\frac{a_5}{24\omega_1^2} = -\frac{a_5 A}{24a_1}. \tag{20b}$$

If $k=1$, Eq. (13) becomes

$$\ddot{x}_2+\omega^2 x_2=\omega^2 x_1-\frac{x_0^3}{1+x_0^2}-\frac{3x_0^2+x_0^4}{(1+x_0^2)^2}(x_1-x_0),$$
$$x_2(0)=A,\quad \dot{x}_2(0)=0. \tag{21}$$

Using Eqs. (7), (15) and (19), we have

$$\begin{aligned}\frac{3x_0^2+x_0^4}{(1+x_0^2)^2}(x_1-x_0)=&\frac{1}{2}[(b_0+b_2-b_4-b_6)c_5\\&+(b_0-b_4)c_3]\cos\theta\\&+\frac{1}{2}[(-b_0+b_2+b_4-b_6)c_3\\&+b_4c_5]\cos 3\theta+\frac{1}{2}[(-b_2+b_4+b_6)c_3\\&+(-b_0+b_4+b_6)c_5]\cos 5\theta+\mathrm{HOH},\end{aligned} \tag{22}$$

where HOH stands for higher-order harmonics. Substituting Eqs. (14), (19) and (22) into Eq. (21) and simplifying the resulting expression yields

$$\begin{aligned}\ddot{x}_2+\omega^2 x_2=&[\omega^2(A+c_3+c_5)-a_1-\frac{1}{2}(b_0+b_2-b_4-b_6)c_5\\&-\frac{1}{2}(b_0-b_4)c_3]\cos\theta\\&-\left[\omega^2c_3+a_3+\frac{1}{2}(-b_0+b_2+b_4-b_6)c_3+\frac{b_4c_5}{2}\right]\cos 3\theta\\&-\left[\omega^2c_5+a_5+\frac{1}{2}(-b_2+b_4+b_6)c_3+\frac{1}{2}(-b_0+b_4+b_6)c_5\right]\cos 5\theta\\&+\mathrm{HOH},\end{aligned}$$

$$x_2(0)=A,\quad \dot{x}_2(0)=0. \tag{23}$$

Secular terms are eliminated by setting the coefficient of cos ωt equal to zero; doing this yields

$$\omega=\omega_2=\left[\frac{a_1+\frac{1}{2}(b_0+b_2-b_4-b_6)c_5+\frac{1}{2}(b_0-b_4)c_3}{A+c_3+c_5}\right]^{1/2}. \tag{24}$$

The corresponding approximate periodic solution $x_2(t)$ is

$$x_2(t)=A\cos\omega t+d_3(\cos\omega t-\cos 3\omega t)+d_5(\cos\omega t-\cos 5\omega t), \tag{25}$$

where ω is given by Eq. (24) and

$$d_3=-\frac{1}{8\omega_2^2}\left[\omega_2^2c_3+a_3+\frac{1}{2}(-b_0+b_2+b_4-b_6)c_3+\frac{b_4c_5}{2}\right], \tag{26a}$$

$$d_5=-\frac{1}{24\omega_2^2}\left[\omega_2^2c_5+a_5+\frac{1}{2}(-b_2+b_4+b_6)c_3+\frac{1}{2}(-b_0+b_4+b_6)c_5\right]. \tag{26b}$$

3. Discussion

Now we compare the above approximate solutions with the exact solution and other approximate solutions. The exact frequency ω_e of Eq. (2) is [4]

$$\omega_e = \frac{\pi}{2\int_0^{\pi/2}\left\{A^2\cos^2\theta/\left[A^2\cos^2\theta + \ln\left(1 - \frac{A^2\cos^2\theta}{1+A^2}\right)\right]\right\}^{1/2}\mathrm{d}\theta}. \tag{27}$$

The second approximate frequency obtained by Lim and Wu [4] is

$$\omega_{L2} = \omega_2(A) = \sqrt{g_L(A) + \sqrt{g_L^2(A) - h_L(A)}}, \tag{28}$$

where

$$g_L(A) = \frac{(b_0 - b_2 - b_4 + b_6)A + 18a_1 + 2a_3}{36A}, \tag{29}$$

$$h_L(A) = \frac{a_1(b_0 - b_2 - b_4 + b_6) + a_3(b_0 - b_4)}{18A}. \tag{30}$$

The corresponding approximate periodic solution is [4]

$$x_{L2} = x_2 = A\cos\omega_{L2}t + x_1(A)(\cos\omega_{L2} - \cos 3\omega_{L2}t), \tag{31}$$

where

$$x_1(A) = -\frac{2a_3}{b_2 + b_4 - b_0 - b_6 + 18\omega_{L2}^2}. \tag{32}$$

By rewriting Eq. (2) as

$$(1 + x^2)\ddot{x} + x^3 = 0, \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{33}$$

Mickens [2] has obtained an approximate frequency

$$\omega_M = \sqrt{\frac{3A^2/4}{1 + 3A^2/4}}. \tag{34}$$

For comparison, the exact frequency ω_e obtained by integrating Eq. (27) and the approximate frequencies computed by Eqs. (18), (24), (28) and (34), respectively, are listed in Table 1 for $0.1 \leqslant A \leqslant 10$. ω_2 (Eq. (24)) is

Table 1
Comparison of the approximate frequencies with the exact frequency ω_e

A	ω_e Eq. (27)	ω_M Eq. (34)	ω_1 Eq. (18)	ω_{L2} Eq. (28)	ω_2 Eq. (24)
0.1	0.0843887	0.0862796	0.0862441	0.0842560	0.0843678
0.2	0.1668303	0.1706640	0.1703930	0.1665626	0.1667969
0.4	0.3194026	0.3273268	0.3255129	0.3188634	0.3193871
0.6	0.4491013	0.4610840	0.4563924	0.4483261	0.4491515
0.8	0.5540680	0.5694948	0.5614401	0.5531399	0.5541943
1	0.6367803	0.6546537	0.6435943	0.6357955	0.6369633
2	0.8476261	0.8660254	0.8506508	0.8470211	0.8477949
3	0.9195998	0.9332565	0.9208966	0.9193277	0.9196820
4	0.9508565	0.9607689	0.9514815	0.9507304	0.9508974
5	0.9669758	0.9743547	0.9673103	0.9669129	0.9669982
10	0.9909163	0.9933993	0.9909541	0.9909118	0.9909194

more accurate than any other approximate frequency in Table 1. Furthermore, we have

$$\lim_{A\to+\infty} \omega_2 = 1, \tag{35}$$

$$\lim_{A\to+0} \frac{\omega_2}{\omega_e} = \lim_{A\to+0} \frac{\omega_2}{\omega_1} \lim_{A\to+0} \frac{\omega_1}{\omega_e} = \sqrt{\frac{22}{23}} \times 1.0222 = 0.9997. \tag{36}$$

The numerical solution $x_{num}(t)$ of Eq. (2) obtained by using Runge–Kutta (R–K) method, the corresponding approximate solutions $x_{L2}(t)$, $x_1(t)$ and $x_2(t)$ computed by Eq. (31), Eq. (19) and Eq. (25), respectively, are listed in Tables 2–4 for $A = 0.1$, 1, and 5. The percentage errors are defined as

Table 2
Comparison of the approximate solutions with the numerical solution ($A = 0.1$, $T_e = 2\pi/\dot{\omega}_e = 74.4553$, $h = T/10$)

t	x_{num}	x_1 (% error)	x_{L2}(% error)	x_2 (% error)
h	0.07577	0.07532(−0.60)	0.07608(0.40)	0.07572(−0.07)
$2h$	0.02622	0.02394(−8.70)	0.02617(−0.19)	0.02617(−0.20)
$3h$	−0.02622	−0.02974(13.39)	−0.02576(−1.76)	−0.02611(−0.44)
$4h$	−0.07577	−0.07993(5.49)	−0.07574(−0.05)	−0.07566(−0.14)
$5h$	−0.10000	−0.09968(−0.32)	−0.10000(0.00)	−0.10000(0.00)
$6h$	−0.07577	−0.07040(−7.08)	−0.07642(0.85)	−0.07577(0.00)
$7h$	−0.02622	−0.01816(−30.76)	−0.02658(1.37)	−0.02624(0.05)
$8h$	0.02622	0.03552(35.46)	0.02535(−3.33)	0.02604(−0.69)
$9h$	0.07577	0.08418(11.10)	0.07540(−0.50)	0.07561(−0.21)
T	0.10000	0.09874(−1.26)	0.09999(−0.01)	0.10000(0.00)

Table 3
Comparison of the approximate solutions with the numerical solution ($A = 1.0$, $T_e = 9.8671$, $h = T/10$)

t	x_{num}	x_1 (% error)	X_{L2} (% error)	x_2 (% error)
h	0.77523	0.77381(−0.18)	0.77385(−0.18)	0.77502(−0.03)
$2h$	0.27305	0.26220(−3.97)	0.27488(0.67)	0.27249(−0.21)
$3h$	−0.27305	−0.29146(6.74)	−0.27069(−0.87)	−0.27327(0.08)
$4h$	−0.77523	−0.79565(2.63)	−0.77063(−0.59)	−0.77562(0.05)
$5h$	−1.00000	−0.99932(−0.07)	−0.99999(−0.00)	−1.00000(0.00)
$6h$	−0.77523	−0.75113(−3.11)	−0.77704(0.23)	−0.77443(−0.10)
$7h$	−0.27305	−0.23291(−14.70)	−0.27908(2.21)	−0.27171(−0.49)
$8h$	0.27305	0.32066(17.44)	0.26649(−2.40)	0.27405(0.37)
$9h$	0.77523	0.81662(5.34)	0.76740(−1.01)	0.77622(0.13)
T	1.00000	0.99729(−0.27)	0.99994(−0.01)	1.00000(0.00)

Table 4
Comparison of the approximate solutions with the numerical solution ($A = 5.0$, $T_e = 6.4978$, $h = T/10$)

t	x_{num}	x_1 (% error)	x_{L2} (% error)	x_2 (% error)
h	4.02161	4.02385(0.06)	4.01226(−0.23)	4.02379(0.05)
$2h$	1.50555	1.50630(0.05)	1.51252(0.46)	1.50766(0.14)
$3h$	−1.50555	−1.51139(0.39)	−1.51156(0.40)	−1.50800(0.16)
$4h$	−4.02161	−4.02712(0.14)	−4.01165(−0.25)	−4.02401(0.06)
$5h$	−5.00000	−5.00000(0.00)	−5.00000(0.00)	−5.00000(0.00)
$6h$	−4.02161	−4.02058(−0.03)	−4.01287(−0.22)	−4.02357(0.05)
$7h$	−1.50555	−1.50121(−0.29)	−1.51348(0.53)	−1.50732(0.12)
$8h$	1.50555	1.51648(0.73)	1.51061(0.34)	1.50834(0.19)
$9h$	4.02161	4.03038(0.22)	4.01103(−0.26)	4.02422(0.07)
T	5.00000	4.99999(-0.00)	5.00000(0.00)	5.00000(0.00)

$100[x_{L2}(x_1,x_2) - x_{\mathrm{num}}]/x_{\mathrm{num}}$. For small x, the equation of motion (2) is that of a Duffing-type nonlinear oscillator (i.e., $\ddot{x} + x^3 \approx 0$), while for large x, the equation of motion (2) approximates that of a linear harmonic oscillator (i.e., $\ddot{x} + x \approx 0$). Therefore, x_1 gives poor accuracy for small amplitudes of oscillation (Table 2). Tables 2–4 show that $x_{L2}(t)$ and $x_2(t)$ give excellent analytical approximate periodic solutions for small as well as large amplitudes.

4. Conclusions

A modified iteration method, which is described by Eq. (12), has been applied to the Duffing-harmonic oscillator. The first approximate frequency ω_1 given in Eq. (18) is identical to the result in Ref. [4]. The ω_2 obtained by the second iteration gives very accurate results. The second approximate periodic solution $x_2(t)$ is in good agreement with the numerical solution. Although formula (12) is identical to formula (10) for the first and second iterations, formula (12) is more convenient than formula (10) if the third iteration is required. This is because computing the expressions $x_1^3/1 + x_1^2$ and $3x_1^2 + x_1^4/(1 + x_1^2)^2$ in formula (10) is not an easy task. For each iteration, x_0 in $(x_k - x_0)$ is not the same. For example, x_0 in $(x_1 - x_0)$ is $A \cos \omega_1 t$, and x_0 in $(x_2 - x_0)$ is $A \cos \omega_2 t$. Since ω_2 is more accurate than ω_1, $x_2 - x_0$ is "smaller" than $x_1 - x_0$.

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