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Multi-objective stochastic optimisation of the suspension system of road vehicles

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Abstract

This paper addresses a new approach for designing automotive suspension systems, based on the theory of multi-objective programming together with the theory of robust design. A two-degrees-of-freedom (2 dof) linear model is used to describe the dynamic behaviour of vehicles running on randomly profiled roads. The road irregularity is considered a Gaussian random process and modelled by means of a simple exponential PSD. The performance indices considered are discomfort, road holding and working space. The design variables to be optimised are the suspension stiffness and damping (passively suspended vehicle) and the controller gains (actively suspended vehicle). The mass of the vehicle's body and the tyre radial stiffness are considered as stochastic parameters, together with the design variables (stochastic design variables). The optimal trade-off solutions (Pareto-optimal solutions) are derived in a stochastic framework and, whenever possible, in a non-dimensional analytical form. The analytical expressions are derived by means of a new method based on the Fritz John necessary condition.

1. Introduction

The paper presents a theoretical investigation on the dynamic behaviour of passively and actively suspended road vehicles. The main aim of the paper is to describe in the simplest possible mathematical way (i.e. in analytical form) the basic relationships between vehicle parameters and vehicle performance indices and to define in a rigorous way the best compromise among these indices within a deterministic and stochastic optimisation framework.

Many Authors (see Refs. [1,2]) have attempted the derivation of basic engineering rules useful for the preliminary design of road vehicle suspension systems (active and/or passive). These are generally based on purely numerical computations, even when dealing with very simple vehicle system models. The analytical derivation of simple formulae for the estimation of the dynamic response of road vehicles running on randomly profiled roads is possible in fact only for simple linear vehicle models with 1 or 2 degrees of freedom (dof). According to Ref. [3] the so-called "quarter-car model" (2 dof) has proved to estimate with reasonable accuracy the dynamic behaviour of an actual road vehicle in terms of discomfort, road holding and working

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Nomenclature

A_b	road irregularity parameter (m)
AS	actively suspended vehicle
CV	coefficient of variation $\text{CV} = \sigma_{f_i}/f_i$
f_i, h_i	mean value of the performance index
g_1	controller gain $(x_2 - x_1)$ (N/m)
g_2	controller gain $\dot{x}_2$ (Ns/m)
j	$\text{j} = \sqrt{-1}$
k_1	tyre radial stiffness (N/m)
k_2	suspension stiffness (N/m)
m_1	unsprung mass (kg)
m_2	sprung mass (kg)
MOP	multi-objective programming
PS	passively suspended vehicle
q	ratio between unsprung and sprung mass
r_2	suspension damping (Ns/m)
v	vehicle speed (m/s)
x_1	body m_1: absolute vertical displacement (m)
x_2	body m_2: absolute vertical displacement (m)
α_i	percentile value $\alpha_i = \Phi^{-1}(\beta_i)$
β_i	tolerable level of risk (failure probability $= 1 - \beta_i$)
ξ	imposed vertical displacement (m)
σ_{F_z}	standard deviation of the road/wheel vertical force (road holding) (N)
$\sigma_{\ddot{x}_2}$	standard deviation of the vehicle body acceleration (discomfort) (m/s^2)
$\sigma_{x_2-x_1}$	standard deviation of the suspension stroke (working space) (m)
$\sigma_{f_i}, \sigma_{h_i}$	standard deviation of the performance index
$\Phi^{-1}(\cdot)$	inverse of the standard normal distribution
ω	circular frequency, (rad/s)

space. In almost all the cited papers the road irregularity has been described by means of a very simple exponential power spectral density (PSD), the so-called "*one slope PSD*". More accurate estimates of the amplitudes of the road irregularity, especially at low excitation frequencies, can be achieved by referring to more complex power spectral densities, see Refs. [4,5]. In Refs. [6,7] the so-called "*two slope PSD*" [2,8] has been used, however, this has shown that an analytical approach is rather impractical. For these reasons in the present paper a 2 dof linear vehicle suspension model has been employed and the simple "*one slope PSD*" has been considered.

The definition of the best compromise among conflicting performance indices is not straightforward and multi-objective programming (MOP) has to be used as the proper theoretical basis. In Refs. [9–11] important analytical relationships are highlighted among vehicle suspension parameters and suspension performance indices. In Refs. [6,7] MOP has been used to find the optimal suspension tuning within a deterministic framework. The results are significant; however, the robustness of the optimal solutions is not guaranteed. The spring stiffness and damping rate may, in fact, vary with respect to the nominal value due to production tolerances and/or wear, ageing... The vehicle body mass and the tyre radial stiffness can have stochastic variations due to the variety of possible vehicle loading conditions and to the uncertainty of the inflating pressure of poorly maintained tyres. The uncertainties in the parameters and/or design variables are transmitted to the performance indices so that they too have a stochastic nature. Therefore, an optimal design based on a deterministic approach may turn out to be not robust, leading to substantial unexpected performance deterioration. As observed by many authors [12], optimal designs based on deterministic approaches are often prone to be the most sensitive to uncertainty in the parameters. For this reason a new optimisation method adapted from Ref. [13], based both on MOP and robust design theory (stochastic multi-objective optimisation) is introduced and applied for the computation of the best trade-off among conflicting performance indices pertaining to the vehicle suspension system. The stochastic multi-objective optimisation method employed in the paper involves the simultaneous optimisation of the mean and standard deviation of the performance indices.

Firstly, the performance indices for the quarter car model (road holding, discomfort, working space) in non-dimensional form are derived as a function of the suspension parameters (stiffness, damping, controller gains). Then, a deterministic and a stochastic multi-objective optimisation method are described. A new approach is introduced to derive, in analytical form, the Pareto-optimal sets. Robust and non-robust design solutions are compared. The results, which are presented, can be useful for academic purposes as well as by designers that may take advantage from the simple and very general theory described.

2. System model: equations of motion and response to stochastic excitation

The adopted quarter-car system model for passively (PS) and actively (AS) suspended road vehicles is shown in Fig. 1. The mass m_1 represents approximately the mass of the wheel plus part of the mass of the suspension arms, m_2 represents approximately $\frac{1}{4}$ of the body mass [3,14] and k_1 is the tyre radial stiffness. k_2 and r_2 are, respectively, the linear stiffness and damping of the suspension and g_1 and g_2 are the controller gains of the actuator. The excitation comes from the stochastic road irregularity ξ. The linear equations of motions of the system are

$$\begin{aligned} &m_1\ddot{x}_1 - r_2(\dot{x}_2 - \dot{x}_1) - k_2(x_2 - x_1) + k_1(x_1 - \xi) - F_{\text{act}} = 0, \\ &m_2\ddot{x}_2 + r_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) + F_{\text{act}} = 0. \end{aligned} \tag{1}$$

The actuator force is assumed to be a function of the relative displacement wheel–vehicle body and of the absolute velocity of the vehicle body mass m_2 (passive spring and simple skyhook damping)

$$F_{\text{act}} = g_1(x_2 - x_1) + g_2\dot{x}_2. \tag{2}$$

Although very simple (in order to allow an analytical approach), the considered model is generally reputed sufficiently accurate for capturing the essential features related to discomfort, road holding and working space (see Ref. [3]).

For the PS model the design variables are k_2 and r_2. The actuator force disappears by setting $g_1 = g_2 = 0$. For the AS model, instead, the design variables are g_1 and g_2, and k_2 and r_2 are set to zero. In particular, k_2 disappears because its effect is actually the same as that of g_1. r_2 is set to zero for simplicity. Specifically, the so-called "Modified Skyhook Damping" strategy ($r_2 \neq 0$) has not been considered due to the complexity of the derived analytical expressions.

The displacement ξ (road irregularity) may be represented by a random variable defined by a stationary and ergodic stochastic process with zero mean value [4,5]. The PSD of the process may be determined on the basis of experimental measurements and in the literature there are many different formulations for it, see Refs. [2,4].

In the present paper for sake of simplicity the following spectrum has been considered

$$S_\xi(\omega) = \frac{A_b v}{\omega^2}. \tag{3}$$

In a log–log scaled plot (abscissa ω), the spectrum of Eq. (3) takes the shape of a line with gradient -2. By adjusting the parameter A_b, Eq. (3) approximates various roads with a satisfactory degree of accuracy (it generally overestimates the amplitudes of the irregularity at low frequencies). A better correlation with measured spectra can be obtained by resorting to more complex expressions as, for example, the so-called "*two slopes spectrum*" (see Ref. [4]). However, by employing a more complex spectrum only a numerical solution can be found.

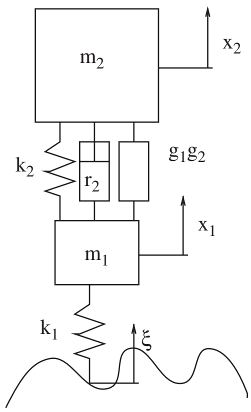

Fig. 1. Simplified vehicle model.

The outputs of the vehicle model are the vertical vehicle body acceleration ($\ddot{x}_2$), the force applied between road and wheel (F_z) and the relative displacement between wheel and vehicle body ($x_2 - x_1$). Discomfort ($\sigma_{\ddot{x}_2}$, standard deviation of $\ddot{x}_2$, see Ref. [8]), road holding (σ_{F_z}, standard deviation of F_z, see Ref. [10]) and working space ($\sigma_{x_2-x_1}$, standard deviation of $x_2 - x_1$, see Ref. [6]) are the *performance indices* to be minimized.

The PSD S_l of the output of an asymptotically stable system can be computed as (see, e.g. Ref. [15])

$$S_l(\omega) = |H_l(\mathrm{j}\omega)|^2 S_\xi(\omega) \quad l = 1, \ldots, 3. \tag{4}$$

for $l = 1$, S_l represents the PSD of the vertical acceleration of the vehicle body and H_l represents the transfer function between ξ and $\ddot{x}_2$, for $l = 2$, S_l is the PSD of the vertical force at the wheel–road interface and H_l is the transfer function between ξ and F_z, and for $l = 3$, S_l represents the PSD of the relative displacement chassis-wheel (suspension stroke) and H_l represents the transfer function between ξ and $x_2 - x_1$.

By definition (see Refs. [15,16]) the variance of a random variable described by a stationary and ergodic stochastic process (whose PSD is S_l) is

$$\sigma_l^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_l(\omega)\, \mathrm{d}\omega, \tag{5}$$

where σ_l refer, respectively, to $\sigma_{\ddot{x}_2}$, σ_{F_z}, $\sigma_{x_2-x_1}$. In this way it is possible to derive the analytical expressions of the three performance indices for both the PS and the AS vehicle models.

2.1. Formulae referring to the passively suspended (PS) vehicle

The analytical formulae giving the performance indices of a PS vehicle ($g_1 = g_2 = 0$) have been already derived and presented in Refs. [6,10,17]. These formulae can be simplified by introducing the following non-dimensional variables

$$q = \frac{m_1}{m_2}, \quad K_x = k_2 \frac{(1+q)^2}{k_1 q}, \quad R_x = r_2 \sqrt{\frac{(1+q)^3}{k_1 m_2 q}}. \tag{6}$$

The performance indices for the PS vehicle model are the following:

- Standard deviation of the relative displacement between wheel and vehicle body $x_2 - x_1$ (working space)

$$\sigma_{x_2-x_1} = \sqrt{1/2 A_b v} f_1, \quad f_1^2 = \sqrt{\frac{m_2 (1+q)^5}{k_1 q}} \left(\frac{1}{R_x} \right). \tag{7}$$

- Standard deviation of the vehicle body acceleration $\ddot{x}_2$ (discomfort)

$$\sigma_{\ddot{x}_2} = \sqrt{1/2 A_b v} f_2, \quad f_2^2 = \sqrt{\frac{k_1^3 q^3}{m_2^3 (1+q)^3}} \left(\frac{K_x^2}{R_x} + \frac{R_x}{q} \right). \tag{8}$$

- Standard deviation of the force acting between road and wheel F_z (road holding)

$$\sigma_{F_z} = \sqrt{1/2 A_b v} f_3, \quad f_3^2 = \sqrt{k_1^3 m_2 q^3 (1+q)} \left(\frac{(K_x - 1)^2}{R_x} + \frac{1}{q} \left(R_x + \frac{1}{R_x} \right) \right). \tag{9}$$

2.2. Formulae referring to the actively suspended (AS) vehicle

The analytical formulae giving the discomfort, road holding and working space for the AS vehicle model ($k_2 = r_2 = 0$) are given in Ref. [7]. These formulae can be written in a more compact expression by introducing the following non-dimensional variables

$$q = \frac{m_1}{m_2}, \quad G_{1x} = g_1 \frac{(1+q)}{k_1}, \quad G_{2x} = g_2 \sqrt{\frac{q}{k_1 m_2}}. \tag{10}$$

The performance indices for the AS vehicle model are the following:

- Standard deviation of the relative displacement between wheel and vehicle body $x_2 - x_1$ (working space)

$$\sigma_{x_2 - x_1} = \sqrt{1/2 A_b v} h_1, \quad h_1^2 = \left(\frac{m_2 q (1+q)^2}{k_1} \right)^{1/2} \frac{1}{G_{1x}} \left(\frac{1}{G_{2x}} + G_{2x} \left(1 + \frac{1}{q} \right) \right). \tag{11}$$

- Standard deviation of the vehicle body acceleration $\ddot{x}_2$ (discomfort)

$$\sigma_{\ddot{x}_2} = \sqrt{1/2 A_b v} h_2, \quad h_2^2 = \left(\frac{k_1^3 q}{m_2^3 (1+q)^2} \right)^{1/2} \frac{G_{1x}}{G_{2x}}. \tag{12}$$

- Standard deviation of the force acting between road and wheel F_z (road holding)

$$\sigma_{F_z} = \sqrt{\frac{1}{2} A_b v} h_3, \quad h_3^2 = (k_1^3 m_2 (1+q)^2 q)^{1/2} \left(\frac{(G_{1x} - 1)^2}{G_{1x} G_{2x}} + \frac{G_{2x}}{G_{1x}} + \frac{1}{G_{2x}(1+q)} \right). \tag{13}$$

3. Multi-objective robust optimisation

3.1. Deterministic formulation

The MOP theory, see Ref. [18] for details, refers to the minimisation of a vector of objective functions ($f = [f_1, f_2, \ldots, f_k]$) that depend on a vector of design variables ($z = [z_1, z_2, \ldots, z_n]$) defined into a feasible domain (Z)

$$\min f(z) = \min(f_1(z), f_2(z), \ldots, f_k(z))$$

with

$$z = [z_1, z_2, \ldots, z_n] \in Z. \tag{14}$$

Generally, the objective functions are conflicting, therefore there is no obvious optimal solutions. There is not a single solution, as in a single objective problem, but a set of optimal or *efficient* solutions, that is called Pareto-optimal set. For a solution belonging to the optimal set it is not possible to improve one objective function without worsening at least another one. The Pareto-optimal set represents the best obtainable compromises between all the conflicting objective functions. So, the designer has to choose one single final solution among this set.

In recent years many authors have proposed numerical methods for the computation of the Pareto-optimal set. They are mainly based on Monte-Carlo search procedures [19,20], on constrained optimisation approaches or on the use of aggregate functions [18,21].

3.2. Stochastic formulation

Deterministic optimisation techniques have been employed to solve a wide variety of mechanical engineering design problems. However, actual systems are subject to variations and uncertainties that arise from a variety of sources, i.e. manufacturing processes, external disturbances, operating conditions. For this reason almost every engineering design should be performed within a stochastic framework.

A stochastic system is described by a mathematical model in which there are some random quantities subject to uncertainty. These quantities may be parameters (c), whose values are not under the designer's control, or design variables (z), whose expected (i.e. mean) value can be freely defined by the designer. The uncertainties on the parameters and the design variables are transmitted to the performance indices and so they have a stochastic nature too. We will indicate with $f_i(z,c)$ the stochastic performance index whose mean value and variance can be estimated as

$$f_i = f_i(\mu_z, \mu_c), \quad \sigma_{f_i}^2 = \sum_{i=1}^{n} \left(\frac{\partial f_i}{\partial z_i}\right)^2 \sigma_{z_i}^2 + \sum_{i=1}^{m} \left(\frac{\partial f_i}{\partial c_i}\right)^2 \sigma_{c_i}^2. \tag{15}$$

When considering a stochastic system, the Pareto-optimal solutions can be obtained by transforming the original stochastic problem into an *equivalent deterministic problem* [13]. The formulation of the equivalent deterministic problem adopted in the present paper is known as the K_β formulation or *β-efficient problem*

$$\begin{aligned} &\text{Given the probabilities } \beta_1, \beta_2, \ldots, \beta_k \\ &\text{find } \min(u_1, u_2, \ldots, u_k) \\ &\text{such that } \mathrm{Prob}(f_i(z,c) \leqslant u_i) \geqslant \beta_i \quad \text{with } i = 1, 2, \ldots, k. \end{aligned} \tag{16}$$

The probabilities β_i describe the tolerable level of risk (fixed by the designer) that the performance of the optimal solutions will be worse than expected (the failure probability is F.P. $= 1 - \beta_i$). This formulation seems to be the best for many engineering problems in which, in general, the levels of risk are fixed by standards or good design practice while little is known about the maximum obtainable performances. If it is assumed that the performance indices have normal distribution, the K_β problem can be converted [13] to the so-called K_α problem or *β-efficient problem with normal distribution*

$$\begin{aligned} &\text{Given the probabilities } \beta_1, \beta_2, \ldots, \beta_k \\ &\text{find } \min(\bar{f}_1(z,c) + \alpha_1 \sigma_{f_1}(z,c),\ \bar{f}_2(z,c) + \alpha_2 \sigma_{f_2}(z,c), \ldots,\ \bar{f}_k(z,c) + \alpha_k \sigma_{f_k}(z,c)) \\ &\text{with } \alpha_i = \Phi^{-1}(\beta_i) \quad \text{for } i = 1, 2, \ldots, k, \end{aligned} \tag{17}$$

where the function $\Phi^{-1}(\cdot)$ is the inverse of the standard normal distribution and the percentile α_i corresponds to the objective function f_i. So the robust design approach [22] involves the optimisation of a weighted sum of mean $\bar{f}_i$ and standard deviation σ_{f_i} of the performance indices.

3.3. Analytical method to find Pareto-optimal solutions

The analytical expression of the Pareto-optimal set can be derived by the *Fritz John necessary condition* for unconstrained Pareto optimality [18]. Let the performance indices of problem (14) be continuously differentiable at a decision vector $z^* \in Z$. A necessary condition for z^* to be Pareto optimal is that there exist a vector $\mathbf{0} < \lambda \in R^k$ such that

$$\sum_{i=1}^{k} \lambda_i \nabla f_i(z^*) = 0. \tag{18}$$

By writing the Jacobian matrix of the objective functions as

$$\nabla F = [\nabla f_1\ \nabla f_2\ \ldots\ \nabla f_k] = \begin{bmatrix} \dfrac{\partial f_1}{\partial z_1} & \cdots & \dfrac{\partial f_k}{\partial z_1} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_1}{\partial z_n} & \cdots & \dfrac{\partial f_k}{\partial z_n} \end{bmatrix} \tag{19}$$

the Fritz–John condition can be simply re-written as

$$\nabla F \lambda = \mathbf{0} \Rightarrow (\nabla F^{\mathrm{T}} \nabla F)\lambda = \mathbf{0} \Rightarrow \det(\nabla F^{\mathrm{T}} \nabla F) = 0. \tag{20}$$

If $n = k = 2$ Eq. (20) leads to

$$\frac{\partial f_1}{\partial z_1}\frac{\partial f_2}{\partial z_2} = \frac{\partial f_1}{\partial z_2}\frac{\partial f_2}{\partial z_1}. \tag{21}$$

The Pareto-optimal set is therefore the curve described by Eq. (21) bounded by the two minima of the two objective functions. This equation allows the computation of the analytical expression of the Pareto-optimal set in a very effective way. This mathematical procedure will be used in Sections 5 and 6.

4. Stochastic performance indices

For the PS vehicle the design variables to be optimised are the stiffness k_2 and the damping r_2 of the suspension system. They are considered as stochastic design variables, that is, their mean value is given but their actual value may vary randomly following a normal distribution. For the AS vehicle, the design variables are the gains g_1 and g_2. They are considered as deterministic quantities. The vehicle body mass m_2 and the tyre radial stiffness k_1 are considered as stochastic parameters due to the variety of possible loading conditions and to the variability of the tyre pressure. The unsprung mass m_1 is instead a deterministic parameter. The variability of all stochastic parameters and design variables are described by the coefficients of variation $\mathrm{CV}_i = \sigma_i/\mu_i$ (σ standard deviation, μ mean value). The values of the coefficients of variation CV_i have been selected according to the data available in literature [23,24] and their values turned out to be 3% for the spring stiffness and 10% for the damping coefficient, the tyre radial stiffness and the sprung mass.

The adopted reference values, variation ranges and CV_i of the design variables and of the parameters are reported in Table 1. The mean values of the performance indices are f_i and h_i (Eqs. (7–9), (11–13)), the standard deviations σ_{f_i} and σ_{h_i} are computed in analytical form by means of Eq. (15).

4.1. Stochastic performance indices for the passively suspended vehicle

The stochastic performance indices for the PS vehicle, derived by means of the K_α formulation (see Eq. (16)), are the following

Working space:

$$\sigma_{x_2-x_1} = \sqrt{1/2A_b v}\cdot\overline{f}_1,$$
$$\overline{f}_1 = f_1 + \alpha_1\sigma_{f_1} = \left(\frac{m_2(1+q)^5}{k_1 q}\right)^{1/4} F_{1x}(R_x, K_x, q, \alpha_1). \tag{22}$$

Discomfort:

$$\sigma_{\ddot{x}_2} = \sqrt{1/2A_b v}\cdot\overline{f}_2,$$
$$\overline{f}_2 = f_2 + \alpha_2\sigma_{f_2} = \left(\frac{k_1^3 q^3}{m_2^3(1+q)^3}\right)^{1/4} F_{2x}(R_x, K_x, q, \alpha_2). \tag{23}$$

Table 1
Design variables and parameters

Units	Reference value	Lower–upper bounds	CV
k_2 (N/m)	25000	0–80 000	.03
r_2 (Ns/m)	1000	0–5000	.10
g_1 (N/m)	—	0–700 000	—
g_2 (Ns/m)	—	0–50 000	—
k_1 (N/m)	120 000	—	.10
m_2 (kg)	229	—	.10
m_1 (kg)	31	—	—

Road holding:

$$\sigma_{F_z} = \sqrt{1/2A_b v} \cdot \bar{f}_3,$$
$$\bar{f}_3 = f_3 + \alpha_3 \sigma_{f_3} = (k_1^3 m_2 q^3 (1+q))^{1/4} F_{3x}(R_x, K_x, q, \alpha_3), \tag{24}$$

where α_i are the percentiles of the three performance indices (Eq. (17)).

Since the functions F_{ix} do not depend directly on k_1, m_1 and m_2, Eq. (21) states that the Pareto-optimal set, in the plane of the non-dimensional variables R_x and K_x, depends only on α_i and q.

4.2. Stochastic performance indices for the actively suspended vehicle

The stochastic performance indices of the AS vehicle, derived by means of the K_α formulation, are the following

Working space:

$$\sigma_{x_2 - x_1} = \sqrt{1/2A_b v} \cdot \bar{h}_1,$$
$$\bar{h}_1 = h_1 + \alpha_1 \sigma_{h_1} = \left(\frac{m_2 q (1+q)^2}{k_1} \right)^{1/4} H_{1x}(G_{1x}, G_{2x}, q, \alpha_1). \tag{25}$$

Discomfort:

$$\sigma_{\ddot{x}_2} = \sqrt{1/2A_b v} \cdot \bar{h}_2,$$
$$\bar{h}_2 = h_2 + \alpha_2 \sigma_{h_2} = \left(\frac{k_1^3 q}{m_2^3 (1+q)^2} \right)^{1/4} H_{2x}(G_{1x}, G_{2x}, q, \alpha_2). \tag{26}$$

Road holding:

$$\sigma_{F_z} = \sqrt{1/2A_b v} \cdot \bar{h}_3,$$
$$\bar{h}_3 = h_3 + \alpha_3 \sigma_{h_3} = (k_1^3 m_2 q (1+q)^2)^{1/4} H_{3x}(G_{1x}, G_{2x}, q, \alpha_3), \tag{27}$$

where α_i are the percentiles of the three performance indices (Eq. (17)).

Since the functions H_{ix} do not depend directly on k_1, m_1 and m_2, the pareto-optimal set, in the plane of the non-dimensional variables G_{1x} and G_{2x}, will depend only on α_i and q.

5. Pareto-optimal set for the PS vehicle

The considered design variables are K_x and R_x, the functions to be minimised are $\bar{f}_1$, $\bar{f}_2$ and $\bar{f}_3$ (Eqs. (22–24)). The following subsections describe the computation of the Pareto-optimal set in the domain of the design variables for the combinations of two performance indices, namely: $\sigma_{\ddot{x}_2}$-σ_{F_z}, $\sigma_{\ddot{x}_2}$-$\sigma_{x_2-x_1}$, σ_{F_z}-$\sigma_{x_2-x_1}$. The Pareto-optimal set in the plane of the performance indices can be easily derived by substituting the optimal design variables values into the performance indices expressions.

5.1. Suspension parameters for optimal $\sigma_{\ddot{x}_2}, \sigma_{F_z}$

The minima of the road holding (σ_{F_z}) in the deterministic case ($\alpha_3 = 0$) is obtained (see Eq. (9)) by setting

$$K_x = 1, \quad R_x = 1. \tag{28}$$

For the stochastic problem the analytical expressions of $\overline{K_x}$ and $\overline{R_x}$ which define the minima of the road holding appear to be too involved to be presented here. Approximate expressions are provided in Appendix.

The minima of the discomfort ($\sigma_{\ddot{x}_2}$) is at the point defined by

$$K_x = 0, \quad R_x = 0 \tag{29}$$

for both the deterministic and the stochastic problems. The analytical expression of the Pareto-optimal set in the deterministic case can be derived by means of Eq. (21)

$$\frac{\partial f_2}{\partial K_x}\frac{\partial f_3}{\partial R_x}=\frac{\partial f_2}{\partial R_x}\frac{\partial f_3}{\partial K_x}. \tag{30}$$

This equation leads to the following compact formula:

$$R_x=\sqrt{(1+q)K_x-qK_x^2} \quad \text{with } 0\leqslant K_x\leqslant 1. \tag{31}$$

For the stochastic problem the full analytical expression of the Pareto-optimal set is, again, very involved. By considering $\alpha_2=\alpha_3=\alpha$, a good approximation has been found and is given below

$$\begin{cases} R_x=\Gamma_{\sigma_{\ddot{x}_2},\sigma_{F_z}}(q,\alpha,K_x), \\ 0\leqslant K_x\leqslant \overline{K}_x \end{cases} \tag{32}$$

the function $\Gamma_{\sigma_{\ddot{x}_2},\sigma_{F_z}}$ is reported in the appendix. The deterministic solutions (31) can be found from the stochastic ones by setting to zero the percentile α.

5.2. Suspension parameters for optimal $\sigma_{\ddot{x}_2},\sigma_{x_2-x_1}$

The minimum of the working space ($\sigma_{x_2-x_1}$) is at

$$R_x\to\infty. \tag{33}$$

The Pareto-optimal set is derived by means of Eq. (21)

$$\frac{\partial \overline{f}_1}{\partial K_x}\frac{\partial \overline{f}_2}{\partial R_x}=\frac{\partial \overline{f}_1}{\partial R_x}\frac{\partial \overline{f}_2}{\partial K_x}. \tag{34}$$

For the deterministic problem the exact solution of this equation is

$$R_x\geqslant 0, \quad K_x=0. \tag{35}$$

For the stochastic problem an approximate solution of Eq. (34) is

$$\begin{cases} R_x\geqslant 0, \quad K_x=\Gamma_{\sigma_{\ddot{x}_2},\sigma_{x_2-x_1}}(q,\alpha_2)\cdot R_x & \text{if} \quad \Gamma_{\sigma_{\ddot{x}_2},\sigma_{x_2-x_1}}(q,\alpha_2)\in\Re, \\ R_x\geqslant 0, \quad K_x=0 & \text{otherwise.} \end{cases} \tag{36}$$

The expression of $\Gamma_{\sigma_{\ddot{x}_2},\sigma_{x_2-x_1}}$ is reported in the Appendix. The optimal set for the stochastic problem is a sloped line starting from the origin, instead for the deterministic model the optimal set is the R_x-axis, see Ref. [6].

5.3. Suspension parameters for optimal $\sigma_{x_2-x_1},\sigma_{F_z}$

The exact analytical expression of the Pareto-optimal set for the deterministic problem is

$$R_x\geqslant 1, \quad K_x=1. \tag{37}$$

For the stochastic problem the Pareto-optimal set is too involved to be useful for technical purposes. The expression has been approximated with

$$R_x\geqslant \overline{R}_x, \quad K_x=\Gamma_{\sigma_{x_2-x_1},\sigma_{F_z}}(q,\alpha_3,R_x). \tag{38}$$

The function $\Gamma_{\sigma_{x_2-x_1},\sigma_{F_z}}$ is reported in the appendix. The Pareto-optimal set for the stochastic problem is a sloped line, whilst for the deterministic problem it is a line parallel to the R_x-axis, see Ref. [6].

The overall Pareto-optimal set for the three performance indices together is the area bounded by the bidimensional Pareto-optimal curves $\sigma_{\ddot{x}_2},\sigma_{F_z}$ (Eq. (31),(32)), $\sigma_{\ddot{x}_2},\sigma_{x_2-x_2}$ (Eq. (35),(36)), $\sigma_{x_2-x_1},\sigma_{F_z}$ (Eq. (37),(38)). These Pareto-optimal curves are limited by the three points which minimise respectively road holding, discomfort and working space. Any combination of non-dimensional stiffness and damping that belongs to these areas, see Figs. 2 and 8, constitutes an optimal trade-off between the three objective functions. For sake of simplicity, in the following we have considered $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$.

Figs. 2 and 8 show the Pareto-optimal sets in the plane of the design variables. Figs. 3 and 4 show the mean performances of the Pareto-optimal solutions in the domain of the performance indices. The length of vertical and horizontal bars at each of the marked points represent the standard deviation of the two performance indices. The influence of the variation of the parameters and variables (given in Table 1) on the performance indices is reported in Table 2. The weak influence of the variable k_2 is manly due to its small coefficient of variation (CV).

In order to illustrate the advantages of the robust optimisation approach, three Pareto-optimal sets have been plotted in Fig. 2 for three different values of the percentiles α. The first Pareto-optimal set has been obtained by setting $\alpha = 0$ that corresponds to a deterministic optimisation, i.e. only the mean values of the objective functions are considered. The deterministic optimisation leads to the best mean performances but with the lowest robustness (F.P. = .5). On the other hand, we have analysed the optimisation of the variances only when $\alpha \to \infty$ which gives the maximum robustness (F.P. $\to$ 0) without taking in account the mean performance. The resulting design is a very stiff and damped suspension with poor mean performances as shown in Figs. 3 and 4. A good compromise between robustness and performance can be achieved by considering the third Pareto-optimal set shown in Fig. 2. This optimal region, derived for $\alpha = 6$, corresponds to a very small failure probability of F.P. = 10^{-9} and it has good mean performances with high robustness. The Pareto-optimal set obtained for $\alpha = 6$ is indeed similar to the deterministic one in terms of expected values of the performance indices as shown in Figs. 3 and 4.

Table 3 shows a comparison of the mean and the standard deviation for road holding and working space between the reference passive suspension (see Table 1) and the points of minimum of the road holding (marked by means of circle, square and diamond) for the three considered values of the percentiles ($\alpha = 0, 6, +\infty$).

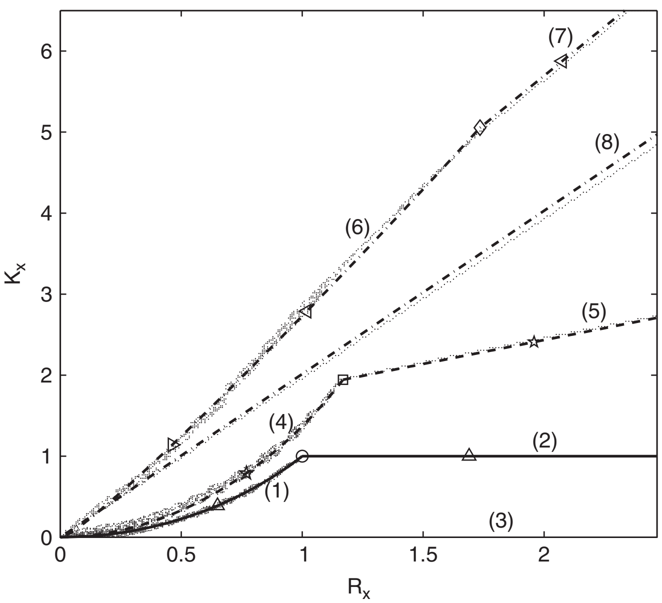

Fig. 2. Pareto-optimal design variables for the PS vehicle in non-dimensional coordinates K_x, R_x (Eq. (6)). Road roughness parameter $A_b = 6.9\text{E-}6$, vehicle speed 20 m/s, vehicle data in Table 1. Solid lines refer to the exact analytical equations for $\alpha = 0$ (F.P. = 0.5). (1) $\sigma_{\ddot{x}_2}, \sigma_{F_z}$ Eq. (31), (2) $\sigma_{x_2-x_1}, \sigma_{F_z}$ Eq. (38), (3) $\sigma_{\ddot{x}_2}, \sigma_{x_2-x_1}$ Eq. (35). Dotted lines refer to the approximated analytical equations for $\alpha = 6$ (F.P. = 1E-9): (4) $\sigma_{\ddot{x}_2}, \sigma_{F_z}$ Eq. (32), (5) $\sigma_{x_2-x_1}, \sigma_{F_z}$ Eq. (38), (3) $\sigma_{\ddot{x}_2}, \sigma_{x_2-x_1}$ Eq. (36). Dash-dotted lines refer to the approximated analytical equations for $\alpha \to \infty$ (F.P. $\to$ 0): (6) $\sigma_{\ddot{x}_2}, \sigma_{F_z}$ Eq. (32), (7) $\sigma_{x_2-x_1}, \sigma_{F_z}$ Eq. (38), (8) $\sigma_{\ddot{x}_2}, \sigma_{x_2-x_1}$ Eq. (36). Points refer to numerical computations. Points marked by symbols correspond to points in Figs. 3 and 4. The Pareto-optimal sets for $\alpha = 0$, $\alpha = 6$ and $\alpha \to \infty$ are shown also in Fig. 8.

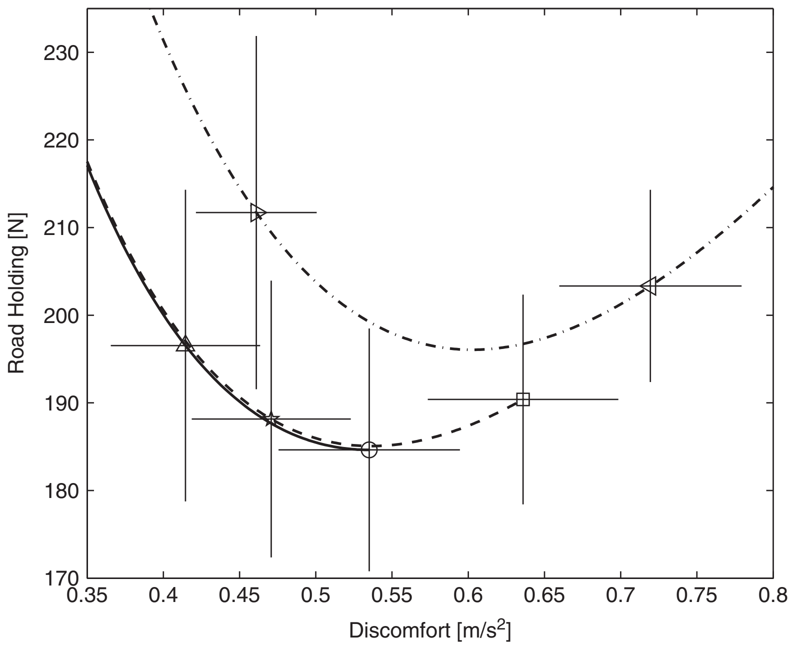

Fig. 3. Pareto-optimal set in the performance indices domain $\sigma_{\ddot{x}_2}, \sigma_{F_z}$ for the PS vehicle: solid line for $\alpha = 0$, dotted line for $\alpha = 6$ and dash-dotted line for $\alpha \rightarrow \infty$. Road roughness parameter $A_b = 6.9\text{E-}6$, vehicle speed 20 m/s, vehicle data in Table 1. Design variables in Fig. 2, points marked by symbols correspond to those in Fig. 2. The vertical and horizontal bars, at each marked point, define the variations of the two performance indices with respect to their mean values due to uncertain parameters (vehicle mass and tyre stiffness) and tolerances on design variables (see Table 1).

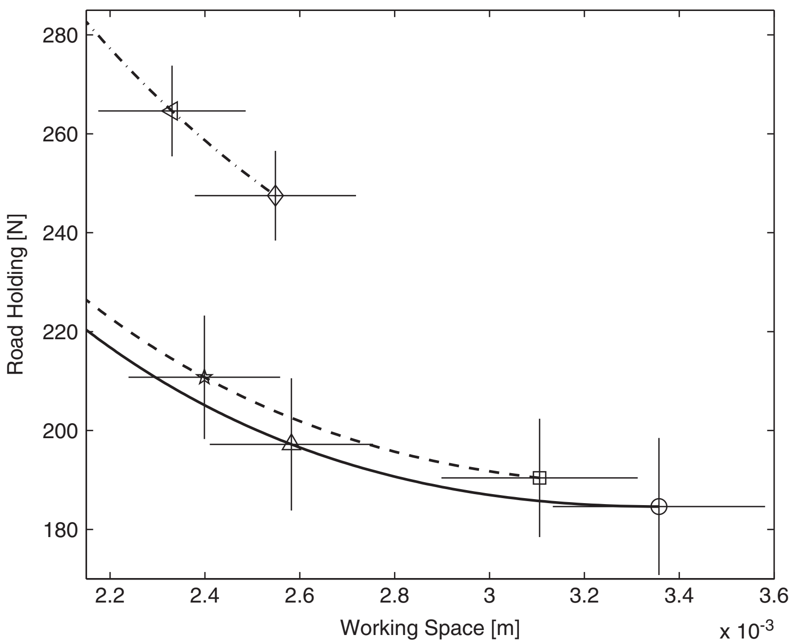

Fig. 4. Pareto-optimal set in the performance indices domain $\sigma_{x_2-x_1}, \sigma_{F_z}$ for the PS vehicle: solid line for $\alpha = 0$, dotted line for $\alpha = 6$ and dash-dotted line for $\alpha \rightarrow \infty$. Road roughness parameter $A_b = 6.9\text{E-}6$, vehicle speed 20 m/s, vehicle data in Table 1. Design variables in Fig. 2, points marked by symbols correspond to those in Fig. 2. The vertical and horizontal bars, at each marked point, define the variations of the two performance indices with respect to their mean values due to uncertain parameters (vehicle mass and tyre stiffness) and tolerances on design variables (see Table 1).

Table 2
Effect of the parameters and variables variations on the deviations of the performance indices

	PS vehicle model			AS vehicle model		
	$\sigma_{x_2-x_1}$	$\sigma_{\ddot{x}_2}$	σ_{F_z}	$\sigma_{x_2-x_1}$	$\sigma_{\ddot{x}_2}$	σ_{F_z}
σ_{k_2}	—	1%	1%	—	—	—
σ_{r_2}	55%	23%	25%	—	—	—
σ_{k_1}	—	25%	68%	20%	50%	70%
σ_{m_2}	45%	51%	6%	80%	50%	30%

Road roughness parameter $A_b = 6.9\text{E-}6$, vehicle speed 20 m/s, vehicle data in Table 1.

Table 3
Comparison of mean and standard deviation of road holding (f_1) and working space (f_2) between the reference passive suspension and the minimum of road holding for PS and AS vehicle models for three different values of the percentiles

Reference PS suspension	f_1 203 (N)	f_2 4.24 (mm)	σ_{f_1} 15.2 (N)	σ_{f_2} .282 (mm)
PS suspension ($\alpha = 0$)	−9.4%	−20.8%	−9.2%	−20.9%
PS suspension ($\alpha = 6$)	−6.4%	−26.9%	−21.7%	−26.9%
PS suspension ($\alpha \to \infty$)	+21.6%	−39.9%	−40.1%	−39.7%
AS suspension ($\alpha = 0$)	−14.8%	−28.3%	−12.5%	−95.6%
AS suspension ($\alpha = 6$)	−13.3%	−34.9%	−23.0%	−95.8%
AS suspension ($\alpha \to \infty$)	−7.9%	−42.7%	−26.9%	−96.3%

Road roughness parameter $A_b = 6.9\text{E-}6$, vehicle speed 20 m/s, vehicle data in Table 1.

6. Pareto-optimal set for the AS model

The considered design variables are G_{1x} and G_{2x}, the functions to be minimised are $\overline{h}_1$, $\overline{h}_2$ and $\overline{h}_3$ (Eq. (25–27)). The following subsections describe the computation of the Pareto-optimal set in the domain of the design variables for the combinations of two performance indices. The Pareto-optimal set in the plane of the performance indices can be derived by substitution.

6.1. *Suspension parameters for optimal* $\sigma_{x_2-x_1}, \sigma_{F_z}$

The minimum of the road holding in the deterministic case ($\alpha_3 = 0$) is simply obtained by setting (see Ref. [7])

$$G_{1x} = \frac{2(1+q)}{1+2q}, \quad G_{2x} = \frac{\sqrt{3+4q}}{1+2q}. \tag{39}$$

By considering the stochastic problem the analytical expressions of $\overline{G}_{1x}$ and $\overline{G}_{2x}$ which define the minimum of the road holding are too complex to be used. Approximate expressions are given in the appendix. The minimum of the working space is obtained at

$$G_{1x} \to \infty. \tag{40}$$

The analytical expression of the Pareto-optimal set in the deterministic case can be derived by means of Eq. (21)

$$\frac{\partial h_2}{\partial G_{1x}}\frac{\partial h_3}{\partial G_{2x}} = \frac{\partial h_2}{\partial G_{2x}}\frac{\partial h_3}{\partial G_{1x}}. \tag{41}$$

This equation leads to

$$G_{2x} = \sqrt{\frac{2qG_{1x}}{1+2q}\left(\frac{G_{1x}-\frac{1+2q}{2(1+q)}}{G_{1x}-\frac{2}{1+2q}}\right)} \quad \text{with } G_{1x} \geqslant \frac{2(1+q)}{1+2q}. \tag{42}$$

By considering the stochastic problem with $\alpha_1 = \alpha_3 = \alpha$, a good numerical approximation has proven to be

$$\begin{cases} G_{1x} \geqslant \overline{G}_{1x}, \quad G_{2x} = \Psi^1_{\sigma_{x_2-x_1},\sigma_{F_z}}(q,\alpha,G_{1x}) \quad \text{if} \quad \alpha < 15, \\ G_{2x} \leqslant \overline{G}_{2x}, \quad G_{1x} = \Psi^2_{\sigma_{x_2-x_1},\sigma_{F_z}}(q,\alpha,G_{2x}) \quad \text{if} \quad \alpha \to \infty. \end{cases} \tag{43}$$

The functions $\Psi^1_{\sigma_{x_2-x_1},\sigma_{F_z}}$ and $\Psi^2_{\sigma_{x_2-x_1},\sigma_{F_z}}$ are reported in the appendix.

6.2. Suspension parameters for optimal $\sigma_{\ddot{x}_2}, \sigma_{x_2-x_1}$

The minimum of the discomfort is

$$G_{2x} \to \infty \quad \text{or} \quad G_{1x} = 0. \tag{44}$$

The Pareto-optimal set has been derived by means of Eq. (21)

$$\frac{\partial \overline{h}_1}{\partial G_{1x}}\frac{\partial \overline{h}_2}{\partial G_{2x}} = \frac{\partial \overline{h}_1}{\partial G_{2x}}\frac{\partial \overline{h}_2}{\partial G_{1x}}. \tag{45}$$

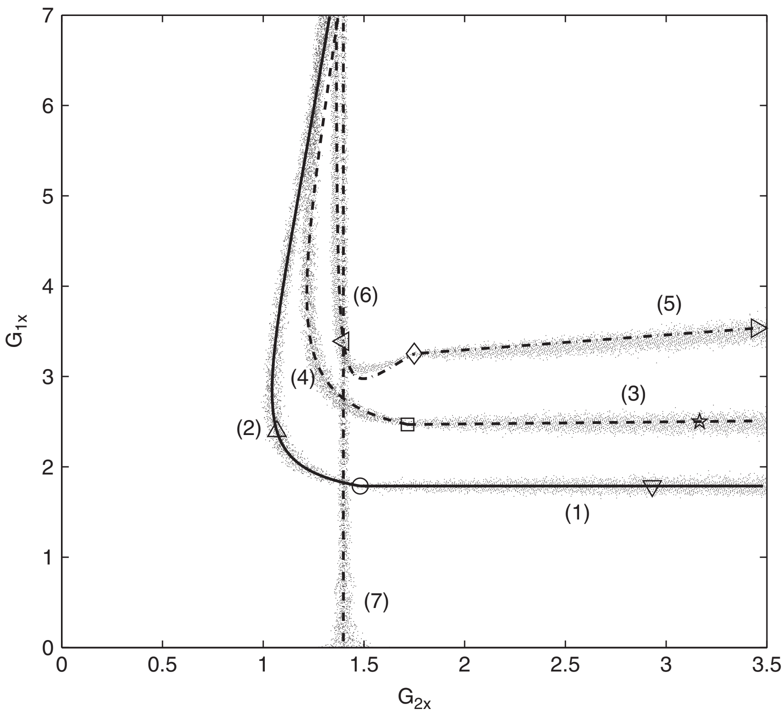

Fig. 5. Pareto-optimal design variables for the AS vehicle in non-dimensional coordinates G_{1x}, G_{2x} (Eq. (10)). Road roughness parameter $A_b = 6.9\text{E-}6$, vehicle speed 20 m/s, vehicle data in Table 1. Solid lines refer to the exact analytical equations for $\alpha = 0$ (F.P. = 0.5): (1) $\sigma_{\ddot{x}_2}, \sigma_{F_z}$ Eq. (48), (2) $\sigma_{x_2-x_1}, \sigma_{F_z}$ Eq. (42). Dotted lines refer to the approximated analytical equations for $\alpha = 6$ (F.P. = 1E-9): (3) $\sigma_{\ddot{x}_2}, \sigma_{F_z}$ Eq. (49), (4) $\sigma_{x_2-x_1}, \sigma_{F_z}$ Eq. (43). Dash-dotted lines refer to the approximated analytical equations for $\alpha \to \infty$ (F.P. $\to$ 0): (5) $\sigma_{\ddot{x}_2}, \sigma_{F_z}$ Eq. (49), (6) $\sigma_{x_2-x_1}, \sigma_{F_z}$ Eq. (43), (7) $\sigma_{\ddot{x}_2}, \sigma_{x_2-x_1}$ Eq. (47). Points refer to numerical computations. Points marked by symbols correspond to points in Figs. 6 and 7. The Pareto-optimal sets for $\alpha = 0$, $\alpha = 6$ and $\alpha \to \infty$ are shown also in Fig. 8.

For the deterministic problem the exact solution of this equation is

$$G_{1x} \geqslant 0, \quad G_{2x} \to \infty. \tag{46}$$

For the stochastic problem an approximate solution of Eq. (45) is

$$\begin{cases} G_{1x} \geqslant 0, \quad G_{2x} = \Psi_{\sigma_{\ddot{x}_2},\sigma_{x_2-x_1}}(q, \alpha_1) & \text{if} \quad \Psi_{\sigma_{\ddot{x}_2},\sigma_{x_2-x_1}} \in \Re, \\ G_{1x} \geqslant 0, \quad G_{2x} \to \infty & \text{otherwise}. \end{cases} \tag{47}$$

The function $\Psi_{\sigma_{\ddot{x}_2},\sigma_{x_2-x_1}}$ is reported in the appendix.

6.3. Suspension parameters for optimal $\sigma_{\ddot{x}_2}, \sigma_{F_z}$

The analytical expression of the Pareto-optimal set for the deterministic problem is

$$G_{2x} \geqslant \frac{\sqrt{3+4q}}{1+2q}, \quad G_{1x} = \frac{2(1+q)}{1+2q}. \tag{48}$$

The analytical expression of the Pareto-optimal set for the stochastic problem is too involved for practical purpose and so it has been approximated with

$$G_{2x} \geqslant \overline{G}_{2x}, \quad G_{1x} = \Psi_{\sigma_{\ddot{x}_2},\sigma_{F_z}}(q, \alpha_1, G_{2x}). \tag{49}$$

The function $\Psi_{\sigma_{\ddot{x}_2},\sigma_{F_z}}$ is reported in the appendix. The optimal set for the stochastic problem is a sloped line while for the deterministic problem it is a line parallel to the G_{2x}-axis [7].

The overall Pareto-optimal set for the three performance indices together is the area bounded by the bidimensional Pareto-optimal curves $\sigma_{\ddot{x}_2}, \sigma_{F_z}$ (Eqs. (48),(49)), $\sigma_{\ddot{x}_2}, \sigma_{x_2-x_2}$ (Eqs. (46),(47)), $\sigma_{x_2-x_1}, \sigma_{F_z}$

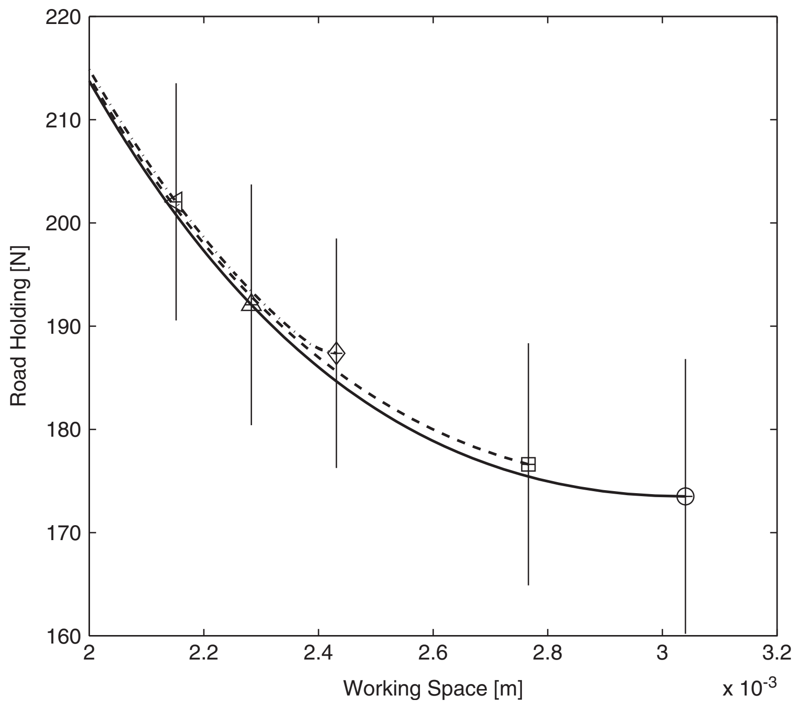

Fig. 6. Pareto-optimal set in the performance indices domain $\sigma_{x_2-x_1}, \sigma_{F_z}$ for the AS vehicle: solid line for $\alpha = 0$, dashed line for $\alpha = 6$ and dash-dotted line for $\alpha \to \infty$. Road roughness parameter $A_b = 6.9\text{E-}6$, vehicle speed 20 m/s, vehicle data in Table 1. Design variables in Fig. 5, points marked by symbols correspond to those in Fig. 5. The vertical and horizontal bars, at each marked point, define the variations of the two performance indices with respect to their mean values due to uncertain parameters and tolerances on design variables (see Table 1).

(Eqs. (42),(43)). These Pareto-optimal curves are limited by the three points which minimise respectively road holding, discomfort and working space. Any combination of non-dimensional controller gains that belongs to these areas, see Figs. 5 and 8, constitutes an optimal trade-off between the three objective functions. For sake of simplicity we have considered $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$.

Figs. 5 and 8 show the Pareto-optimal sets in the plane of the design variables, Figs. 6 and 7 show instead the mean and standard deviation of the performances indices for the Pareto-optimal solutions. Table 2 shows that the variation of the working space is mainly due to the variation of the sprung mass m_2 while the variation of the road holding is mainly due to the variation of the tyre radial stiffness k_1.

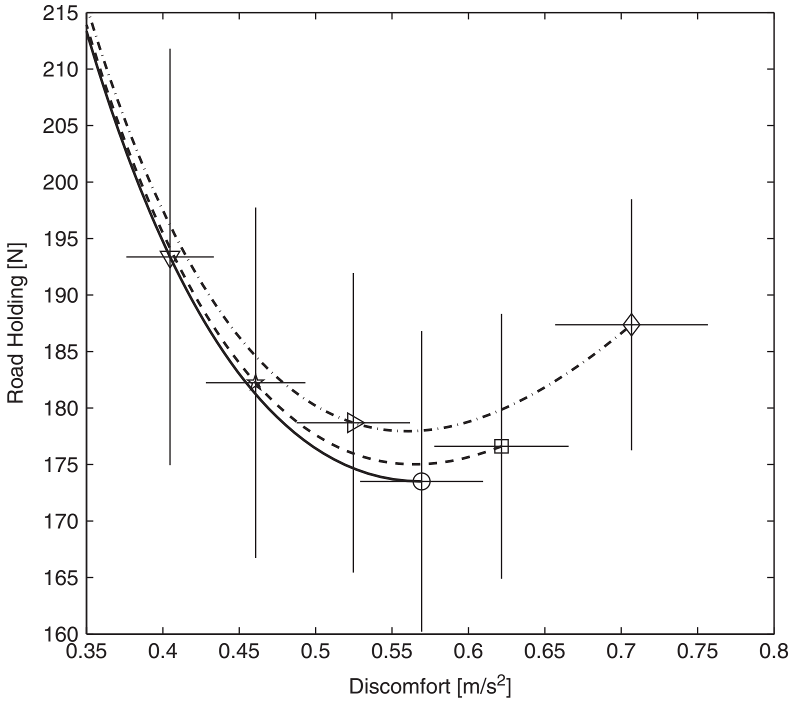

Fig. 7. Pareto-optimal set in the performance indices domain $\sigma_{\ddot{x}_2}, \sigma_{F_z}$ for the AS vehicle: solid line for $\alpha = 0$, dashed line for $\alpha = 6$ and dash-dotted line for $\alpha \to \infty$. Road roughness parameter $A_b = 6.9\text{E-}6$, vehicle speed 20 m/s, vehicle data in Table 1. Design variables in Fig. 5, points marked by symbols correspond to those in Fig. 5. The vertical and horizontal bars, at each marked point, define the variations of the two performance indices with respect to their mean values due to uncertain parameters and tolerances on design variables (see Table 1).

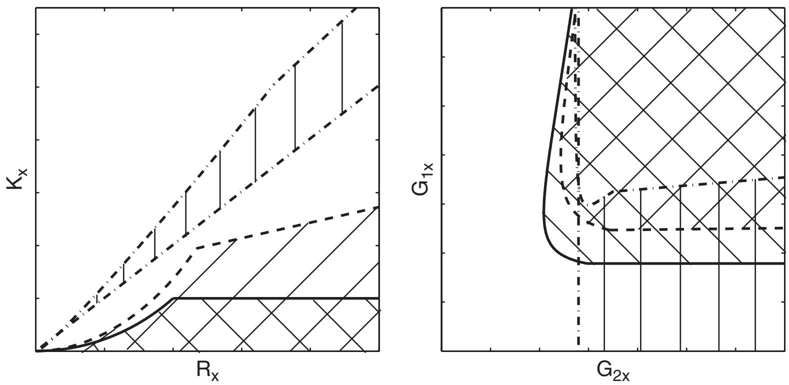

Fig. 8. Pareto-optimal sets in the design variables domain for the PS (left) and AS (right) vehicles. Pareto-optimal design variables are derived for $\alpha = 0$ (−45° hatch), $\alpha = 6$ (+45° hatch) and $\alpha \to \infty$ (vertical hatch).

In Fig. 5 three Pareto-optimal sets have been plotted for three different values of the percentiles α. The deterministic ($\alpha = 0$) and robust ($\alpha \to \infty$) Pareto-optimal sets have proved to be completely different. By increasing the robustness from $\alpha = 0$ to $\alpha = 6$, the resulting Pareto-optimal set is instead very similar to the deterministic one by moving to a higher value of G_{1x}, see Fig. 8.

A comparison in terms of mean and standard deviation of road holding and working space between the reference passive suspension and the points of minimum of road holding (marked by means of circle, square and diamond) for the three analysed values of the percentiles is shown in Table 3.

7. Conclusion

The paper presents and applies a multi-objective stochastic optimisation method. A 2 dof linear model has been used to describe in analytical form the dynamic behaviour of vehicles while running on randomly profiled roads. The road irregularity is considered a Gaussian random process and modelled by means of a simple exponential PSD. Discomfort, road holding and working space are the considered performance indices. The design variables are the suspension stiffness and damping (PS vehicle) and the controller gains (AS vehicle). The uncertainties and variations in parameters and design variables have been taken in account by means of a robust approach which involves the minimisation of a weighted sum of mean and standard deviation of each objective function.

The analytical formulae relating the performance indices to the design variables have been derived in an effective non-dimensional form. Additionally, the Fritz–John necessary condition for Pareto-optimality has been re-formulated allowing to derive analytically (whenever possible) the expression of the Pareto-optimal set. The optimal trade-off solutions have been derived in a deterministic and in a stochastic framework. These results constitute useful analytical rules for the preliminary design of road vehicle suspension systems.

A comparison between the three different optimisation approaches—namely deterministic, robust and stochastic—has been performed. It is shown that the solutions computed by means of a deterministic optimisation have the best obtainable mean performance but are prone to be the most sensitive to parameter uncertainty. In contrast, the optimal solutions computed accounting only for robustness are completely different from the deterministic ones and so they are not efficient in terms of expected performances. Moreover, the advantage of the robust design approach ($\alpha \to \infty$), in terms of failure probability, is negligible in comparison with the loss of mean performance. The optimisation of the mean and of the standard deviation ($\alpha = 6$) leads to a good compromise between robustness and performance, so it should be considered as the standard optimisation technique for complex systems design.

Appendix

Minimum of the road holding for the PS vehicle:

$$\begin{cases} \overline{K}_x = \left[5 + .185\dfrac{\alpha_3^2}{\alpha_3^2 + 10}\left(\dfrac{1}{q} - 7\right)\right]^{\left(\frac{\alpha_3^{1.27}}{\alpha_3^{1.27}+14}\right)}, \\ \overline{R}_x = 1.77^{\left(\frac{\alpha_3^{.84}}{\alpha_3^{.84}+12}\right)}. \end{cases} \tag{50}$$

Minimum of the road holding for the AS vehicle:

$$\begin{cases} \overline{G}_{1x} = \dfrac{2(1+q)}{1+2q}\left(1 + \dfrac{.64\alpha_3(1+2.15q)}{\alpha_3 + 7}\right), \\ \overline{G}_{2x} = \dfrac{\sqrt{3+4q}}{1+2q}\left[1 + \left(.203 - \dfrac{.0029}{q}\right)(1 + .0083\mathrm{e}^{(1.98-.005\alpha_3)} - .7\mathrm{e}^{(.4-.29\alpha_3)})\right]. \end{cases} \tag{51}$$

Approximate functions for the computation of the Pareto-optimal sets:

In Eq. (32):

$$\Gamma_{\sigma_{\ddot{x}_2},\sigma_{F_z}} = \sqrt{\left(\frac{1}{1+\alpha/10}\right)((1+q)K_x - qK_x^2)} + \left(\frac{\overline{R}_x}{\overline{K}_x} - \sqrt{\left(\frac{1}{1+\alpha/10}\right)\left(\frac{1+q}{\overline{K}_x} - q\right)}\right)K_x. \tag{52}$$

In Eq. (36):

$$\Gamma_{\sigma_{\ddot{x}_2},\sigma_{x_2-x_1}} = \left(\frac{1}{q}\left(\frac{1.56\sqrt{\alpha_2^2 + 3.8\alpha_2 + 24.5} - \alpha_2 - 11.5}{\alpha_2 + 3.8}\right)\right)^{1/2}. \tag{53}$$

In Eq. (38):

$$\Gamma_{\sigma_{x_2-x_1},\sigma_{F_z}} = \left(\frac{.92}{\sqrt{q}}\mathrm{e}^{(-7.4(\ln(\alpha_3))^{-2.8})}\right)(R_x - \overline{R}_x) + \overline{K}_x. \tag{54}$$

In Eq. (43):

$$\Psi^1_{\sigma_{x_2-x_1},\sigma_{F_z}} = \left(\sqrt{\frac{4q(1+q)^2}{3+4q}\frac{G_{1x}}{\overline{G}_{1x}}\frac{\frac{G_{1x}}{\overline{G}_{1x}} - \left(\frac{1+2q}{2(1+q)}\right)^2}{\frac{G_{1x}}{\overline{G}_{1x}}(1+q) - 1}} - 1\right)(\alpha^{.01}(6q^2 - .123) + 1)\overline{G}_{2x} + \overline{G}_{2x}, \tag{55}$$

$$\Psi^2_{\sigma_{x_2-x_1},\sigma_{F_z}} = \sqrt{\frac{8\left(\frac{G_{2x}-1.36}{\overline{G}_{2x}-1.36}\right)^2 - 2\left(\frac{G_{2x}-1.36}{\overline{G}_{2x}-1.36}\right) + 1}{7\frac{G_{2x}-1.36}{\overline{G}_{2x}-1.36}}} + \overline{G}_{1x} - 1. \tag{56}$$

In Eq. (47):

$$\Psi_{\sigma_{\ddot{x}_2},\sigma_{x_2-x_1}} = \left[\frac{2\alpha_1(1+q) - 10 + \sqrt{2\alpha_1(2+6q+5q^2)(\alpha_1 - 10) + 100(1+2q)^2}}{\alpha_1(2+3q) - 20(1+q)}\right]^{1/2}. \tag{57}$$

In Eq. (49):

$$\Psi_{\sigma_{\ddot{x}_2},\sigma_{F_z}} = \left(.138\frac{\alpha_1^{1.38}}{\alpha_1^{1.38} + 75}\right)(G_{2x} - \overline{G}_{2x}) + \overline{G}_{1x}. \tag{58}$$

References

[1] D. Hrovat, Applications of optimal control to advanced automotive suspension design, *Transactions of the ASME*, *Journal of Dynamic Systems Measurement and Control* 115 (1993).

[2] M. Mitschke, *Dynamik der Kraftfahrzeuge*, Springer, Berlin, 1990.

[3] R. Sharp, D. Crolla, Road vehicle suspension system design—a review, *Vehicle System Dynamics* 16 (3) (1987).

[4] C. Dodds, J. Robson, The description of road surface roughness, *Journal of Sound and Vibration* 2 (31) (1973) 175–183.

[5] K. Kamash, J.D. Robson, The application of isotropy in road surface modelling, *Journal of Sound and Vibration* 1 (57) (1978) 89–100.

[6] M. Gobbi, G. Mastinu, Analytical description and optimization of the dynamic behaviour of passively suspended road vehicles, *Journal of Sound and Vibration* 3 (245) (2001) 457–481.

[7] M. Gobbi, G. Mastinu, Symbolical multi-objective optimisation of the dynamic behaviour of actively suspended road vehicles, *International Journal of Vehicle Design* 28 (1–3) (2002) 189–213.

[8] P. Venhovens, *Optimal Control of Vehicle Suspensions*, Ph.D. Thesis, Delft University of Technology, Delft, The Netherlands, 1994.

[9] J. Dixon, *Tyres, Suspension and Handling*, Cambridge University Press, Cambridge, 1991.

[10] A. Thompson, Suspension design for optimum road-holding, *SAE Paper* 830663, 1983.

[11] G. Mastinu, Automotive suspension design by multi-objective programming, *Proceedings of the International Symposium on Advanced Vehicle Control AVEC94*, JSAE, 1994.

[12] M. Gobbi, G. Mastinu, C. Doniselli, Optimal and robust design of a road vehicle suspension system, *Vehicle System Dynamics* (Supplement 33) (1999).

[13] R. Caballero, et al., Efficient solution concepts and their relations in stochastic multiobjective programming, *Journal of Optimization Theory and Applications* 1 (110) (2001) 53–74.
[14] J. Pauwelussen, H. Pacejka (Eds.), *Smart Vehicles*, Swets Zeitlinger Publishers, 1995.
[15] P. Mueller, W. Schiehlen, *Linear Vibrations*, Martinus Nijoff, Dordrecht, 1985.
[16] A. Papoulis, *Probability Random Variables, and Stochastic Processes*, McGraw-Hill, New York, 1985.
[17] G. Mastinu, Passive automobile suspension parameter adaptation, *Proceedings of the IMechE Conference—Advanced Suspensions*, Institution of Mechanical Engineers, London, 1988.
[18] K. Miettinen, *Nonlinear Multiobjective Optimization*, Kluwer Academic Publishers, Boston, 1999.
[19] R. Statnikov, J. Matusov, *Multicriteria Optimization and Engineering*, Chapman & Hall, New York, 1995.
[20] F. Levi, M. Gobbi, G. Mastinu, An application of multi-objective stochastic optimisation to structural design, *Journal of Structural Optimisation* (2004).
[21] P.Y. Papalambros, D.J. Wilde, *Principles of Optimal Design*, Cambridge University Press, Cambridge, 2000.
[22] M.S. Phadke, *Quality Engineering using Robust Design*, Prentice-Hall, Englewood Cliffs, NJ, 1989.
[23] J. Reimpell, H. Stoll, *Fahrwerktechnik : Stoß-und Schwingungsdämpfer*, VOGEL Buchverlag Würzburg, 1989.
[24] J. Reimpell, *Fahrwerktechnik: Grundlagen*, VOGEL Buchverlag Würzburg, 1988.