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Coupled torsional and vertical oscillations of a beam subjected to boundary damping

M.A. Zarubinskaya, W.T. van Horssen*

Delft Institute of Applied Mathematics, Faculty of Electrical Engineering, Mathematics and Computer Science, Delft University of Technology, Mekelweg 4, 2628 CD Delft, The Netherlands

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Abstract

In this paper coupled torsional and vertical oscillations of a beam in a wind-field are studied. Different kinds of dampers are added to the beam to suppress undesirable oscillations. Using a two times scales perturbation method, the relationship between the beam parameters and the damping rates are obtained analytically.

1. Introduction

During the last decades a lot of suspension bridges and cable stay bridges have been constructed. Compared to the old bridges the new ones are usually longer and are built with fewer pillars and stays. One of the longest suspension bridges in the world at the moment is the Akashi Kaikyo Bridge in Japan with a span length of 12 828 feet. This relatively high bridge connects the city of Kobe to one of its neighboring islands. It is very important to investigate the behavior of such suspension bridges in airflow. It is known that a suspension bridge can undergo dangerous oscillations under strong wind. These oscillations can be vertical and torsional ones. Moreover, the torsional oscillations are very sensitive to the nonlinear behavior of the cables and the hangers connecting the roadbed to the main suspension cables. Without special design tricks flutter instabilities will occur at wind speeds below the required critical wind speed for these span lengths. It is known that flutter produces motions often in the form of torsional oscillations. It is believed nowadays that flutter caused the collapse of the Tacoma Narrow Bridge in 1940. Hence it is so important to investigate dynamic oscillations in suspension bridges, especially the destructive large-amplitude oscillations, and to develop design techniques to prevent such destructive oscillations.

Simple models for such oscillations are described with second- and fourth-order partial differential equations. Usually asymptotic methods can be used to construct approximations for the solutions of these wave, beam or plate equations. For a long time initial-boundary value problems for weakly nonlinear wave equations have been studied. For example in Ref. [1] a forced nonlinear wave equation on a bounded domain

*Corresponding author. Tel.: +31 15 2783524; fax: +31 15 2787295.
E-mail addresses: mzarubinska@ewi.tudelft.nl (M.A. Zarubinskaya), W.T.vanHorssen@ewi.tudelft.nl (W.T. van Horssen).

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has been considered which (under certain physical assumptions) models the torsional oscillations of the main deck of a suspension bridge.

The analysis becomes more complicated for beam equations (see for instance Refs. [2–7]). In the last decades Lazer and McKenna proposed mathematical models describing oscillations in suspension bridges, which are based upon the observation that the fundamental nonlinearity in suspension bridges is that the stays connecting the supporting cables and the roadbed resist expansion but do not resist compression. These models are described by systems of coupled nonlinear partial differential equations. The vertical and torsional motions are coupled through nonlinear terms. These nonlinearities arise from the loss of tension in the vertical cables supporting the deck. The impact of wind forces on the stability of motion in this system is considered for cases with and without viscous and structural damping. However, the case of coupled vertical and torsional oscillations is not completely studied and understood. Multiple large-amplitude periodic oscillations have been found theoretically and numerically in the single Lazer–McKenna suspension bridge equation (see Refs. [2,3,8–12]). Recently for the Lazer–McKenna suspension bridge system governed by coupled nonlinear beam and wave equations multiple periodic oscillations have been found [13,14].

In most of the papers as mentioned before the authors used numerical approaches to study and to describe the vertical or/and torsional oscillations, and also assumptions have been introduced to decouple the system of nonlinear differential equations. In this paper a model describing both torsional and vertical oscillations of suspension bridges will be presented. An engineering approach will be used, that is, different dampers will be added to the suspension bridge to diminish undesirable oscillations. The system will be linearized around the most critical regimes where the oscillations can occur. The analysis of the linearized problems will be presented in this paper.

The outline of this paper is as follows. In Section 2 the derivation of the model will be given. In Section 3 several kinds of damping mechanisms are introduced and their influence on the vertical oscillations of the system are studied. Three types of damping mechanisms to diminish the coupled torsional and vertical oscillations of the system are considered and studied in Section 4. Finally some conclusions will be drawn in Section 5.

2. Mathematical model of coupled torsional and vertical oscillations of a beam in a wind field

In this section a simple model describing both vertical and torsional oscillations of suspension bridges will be derived. To derive the equations of motion for an elastic beam part of the analysis as given in Refs. [7,15] will be followed. An elastic beam of length l will be considered. The x-axis is taken along the beam axis, such that the left end of the beam corresponds with $x = 0$. The z-axis is taken vertically. It is assumed that the beam can rotate around the x-axis. Using Kirchhoff's approach the horizontal movement in x-direction can be eliminated. It is known that the galloping oscillations (which are considered in this section) produce almost purely vertical motion of an elastic structure in a wind-field. So it is assumed that the movement of the beam in the y-direction can be neglected, and only vertical (that is, in the z-direction) and torsional oscillations of the beam around the x-axis are considered.

Coupled flexural and torsional vibrations will occur for this beam due to wind forces. We will consider low frequencies and a large amplitude phenomenon involving vertical and torsional oscillations of a beam on which for instance ice has been accumulated. The frequencies involved are so low that the assumption can be made that the aerodynamic forces are as in steady flow. Another consequence of these low frequencies is that structural damping may be neglected. To model such vibrations a (weakly) nonsymmetrical cross-section perpendicular to the x-axis of the beam (for example with an ice ridge) will be considered. It is assumed that every cross-section perpendicular to the x-axis oscillates in the (y, z)-plane (see Fig. 1). Along the axis of symmetry of a cross-section a vector $\mathbf{e}_s$ is defined to be directing away from the ice ridge and starting in the center of the cross-section. On each coordinate axis a unit vector is fixed: on the x-axis the vector $\mathbf{e}_x$, on the y-axis the vector $\mathbf{e}_y$ and on the z-axis the vector $\mathbf{e}_z$, which has a direction opposite to gravity. Let the static angle of attack α_s (assumed to be constant and identical for all cross-sections) to be the angle between $\mathbf{e}_s$ and the uniform airflow $\mathbf{v}_\infty$, with $|\alpha_s| \leqslant \pi$. In this uniform airflow with flow velocity $\mathbf{v}_\infty = v_\infty \mathbf{e}_y$ $(v_\infty > 0)$ the beam may oscillate due to the lift force $L\mathbf{e}_L$, the drag force $D\mathbf{e}_D$, and the moment $M\mathbf{e}_M$. It should be noted that the drag force $D\mathbf{e}_D$ has the direction of the virtual wind velocity $D\mathbf{v}_s \equiv \mathbf{v}_\infty - (\partial w/\partial t)\mathbf{e}_z$, and the lift force $L\mathbf{e}_L$ has

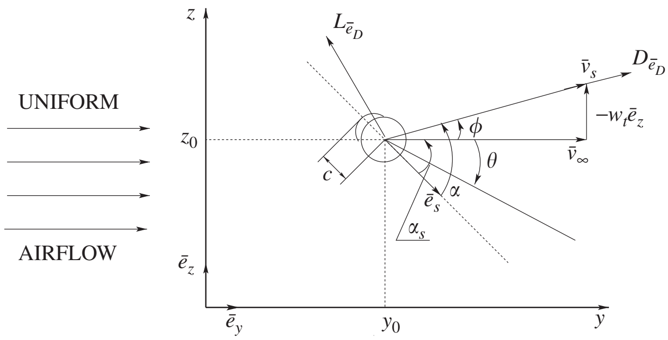

Fig. 1. Cross-section at $x = x_0$ of the circular beam with ice ridge in the (y, z)-plane.

a direction perpendicular to the virtual windvelocity $\mathbf{v}_s$. In Fig. 1 the forces $L\mathbf{e}_L$ and $D\mathbf{e}_D$ acting on the cross-section are given. The equations describing the vertical and the torsional motions of the beam are given by

$$\rho A w_{tt} - \frac{EA}{2l}\int_0^l w_x^2 \,\mathrm{d}x\, w_{xx} + EIw_{xxxx} = -\rho A g + D\sin(\phi - \theta) + L\cos(\phi - \theta), \tag{1}$$

$$\rho I_p \theta_{tt} - R\theta_{xx} = -\rho A c(w - c\theta)_{tt} + M, \tag{2}$$

where the magnitudes of the drag force, the lift force, and the moment in the (y, z)-plane acting on the beam per unit length of the beam are D, L, and M, respectively, the moment M acts in the (y, z)-plane around the x-axis, ρ is the mass density of the beam (including the small ice ridge), A is the constant cross-sectional area of the beam (including the small ice ridge), ϕ is the angle between $\mathbf{v}_\infty$ and $\mathbf{v}_s$ (with $| \phi | \leqslant \pi$), g is the gravitational acceleration, θ is the angle of torsion in the (y, z)-plane around the axis of the beam, E is Young's modulus, I is the moment of inertia of the cross-section (including the small ice ridge), I_p is the polar moment of inertia of the cross-section (including the small ice ridge), $R = EId_1/2(1 - v)$ is the torsional rigidity, d_1 is a diameter of the cross-section of the beam, v is Poisson's ratio, and w is the displacement in the vertical direction, c is the distance from the centroid of the beam to the outside of the beam. In Eq. (2) the term $\rho A c(w - c\theta)_{tt}$ represents the transverse inertial force and the term $\rho I_p \theta_{tt}$ the inertial torque [16]. The magnitudes D, L and M of the aerodynamic forces may be given by

$$D = \tfrac{1}{2}\rho_a d_1 c_D(\alpha) v_s^2, \quad L = \tfrac{1}{2}\rho_a d_1 c_L(\alpha) v_s^2, \quad M = \tfrac{1}{2}\rho_a d_1 c_M(\alpha) v_s^2, \tag{3}$$

where ρ_a is the density of the air, $v_s = | \mathbf{v}_s |$, α is the angle between $\mathbf{e}_s$ and $\mathbf{v}_s$ with ($| \alpha | \leqslant \pi$), and $c_D(\alpha)$, $c_L(\alpha)$ and $c_M(\alpha)$ are the quasi-steady drag-, lift- and moment-coefficients, which may be obtained from wind-tunnel measurements. For a certain range of values of v_∞ some characteristic results from wind-tunnel experiments are given in Ref. [17]. From these experimental results the drag-, lift- and moment-coefficients can be approximated for low velocities of the beam by (see also Ref. [15]):

$$c_D(\alpha) = (\alpha - \alpha_1)c_D, \quad c_L(\alpha) = c_{L_1}(\alpha - \alpha_1) + c_{L_3}(\alpha - \alpha_1)^3,$$

$$c_M(\alpha) = c_{M_1}(\alpha - \alpha_1) + c_{M_3}(\alpha - \alpha_1)^3, \tag{4}$$

where c_D, c_{L1}, c_{L3}, c_{M1}, and c_{M3} are constants, α_1 is a critical value such that (according to the Den Hartog criterion) galloping may set in. For galloping oscillations which are low-frequency oscillations it can be assumed that $|\phi| \ll 1$. The right-hand side of Eqs. (1)–(2) can be expanded near $w_t/v_\infty = 0$ and $\theta = 0$. Using the fact that $\phi = \arctan(-w_t/v_\infty)$ and neglecting terms of degree four and higher one obtains after some elementary calculations

$$\rho A w_{tt} - \frac{EA}{2l}\int_0^l w_x^2\,\mathrm{d}x\, w_{xx} + EI w_{xxxx} = -\rho A g + \frac{1}{2}\rho_a d_1 v_\infty^2 \left(a_0 + a_1\left(\frac{w_t}{v_\infty} + \theta\right) + a_2\theta^2 + a_3\frac{w_t^2}{v_\infty^2} + a_4\frac{w_t}{v_\infty}\theta + a_5\frac{w_t^3}{v_\infty^3} + a_6\frac{w_t^2}{v_\infty^2}\theta + a_7\frac{w_t}{v_\infty}\theta^2 + a_8\theta^3\right), \tag{5}$$

$$\rho I_p \theta_{tt} - R\theta_{xx} = -\rho A c(w - c\theta)_{tt} + \left(b_0 + b_1\left(\frac{w_t}{v_\infty} + \theta\right) + b_2\theta^2 + b_3\frac{w_t^2}{v_\infty^2} + b_4\frac{w_t}{v_\infty}\theta + b_5\frac{w_t^3}{v_\infty^3} + b_6\frac{w_t^2}{v_\infty^2}\theta + b_7\frac{w_t}{v_\infty}\theta^2 + b_8\theta^3\right), \tag{6}$$

where

$$\begin{gathered}
a_0 = (\alpha_s - \alpha_1)(c_{L1} + c_{L3}((\alpha_s - \alpha_1)^2),\\
a_1 = -(c_{L1} + 3c_{L3}(\alpha_s - \alpha_1)^2 + c_D(\alpha_s - \alpha_1)),\\
a_2 = (\alpha_s - \alpha_1)(\tfrac{1}{2}c_{L1} + \tfrac{1}{2}c_{L3}(6 + (\alpha_s - \alpha_1)^2)),\\
a_3 = -(c_{L1}(\alpha_s - \alpha_1) + (\alpha_s - \alpha_1)c_{L3}((\alpha_s - \alpha_1)^2 - 6) - 2c_D),\\
a_4 = (-\tfrac{1}{2}c_{L1}(\alpha_s - \alpha_1) - \tfrac{1}{2}c_{L3}(\alpha_s - \alpha_1)((\alpha_s - \alpha_1)^2 - 6) + c_D),\\
a_5 = -(\tfrac{1}{6}c_{L1} + c_{L3}(\tfrac{1}{2}(\alpha_s - \alpha_1)^2 + 1) + \tfrac{1}{2}(\alpha_s - \alpha_1)c_D),\\
a_6 = (\tfrac{1}{2}c_{L1} + \tfrac{3}{2}c_{L3}((\alpha_s - \alpha_1)^2 - 6) - \tfrac{1}{2}(\alpha_s - \alpha_1)c_D),\\
a_7 = \tfrac{1}{2}(3c_{L1} + 3c_{L3}(3(\alpha_s - \alpha_1)^2 - 2) + c_D(\alpha_s - \alpha_1)),\\
a_8 = \tfrac{1}{2}(c_{L1} + c_{L3}(3(\alpha_s - \alpha_1)^2 - 2) + \tfrac{1}{3}c_D(\alpha_s - \alpha_1)),\\
b_0 = c_{M1}(\alpha_s - \alpha_1) + c_{M3}(\alpha_s - \alpha_1)^3, \quad b_1 = -(c_{M1} + 3c_{M3}(\alpha_s - \alpha_1)^2),\\
b_2 = (\alpha_s - \alpha_1)(c_{M1} + (\alpha_s - \alpha_1)c_{M3}(3 + \alpha_s - \alpha_1)),\\
b_3 = 6c_{M3}(\alpha_s - \alpha_1), \quad b_4 = 3c_{M3}(\alpha_s - \alpha_1),\\
b_5 = -(\tfrac{2}{3}c_{M1} + c_{M3}(2(\alpha_s - \alpha_1)^2 + 1)), \quad b_8 = -c_{M3},\\
b_6 = -(c_{M1} + 3c_{M3}((\alpha_s - \alpha_1)^2 + 1)), \quad b_7 = -3c_{M3}.
\end{gathered} \tag{7}$$

Eq. (5) will be simplified by eliminating the term $-\rho A g$ by introducing the transformation $w = \tilde{w} - Ags(x)$, where $s(x)$ satisfies the following time-independent boundary value problem:

$$s_{xxxx} - \frac{A^3 g^2}{2lI}\int_0^l s_x^2\,\mathrm{d}x\, s_{xx} + \frac{\rho}{EI} = 0,$$

$$s(0) = s(l) = 0, \quad s_{xx}(0) = s_{xx}(l) = 0.$$

The term $-Ags(x)$ represents the deflection of the beam in static state due to gravity. Using the dimensionless variables $\bar{t} = 1/l^2\sqrt{(EI/\rho A)}t$, $\bar{w} = v_\infty/l^2\sqrt{(EI/\rho A)}\tilde{w}$, $\bar{x} = x/l$ and $\bar{s} = EIk^2/\rho$, where $k = A^3 g^2/2lI\int_0^l s_x^2\,\mathrm{d}x$, and assuming that the deflection of the beam in static state due to gravity, $gAs(x)$, is small with respect to the vertical displacement $\bar{w}$, which is of order 1, Eqs. (5)–(6) become

$$\bar{w}_{\bar{t}\bar{t}} + \bar{w}_{\bar{x}\bar{x}\bar{x}\bar{x}} = \varepsilon(a_0 + a_1\bar{w}_{\bar{t}} + a_1\theta + a_3\bar{w}_{\bar{t}}^2 + a_4\bar{w}_{\bar{t}}\theta + a_5\bar{w}_{\bar{t}}^3 + a_6\bar{w}_{\bar{t}}^2\theta + a_7\bar{w}_{\bar{t}}\theta^2 + a_8\theta^3) + \mathcal{O}(\varepsilon^n), \tag{8}$$

$$\theta_{\bar{t}\bar{t}} - b^2\theta_{xx} = -\varepsilon(\bar{b}_0 + \bar{b}_1\bar{w}_{\bar{t}} + \bar{b}_1\theta + \bar{b}_3\bar{w}_{\bar{t}}^2 + \bar{b}_4\bar{w}_{\bar{t}}\theta + \bar{b}_5\bar{w}_{\bar{t}}^3 + \bar{b}_6\bar{w}_{\bar{t}}^2\theta + \bar{b}_7\bar{w}_{\bar{t}}\theta^2 + \bar{b}_8\theta^3) + \mathcal{O}(\varepsilon^n), \tag{9}$$

with $n>1$, $\bar{b}_i = (Al^2 v_\infty\sqrt{\rho A}/(Ac^2 + I_p)EI)b_i$, $b^2 = Al^2/2(1-v)(Ac^2 + I_p)$, and $\varepsilon = \rho_a d v_\infty l^2/2\sqrt{EI\rho_c A}$ is a small parameter. Assuming that the static angle of attack α_s is such that galloping may set in according to the instability criterion of den Hartog [18], that is, assuming that $\alpha_s = \alpha_1 + \mathcal{O}(\varepsilon)$, the partial differential equations describing up to $\mathcal{O}(\varepsilon^n)$, $n>1$, the vertical and torsional displacement of an elastic beam in a uniform airflow will be

$$\bar{w}_{\bar{t}\bar{t}} + \bar{w}_{\bar{x}\bar{x}\bar{x}\bar{x}} = \varepsilon(a_1 w_{\bar{t}} + a_1\theta + a_3\bar{w}_{\bar{t}}^2 + a_4\bar{w}_{\bar{t}}\theta + a_5\bar{w}_{\bar{t}}^3 + a_6\bar{w}_{\bar{t}}^2\theta + a_7\bar{w}_{\bar{t}}\theta^2 + a_8\theta^3), \quad 0<\bar{x}<1, \ \ \bar{t}>0, \tag{10}$$

$$\theta_{\bar{t}\bar{t}} - b^2\theta_{\bar{x}\bar{x}} = -\varepsilon(\bar{b}_1\bar{w}_{\bar{t}} + \bar{b}_1\theta + \bar{b}_3\bar{w}_{\bar{t}}^2 + \bar{b}_4\bar{w}_{\bar{t}}\theta + \bar{b}_5\bar{w}_{\bar{t}}^3 + \bar{b}_6\bar{w}_{\bar{t}}^2\theta + \bar{b}_7\bar{w}_{\bar{t}}\theta^2 + \bar{b}_8\theta^3), \quad 0<\bar{x}<1, \ \ \bar{t}>0. \tag{11}$$

When a simply supported beam is considered which cannot rotate around the x-axis at $x=0$ and 1, the following boundary conditions should be introduced

$$\bar{w}(0,\bar{t}) = \bar{w}(1,\bar{t}) = \bar{w}_{\bar{x}\bar{x}}(0,\bar{t}) = \bar{w}_{\bar{x}\bar{x}}(1,\bar{t}) = 0, \quad \bar{t}\leqslant 0, \tag{12}$$

$$\theta(0,\bar{t}) = \theta(1,\bar{t}) = 0, \quad \bar{t}\leqslant 0. \tag{13}$$

For convenience all bars will be dropped in the further analysis. These boundary conditions (12)–(13) imply that the functions w and θ can be extended as odd, 2-periodic functions in x, i.e. w and θ can be written in Fourier sine-series in x:

$$w(x,t) = \sum_{n=1}^{\infty} q_n(t)\sin(n\pi x), \quad \theta(x,t) = \sum_{m=1}^{\infty} f_m(t)\sin(m\pi x). \tag{14}$$

Since the right-hand sides of Eqs. (10)–(11) contain a small parameter ε perturbation method are usually applied to construct approximations of the functions $w(x,t)$ and $\theta(x,t)$.

3. Boundary damping

When a perturbation is used terms that give rise to secular terms may occur in the right-hand sides of Eqs. (10)–(11). Usually to eliminate these terms a multiple time scales perturbation method is introduced. However, by substituting expressions (14) for w and θ into system (10)–(13) and by using the perturbation method (see also Ref. [19]), an infinite dimensional system of coupled ODEs will be obtained which would be hard (if not impossible) to analyze because of its complexity. Since all vibration modes have to be considered due to the existing infinitely, many internal resonances it will be unclear how a truncation method can be applied. On the other hand only the low-frequency oscillations are important to describe the galloping oscillations of a beam in a windfield. For that reason a more engineering approach will be used. In practice dampers are added to the beam (or to the elastic structures such as bridges) to diminish undesirable oscillations. It will be assumed that the oscillation amplitudes are sufficiently small, such that the nonlinear terms in Eqs. (8)–(9) are of higher order compared to the linear terms. So, it will be assumed that the linear terms are far more important than the (small) nonlinear terms, and this will imply that the internal resonances due to the nonlinear terms can be left out in the analysis. Then, an analysis of the linearized problem (10)–(13) (including the dampers) is usually sufficient to consider. For that reason system (10)–(11) will be linearized first, yielding

$$w_{tt} + w_{xxxx} = \varepsilon a_1(w_t + \theta), \tag{15}$$

$$\theta_{tt} - b^2\theta_{xx} = -\varepsilon b_1(w_t + \theta), \tag{16}$$

and then several damping aspects for the linearized problem will be considered.

As a first simplification in the investigation of coupled torsional and vertical vibrations of an elastic beam only the vertical oscillations of the beam are considered. In Section 4 both torsional and vertical vibrations will be studied. First a cantilevered beam with two types of dampers attached to the free end will be considered (see Fig. 2). The vertical oscillations of such a beam can be described by the following boundary value problem:

$$w_{tt} + w_{xxxx} = \varepsilon a_1 w_t, \quad 0<x<1, \ t>0, \tag{17}$$

$$w = w_x = 0, \quad x = 0, \ t>0, \tag{18}$$

$$w_{xx} = -\varepsilon\beta w_{xt}, \quad x = 1, \ t>0, \tag{19}$$

$$w_{xxx} = \varepsilon\alpha w_t, \quad x = 1, \ t>0, \tag{20}$$

where α and β are positive damping constants. The boundary condition (19) and (20) describe rotational and vertical damping respectively, and can be obtained by applying Newton's second law. For instance to obtain boundary condition (20) it can be assumed that a small mass m is added to the right end of the beam at $x = 1$. Newton's second law then implies $mw_{tt}(1,t) = w_{xxx}(1,t) - \varepsilon\alpha w_t(1,t)$, where $w_{xxx}(1,t)$ represents the shear force at $x = 1$, and $\varepsilon\alpha w_t(1,t)$ the damping force. By letting $m \longrightarrow 0$ boundary conditions (20) will follow. Similarly, boundary condition (19) can be obtained by studying the angular acceleration.

Two time scales are introduced $t = t_0$ and $\tau = \varepsilon t$, and it is assumed that $w(x,t)$ can be expanded in a formal power series in ε, that is, $w(x,t) = w_0(x,t_0,\tau) + \varepsilon w_1(x,t_0,\tau) + \varepsilon^2 w_2(x,t_0,\tau) + \cdots$. Substituting this into the boundary value problem (17)–(20) and collecting equal powers in ε, yields the following $\mathcal{O}(\varepsilon^0)$-problem

$$\frac{\partial^2 w_0}{\partial t_0^2} + \frac{\partial^4 w_0}{\partial x^4} = 0,$$

$$w_0 = \frac{\partial w_0}{\partial x} = 0, \quad x = 0,$$

$$\frac{\partial^2 w_0}{\partial x^2} = \frac{\partial^3 w_0}{\partial x^3} = 0, \quad x = 1. \tag{21}$$

By using the method of separation of variables solutions of the well-known problem (21) for the cantilevered beam can readily be constructed. The following for w_0 is the finally obtained

$$w_0(x,t_0,\tau) = \sum_{n=1}^{\infty} \left(A_n(\tau)\sin(\lambda_n^2 t_0) + B_n(\tau)\cos(\lambda_n^2 t_0)\right)\phi_n(x), \tag{22}$$

where $\phi_n(x) = \sin(\lambda_n x) - \sinh(\lambda_n x) + \gamma(\cosh(\lambda_n x) - \cos(\lambda_n x))$, $\gamma = \sin(\lambda_n) + \sinh(\lambda_n)/\cos(\lambda_n) + \cosh(\lambda_n)$, and λ_n $n = 0, 1, 2, 3\ldots$ is the nth zero of the transcendental equation $\cosh(\lambda_n)\cos(\lambda_n) + 1 = 0$.

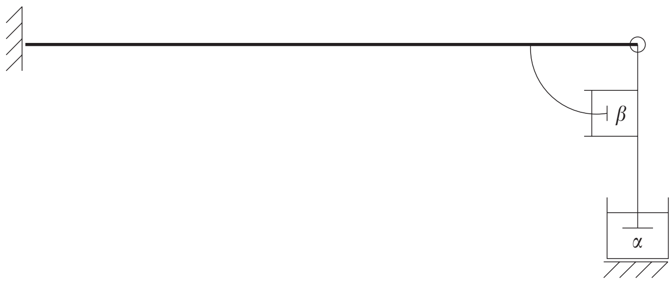

Fig. 2. A beam clamped at one end and with lateral and torsional dampers at the other.

Next the $\mathcal{O}(\varepsilon^1)$-problem is considered

$$\frac{\partial^2 w_1}{\partial t_0^2} + \frac{\partial^4 w_1}{\partial x^4} = a_1 \frac{\partial w_0}{\partial t_0} - 2\frac{\partial^2 w_0}{\partial t_0 \partial \tau},$$

$$w_1 = \frac{\partial w_1}{\partial x} = 0, \quad x = 0,$$

$$\frac{\partial^2 w_1}{\partial x^2} = -\beta \frac{\partial^2 w_0}{\partial x \partial t_0}, \quad x = 1,$$

$$\frac{\partial^3 w_1}{\partial x^3} = \alpha \frac{\partial w_0}{\partial t_0}, \quad x = 1. \tag{23}$$

To solve the boundary value problem (23) for w_1 it is convenient to make the boundary conditions in Eq. (23) at $x = 1$ homogeneous by introducing the following transformation:

$$w_1(x, t_0, \tau) = u_1(x, t_0, \tau) - \left(\frac{\alpha}{2}\frac{\partial w_0(1, t_0, \tau)}{\partial t_0} + \frac{\beta}{2}\frac{\partial^2 w_0(1, t_0, \tau)}{\partial x \partial t_0}\right)x^2 + \frac{\alpha}{6}\frac{\partial w_0(1, t_0, \tau)}{\partial t_0}x^3. \tag{24}$$

The boundary value problem (23) then becomes

$$\frac{\partial^2 u_1}{\partial t_0^2} + \frac{\partial^4 u_1}{\partial x^4} = a_1 \frac{\partial w_0}{\partial t_0} - 2\frac{\partial^2 w_0}{\partial t_0 \partial \tau} + \left(\frac{\alpha}{2}\frac{\partial^3 w_0(1, t_0, \tau)}{\partial t_0^3} + \frac{\beta}{2}\frac{\partial^4 w_0(1, t_0, \tau)}{\partial x \partial t_0^3}\right)x^2 - \frac{\alpha}{6}\frac{\partial^3 w_0(1, t_0, \tau)}{\partial t_0^3}x^3, \tag{25}$$

$$u_1 = \frac{\partial u_1}{\partial x} = 0, \;\; x = 0; \quad \frac{\partial^2 u_1}{\partial x^2} = \frac{\partial^3 u_1}{\partial x^3} = 0, \;\; x = 1. \tag{26}$$

Now u_1 can be written as $u_1 = \sum_n^\infty f_n(t_0, \tau)\phi_n(x)$ and by substituting this into Eq. (25) the following is obtained

$$\sum_{n=1}^{\infty}\left(\frac{\partial^2 f_n}{\partial t_0} + \lambda_n^4 f_n\right)\phi_n(x) = \sum_{n=1}^{\infty}\lambda_n^2\left(a_1 G_n(t_0, \tau) - 2\frac{\partial G_n(t_0, \tau)}{\partial \tau}\right)\phi_n(x) + \frac{\alpha}{2}\sum_{n=1}^{\infty}\lambda_n^6 G_n(t_0, \tau)\phi_n(1)(x^2 - x^3) - \frac{\beta}{2}\sum_{n=1}^{\infty}\lambda_n^7 G_n(t_0, \tau)\phi_n'(1)x^2,$$

where $G_n(t_0, \tau) = A_n(\tau)\cos(\lambda_n^2 t_0) - B_n(\tau)\sin(\lambda_n^2 t_0)$. Using the orthogonality properties of the functions $\phi_n(x)$ it then follows from the last equation that f_k has to satisfy

$$\left(\frac{\partial^2 f_k}{\partial t_0} + \lambda_k^4 f_k\right)\gamma_1 = \lambda_k^2\left(a_1 G_k(t_0) - \frac{\partial G_k(t_0, \tau)}{\partial \tau}\right)\gamma_1 - \sum_{n=1}^{\infty}\frac{\alpha}{6}\lambda_n^6 G_n(t_0, \tau)\phi_n(1)(-3\gamma_2 + \gamma_3) - \sum_{n=1}^{\infty}\frac{\beta}{2}\lambda_n^7 G_n(t_0, \tau)\phi_n'(1)\gamma_2, \tag{27}$$

where

$$\gamma_1 = \int_0^1 \phi_k^2(x)\,\mathrm{d}x, \quad \gamma_2 = \int_0^1 x^2\phi_k(x)\,\mathrm{d}x, \quad \gamma_3 = \int_0^1 x^3\phi_k(x)\,\mathrm{d}x.$$

Since $\cos(\lambda_k^2 t_0)$ and $\sin(\lambda_k^2 t_0)$ are part of the homogeneous solution of u_1, it follows that the coefficients of $\cos(\lambda_k^2 t_0)$ and $\sin(\lambda_k^2 t_0)$ in the right-hand side of Eq. (27) should be equal to zero (elimination of secular terms).

This gives us differential equations for A_k and B_k

$$\frac{\partial A_k}{\partial \tau} = \left(-\alpha f(\lambda_k) - \beta g(\lambda_k) + \frac{a_1}{2}\right) A_k,$$

$$\frac{\partial B_k}{\partial \tau} = \left(-\alpha f(\lambda_k) - \beta g(\lambda_k) + \frac{a_1}{2}\right) B_k, \tag{28}$$

where

$$\begin{aligned} f(\lambda_k) = \frac{\lambda_k \phi_k(1)}{3} & \big((2\lambda_k^3 + 3\gamma\lambda_k^2 - 6\gamma)\cosh(\lambda_k) - (2\gamma\lambda_k^3 + 3\lambda_k^2 - 6)\sinh(\lambda_k) \\ & + (2\lambda_k^3 + 3\gamma\lambda_k^2 + 6\gamma)\cos(\lambda_k) + (2\gamma\lambda_k^3 - 3\lambda_k^2 - 6)\sin(\lambda_k)\big)/ \\ & \big((\gamma^2+1)\cosh(\lambda_k)(\sinh(\lambda_k) - 2\sin(\lambda_k)) - (\gamma^2 - 1)\cos(\lambda_k) \\ & \times (2\sinh(\lambda_k) - \sin(\lambda_k)) - 2\gamma(\cosh^2(\lambda_k) - \cos^2(\lambda_k)) \\ & + 4\gamma\sinh(\lambda_k)\sin(\lambda_k) + 2\gamma\lambda_k\big), \end{aligned}$$

$$\begin{aligned} g(\lambda_k) = \lambda_k^2 \phi_k'(1) & \big((\gamma\lambda_k^2 + 2\lambda_k + 2\gamma)\sinh(\lambda_k) - (\lambda_k^2 + 2\gamma\lambda_k + 2)\sinh(\lambda_k) \\ & - (\gamma\lambda_k^2 - 2\lambda_k - 2\gamma)\sin(\lambda_k) - (\lambda_k^2 + 2\gamma\lambda_k - 2)\cos(\lambda_k)\big)/ \\ & \big((\gamma^2+1)\cosh(\lambda_k)(\sinh(\lambda_k) - 2\sin(\lambda_k)) - (\gamma^2 - 1)\cos(\lambda_k) \\ & \times (2\sinh(\lambda_k) - \sin(\lambda_k)) - 2\gamma(\cosh^2(\lambda_k) - \cos^2(\lambda_k)) \\ & \times 4\gamma\sinh(\lambda_k)\sin(\lambda_k) + 2\gamma\lambda_k\big). \end{aligned}$$

In Table 1 the first ten values of the coefficients $f(\lambda_k)$ and $g(\lambda_k)$ are given.

From Eq. (28) and Table 1 it can be concluded that for sufficiently large (positive) values of α and β the beam can be damped. To be more specific to have damping, the coefficient $-\alpha f(\lambda_k) - \beta g(\lambda_k) + a_1/2$ should be negative. And since the function $f(\lambda_k) \approx 2.0$ and the increasing function $g(\lambda_k) > 9$ it follows that for a given a_1 when α and β are chosen such that $-2\alpha - 9\beta + a_1/2 < 0$ damping will always occur.

As a next step in the investigation the vertical oscillations of a simply supported beam with two different types of dampers attached at a distance d from the left end of the beam is considered (see Fig. 3). The vertical oscillations of such a beam can be described by the following boundary value problem:

$$w_{tt}^i + w_{xxxx}^i = \varepsilon a_1 w_t^i, \quad 0 < x < 1, \quad t > 0, \tag{29}$$

$$w^{\mathrm{I}} = w_{xx}^{\mathrm{I}} = 0, \quad x = 0, \quad t > 0, \tag{30}$$

$$w^{\mathrm{II}} = w_{xx}^{\mathrm{II}} = 0, \quad x = 1, \quad t > 0, \tag{31}$$

Table 1
The first ten values of the coefficients $f(\lambda_k)$ and $g(\lambda_k)$

n	λ_n	$f(\lambda_n)$	$g(\lambda_n)$
1	21.875	2.0	9.141
2	4.694	2.0	57.287
3	7.855	2.0	160.423
4	10.996	2.0	314.371
5	14.137	2.0	519.622
6	17.279	2.0	776.260
7	20.420	2.0	1084.139
8	23.562	2.0	1443.436
9	26.704	2.0	1854.069
10	29.845	2.0	2315.882

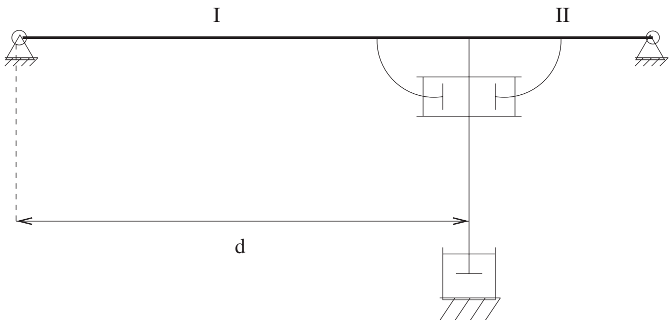

Fig. 3. A beam simply supported at both ends, and with an intermediate lateral and torsional damper.

$$w^{\mathrm{I}} = w^{\mathrm{II}}, \quad w_x^{\mathrm{I}} = w_x^{\mathrm{II}}, \;\; x = d, \tag{32}$$

$$w_{xx}^{\mathrm{II}} - w_{xx}^{\mathrm{I}} = \varepsilon \beta w_{xt}, \quad x = d, \tag{33}$$

$$w_{xxx}^{\mathrm{II}} - w_{xxx}^{\mathrm{I}} = -\varepsilon \alpha w_t, \quad x = d, \tag{34}$$

where α and β are positive damping parameters and i defines the section of the beam (see Fig. 3), that is, w^{I} and w^{II} are the vertical displacements of the beam for $0<x<d$ and for $d<x<1$, respectively.

Again two time scales, $t = t_0$ and $\tau = \varepsilon t$ are introduced, and it is assumed that $w(x,t)$ can be expanded in a formal power series in ε, that is, $w(x,t) = w_0(x,t_0,\tau) + \varepsilon w_1(x,t_0,\tau) + \varepsilon^2 w_2(x,t_0,\tau) + \cdots$. By substituting this expressions into the boundary value problem (29)–(34) and by collecting equal powers in ε then the following $\mathcal{O}(\varepsilon^0)$-problem is obtained

$$\frac{\partial^2 w_0}{\partial t_0^2} + \frac{\partial^4 w_0}{\partial x^4} = 0, \quad 0<x<1, \tag{35}$$

$$w_0 = \frac{\partial^2 w_0}{\partial x^2} = 0, \quad x = 0, \;\; x = 1, \tag{36}$$

and w_0, w_{0x}, w_{0xx}, and w_{0xxx} are continuous at $x = d$. By using the method of separation of variables the solution of this well-known problem for the free oscillations of a simply supported beam can readily be obtained, yielding

$$w_0(x,t_0,\tau) = \sum_{n=1}^{\infty} \left(A_n(\tau)\sin(n^2\pi^2 t_0) + B_n(\tau)\cos(n^2\pi^2 t_0) \right) \sin(n\pi x). \tag{37}$$

Next the $\mathcal{O}(\varepsilon^1)$-problem will be considered

$$\frac{\partial^2 w_1^i}{\partial t_0^2} + \frac{\partial^4 w_1^i}{\partial x^4} = a_1 \frac{\partial w_0^i}{\partial t_0} - 2\frac{\partial^2 w_0^i}{\partial t_0 \partial \tau}, \quad 0<x<d, i=1, \;\; d<x<l, \;\; i=2, \tag{38}$$

$$w_1^{\mathrm{I}} = \frac{\partial^2 w_1^{\mathrm{I}}}{\partial x^2} = 0, \;\; x = 0; \quad w_1^{\mathrm{II}} = \frac{\partial^2 w_1^{\mathrm{II}}}{\partial x^2} = 0, \;\; x = 1,$$

$$w_1^{\mathrm{I}} = w_1^{\mathrm{II}}, \quad \frac{\partial w_1^{\mathrm{I}}}{\partial x} = \frac{\partial w_1^{\mathrm{II}}}{\partial x}, \;\; x = d,$$

$$\frac{\partial^2 w_1^{\mathrm{II}}}{\partial x^2} - \frac{\partial^2 w_1^{\mathrm{I}}}{\partial x^2} = \beta \frac{\partial^2 w_0^2}{\partial x \partial t_0}, \quad x = d,$$

$$\frac{\partial^3 w_1^{\mathrm{II}}}{\partial x^3} - \frac{\partial^3 w_1^{\mathrm{I}}}{\partial x^3} = -\alpha \frac{\partial w_0^2}{\partial t_0}, \quad x = d. \tag{39}$$

To solve the boundary value problem (38)–(39) for w_1 it is convenient to make the boundary conditions in Eq. (39) at $x = d$ homogeneous by introducing the following transformations:

$$\begin{aligned} w_1^{\mathrm{I}}(x, t_0, \tau) = {} & u_1(x, t_0, \tau) - \frac{\alpha d(d^2 - 3d + 2)}{6} \frac{\partial w_0(d, t_0, \tau)}{\partial t_0} x \\ & - \frac{\beta(3d^2 - 6d + 2)}{6} \frac{\partial^2 w_0(d, t_0, \tau)}{\partial x \partial t_0} x + \frac{\alpha(1-d)}{6} \frac{\partial w_0(d, t_0, \tau)}{\partial t_0} x^3 \\ & - \frac{\beta}{6} \frac{\partial^2 w_0(d, t_0, \tau)}{\partial x \partial t_0} x^3, \end{aligned} \tag{40}$$

$$\begin{aligned} w_1^{\mathrm{II}}(x, t_0, \tau) = {} & u_1(x, t_0, \tau) + \frac{\alpha d^3}{6} \frac{\partial w_0(d, t_0, \tau)}{\partial t_0} + \frac{\beta d^2}{2} \frac{\partial^2 w_0(d, t_0, \tau)}{\partial x \partial t_0} \\ & - \frac{\alpha d(d^2 + 2)}{6} \frac{\partial w_0(d, t_0, \tau)}{\partial t_0} x - \frac{\beta(3d^2 + 2)}{6} \frac{\partial^2 w_0(d, t_0, \tau)}{\partial x \partial t_0} x \\ & + \frac{\alpha d}{2} \frac{\partial w_0(d, t_0, \tau)}{\partial t_0} x^2 + \frac{\beta}{2} \frac{\partial^2 w_0(d, t_0, \tau)}{\partial x \partial t_0} x^2 - \frac{\alpha d}{6} \frac{\partial w_0(d, t_0, \tau)}{\partial t_0} x^3 \\ & - \frac{\beta}{6} \frac{\partial^2 w_0(d, t_0, \tau)}{\partial x \partial t_0} x^3. \end{aligned} \tag{41}$$

By substituting this transformation into Eqs. (38)–(39) and by putting $u_1(x, t_0, \tau) = \sum_{n=1}^{\infty} f_n(t_0, \tau) \sin(n\pi x)$, and by using the orthogonality properties of the sine functions the following equation for $f_k(t_0, \tau)$ is obtained

$$\begin{aligned} \left(\frac{\partial^2 f_k}{\partial t_0} + (k\pi)^4 f_k \right) / 2 = {} & (k\pi)^2 \left(a_1 G_k(t_0) - 2 \frac{\partial G_k(t_0, \tau)}{\partial \tau} \right) / 2 \\ & - \sum_{k=1}^{\infty} \frac{\alpha}{6} k^6 \pi^6 (-d(d^2 - 3d + 2)\bar{\gamma}_1 + (1-d)\bar{\gamma}_3 + d^3 \gamma_0 - 3d(d^2 + 1)\gamma_1 \\ & - 3d\gamma_2 - d\gamma_3) G_k(t_0, \tau) \sin(k\pi d) + \sum_{k=1}^{\infty} \frac{\beta}{6} n^7 \pi^7 \big((3d^2 - 6d + 2)\bar{\gamma}_1 \\ & - \bar{\gamma}_3 + 3d^2 \gamma_0 + (3d^2 + 2)\gamma_1 + 3\gamma_2 - \gamma_3 \big) G_k(t_0, \tau) \cos(k\pi d), \end{aligned} \tag{42}$$

where $G_k(t_0, \tau) = A_k(\tau)\cos(k^2\pi^2 t_0) - B_k(\tau)\sin(k^2\pi^2 t_0)$,

$$\bar{\gamma}_1 = \int_0^d x \sin(k\pi x)\,\mathrm{d}x, \quad \bar{\gamma}_3 = \int_0^d x^3 \sin(k\pi x)\,\mathrm{d}x, \quad \gamma_0 = \int_d^1 \sin(k\pi x)\,\mathrm{d}x,$$

$$\gamma_1 = \int_d^1 x \sin(k\pi x)\,\mathrm{d}x, \quad \gamma_2 = \int_d^1 x^2 \sin(k\pi x)\,\mathrm{d}x, \quad \gamma_3 = \int_d^1 x^3 \sin(k\pi x)\,\mathrm{d}x.$$

Since $\cos(\lambda_k^2 t_0)$ and $\sin(\lambda_k^2 t_0)$ are part of the homogeneous solution of u_1, the coefficients of $\cos(\lambda_k^2 t_0)$ and $\sin(\lambda_k^2 t_0)$ in the right-hand side of Eq. (42) should be set equal to zero (elimination of secular terms). The following differential equations for A_k and B_k are then obtained

$$\dot{A}_k = \left(-\alpha \sin^2(k\pi d) - \beta k^2 \pi^2 \cos^2(k\pi d) + \frac{a_1}{2} \right) A_k, \tag{43}$$

$$\dot{B}_k = \left(-\alpha \sin^2(k\pi d) - \beta k^2 \pi^2 \cos^2(k\pi d) + \frac{a_1}{2} \right) B_k. \tag{44}$$

For sufficiently large values of the damping parameters α and β the expression between brackets in Eqs. (43)–(44) is always negative. So, with this type of damping device the flow-induced vibrations of the beam can be damped, that is, all oscillation modes will tend to zero.

4. Coupled torsional and vertical vibrations

In this section the coupled torsional and vertical vibrations of a simply supported beam will be studied. The suspension bridge in this section will be modeled by a system as given in Fig. 3. An additional torsional damper is attached at $x = d$ such that the torsional vibrations of the beam around the beam axis are also damped. This damping is assumed to be proportional to the torsional velocity θ_t. This system can be described by the following initial-boundary value problem

$$w^i_{tt} + w^i_{xxxx} = \varepsilon a_1(w^i_t + \theta^i),$$

$$\theta^i_{tt} - b^2\theta^i_{xx} = -\varepsilon b_1(w^i_t + \theta^i),$$

$$w^{\mathrm{I}} = w^{\mathrm{I}}_{xx} = 0, \ \ x = 0; \quad w^{\mathrm{II}} = w^{\mathrm{II}}_{xx} = 0, \ \ x = 1,$$

$$w^{\mathrm{I}} = w^{\mathrm{II}}, \quad w^{\mathrm{I}}_x = w^{\mathrm{II}}_x, \ \ x = d,$$

$$w^{\mathrm{II}}_{xx} - w^{\mathrm{I}}_{xx} = \varepsilon\beta w_{xt}, \quad x = d,$$

$$w^{\mathrm{II}}_{xxx} - w^{\mathrm{I}}_{xxx} = -\varepsilon\alpha w_t, \quad x = d,$$

$$\theta^i = 0, \quad x = 0, \ \ x = d; \ \ \theta^1 = \theta^2, \ \ x = d,$$

$$\theta^{\mathrm{II}}_x - \theta^{\mathrm{I}}_x = \varepsilon\delta\theta_t, \quad x = d, \tag{45}$$

where α, β, and δ are positive damping parameters, and where the last boundary conditions in Eq. (45) again can be derived by applying Newton's second law to an element at $x = d$. The beam is divided (by dampers) into two sections. The parameter $i = \mathrm{I}$ corresponds to the section of the beam with $0 < x < d$, and the parameter $i = \mathrm{II}$ corresponds to the section with $d < x < 1$. A two time scales perturbation method will be used to solve this problem approximately. To investigate the influence of the wind-forces and the dampers a fast time t_0 and a slow time τ are introduced. Also the functions w and θ are expanded in power series in ε, that is, $w(x,t) = w_0(x,t_0,\tau) + \varepsilon w_1(x,t_0,\tau) + \cdots$ and $\theta(x,t) = \theta_0(x,t_0,\tau) + \varepsilon\theta_1(x,t_0,\tau) + \cdots$. Then the $\mathcal{O}(1)$ and $\mathcal{O}(\varepsilon)$ problems are studied. The $\mathcal{O}(1)$-problem becomes

$$\frac{\partial^2 w_0}{\partial t_0^2} + \frac{\partial^4 w_0}{\partial x^4} = 0,$$

$$\frac{\partial^2 \theta_0}{\partial t_0^2} - b^2\frac{\partial^2 \theta_0}{\partial x^2} = 0,$$

$$w_0 = \frac{\partial^2 w_0}{\partial x^2} = 0, \quad x = 0, \ \ x = 1,$$

$$\theta_0 = 0, \quad x = 0, \ \ x = 1,$$

and w_0, w_{0x}, w_{0xx}, w_{0xxx}, θ_0, and θ_{0x} are continuous at $x = d$. This problem is well-known and can easily be solved, yielding

$$w_0(x,t_0,\tau) = \sum_{n=1}^{\infty}\big(A_n(\tau)\sin(n^2\pi^2 t_0) + B_n(\tau)\cos(n^2\pi^2 t_0)\big)\sin(n\pi x), \tag{46}$$

$$\theta_0(x,t_0,\tau) = \sum_{m=1}^{\infty}(C_m(\tau)\sin(m\pi b t_0) + D_m(\tau)\cos(m\pi b t_0))\sin(m\pi x). \tag{47}$$

The $\mathcal{O}(\varepsilon)$-problem now becomes

$$\frac{\partial^2 w_1}{\partial t_0^2} + \frac{\partial^4 w_1}{\partial x^4} = a_1\left(\frac{\partial w_0}{\partial t_0} + \theta_0\right) - 2\frac{\partial^2 w_0}{\partial t_0 \partial \tau}, \tag{48}$$

$$\frac{\partial^2 \theta_1}{\partial t_0^2} - b^2\frac{\partial^2 \theta_1}{\partial x^2} = -b_1\left(\frac{\partial w_0}{\partial t_0} + \theta_0\right) - 2\frac{\partial^2 \theta_0}{\partial t_0 \partial \tau}, \tag{49}$$

$$w_1^1 = \frac{\partial^2 w_1^1}{\partial x^2} = 0,\ \ x = 0, \quad w_1^2 = \frac{\partial^2 w_1^2}{\partial x^2} = 0,\ \ x = 1, \tag{50}$$

$$w_1^{\mathrm{I}} = w_1^{\mathrm{II}}, \quad \frac{\partial w_1^{\mathrm{I}}}{\partial x} = \frac{\partial w_1^{\mathrm{II}}}{\partial x}, \quad x = d, \tag{51}$$

$$\frac{\partial^2 w_1^{\mathrm{II}}}{\partial x^2} - \frac{\partial^2 w_1^{\mathrm{I}}}{\partial x^2} = \beta\frac{\partial^2 w_0}{\partial t_0 \partial x}, \quad x = d, \tag{52}$$

$$\frac{\partial^3 w_1^{\mathrm{II}}}{\partial x^3} - \frac{\partial^3 w_1^{\mathrm{I}}}{\partial x^3} = -\alpha\frac{\partial w_0}{\partial t_0}, \quad x = d, \tag{53}$$

$$\theta_1 = 0,\ \ x = 0,\ \ x = 1, \quad \theta_1^{\mathrm{I}} = \theta_1^{\mathrm{II}},\ \ x = d, \tag{54}$$

$$\frac{\partial \theta_1^{\mathrm{II}}}{\partial x} - \frac{\partial \theta_1^{\mathrm{I}}}{\partial x} = \delta\frac{\partial \theta_0}{\partial t_0}, \quad x = d. \tag{55}$$

The boundary conditions (51)–(55) are nonhomogeneous ones. To solve the boundary value problem (48)–(55) the following transformations are introduced to make the boundary conditions homogeneous:

$$\begin{aligned} w_1^{\mathrm{I}} = u_1 &+ \left(-\frac{\alpha d(d^2 - 3d + 2)}{6}\frac{\partial w_0(t_0,\tau,d)}{\partial t_0} - \frac{\beta(3d^2 - 6d + 2)}{6}\frac{\partial^2 w_0(t_0,\tau,d)}{\partial x \partial t_0}\right)x \\ &+ \left(\frac{\alpha(1-d)}{6}\frac{\partial w_0(t_0,\tau,d)}{\partial t_0} - \frac{\beta}{6}\frac{\partial^2 w_0(t_0,\tau,d)}{\partial x \partial t_0}\right)x^3, \end{aligned}$$

$$\begin{aligned} w_1^{\mathrm{II}} = u_1 &+ \frac{\alpha d^3}{6}\frac{\partial w_0(d)}{\partial t_0} + \frac{\beta d^2}{2}\frac{\partial^2 w_0(d)}{\partial x \partial t_0} + \left(-\frac{\alpha d(d^2+2)}{6}\frac{\partial w_0(d)}{\partial t_0}\right. \\ &\left. - \frac{\beta(3d^2+2)}{6}\frac{\partial^2 w_0(t_0,\tau,d)}{\partial x \partial t_0}\right)x + \left(\frac{\alpha d}{2}\frac{\partial w_0(t_0,\tau,d)}{\partial t_0} + \frac{\beta}{2}\frac{\partial^2 w_0(t_0,\tau,d)}{\partial x \partial t_0}\right)x^2 \\ &+ \left(-\frac{\alpha d}{6}\frac{\partial w_0(t_0,\tau,d)}{\partial t_0} - \frac{\beta}{6}\frac{\partial^2 w_0(t_0,\tau,d)}{\partial x \partial t_0}\right)x^3, \end{aligned}$$

$$\theta_1^1 = \eta_1 + (d-1)\delta\frac{\partial \theta_0(t_0,\tau,d)}{\partial t_0}x, \quad \theta_1^2 = \eta_1 - d\delta\frac{\partial \theta_0(t_0,\tau,d)}{\partial t_0} + d\delta\frac{\partial \theta_0(t_0,\tau,d)}{\partial t_0}x,$$

where $u_1 = u_1(x, t_0, \tau)$ and $\eta_1 = \eta_1(x, t_0, \tau)$. By using these transformations, by substituting the expressions for w_0 and θ_0 into (48)–(55), by using the orthogonality properties of the sine-series, and by taking into consideration that the functions $u_1(x, t_0, \tau)$ and $\eta_1(x, t_0, \tau)$ have the following form:

$$u_1(x, t_0, \tau) = \sum_{n=1}^{\infty} f_n(t_0, \tau)\sin(n\pi x),$$

$$\eta_1(x, t_0, \tau) = \sum_{m=1}^{\infty} g_m(t_0, \tau)\sin(m\pi x),$$

it follows that the boundary value problem (48)–(55) can be rewritten as a problem for $f_n(t_0,\tau)$ and $g_n(t_0,\tau)$

$$\begin{aligned}\frac{1}{2}\left(\frac{\partial^2 f_k}{\partial t_0^2}+k^4\pi^4 f_k\right) = & -k^2\pi^2\left(\frac{\partial A_k(\tau)}{\partial\tau}\cos(k^2\pi^2 t_0)-\frac{\partial B_k(\tau)}{\partial\tau}\sin(k^2\pi^2 t_0)\right)\\ & +\frac{a_1}{2}k^2\pi^2\big(A_k(\tau)\cos(k^2\pi^2 t_0)-B_k(\tau)\sin(k^2\pi^2 t_0)\big)\\ & +\frac{a_1}{2}(C_k(\tau)\sin(k\pi b t_0)+D_k(\tau)\cos(k\pi b t_0))\\ & \times\frac{\alpha}{6}k^6\pi^6\big(-A_k(\tau)\cos(k^2\pi^2 t_0)+B_k(\tau)\sin(k^2\pi^2 t_0)\big)\\ & \times\sin(k\pi d)\gamma_k-\frac{\beta}{6}k^7\pi^7\big(-A_k(\tau)\cos(k^2\pi^2 t_0)\\ & +B_k(\tau)\sin(k^2\pi^2 t_0)\big)\cos(k\pi d)\sigma_k,\end{aligned}\tag{56}$$

$$\begin{aligned}\frac{1}{2}\left(\frac{\partial^2 g_k}{\partial t_0^2}+k^2\pi^2 b^2 g_k\right) = & -k\pi b\left(\frac{\partial C_k(\tau)}{\partial\tau}\cos(k\pi b t_0)-\frac{\partial D_k(\tau)}{\partial\tau}\cos(k\pi b t_0)\right)\\ & -\frac{b_1}{2}k^2\pi^2\big(A_k(\tau)\cos(k^2\pi^2 t_0)-B_k(\tau)\sin(k^2\pi^2 t_0)\big)\\ & -\frac{b_1}{2}(C_k(\tau)\sin(k\pi b t_0)+D_k(\tau)\cos(k\pi b t_0))\\ & +\delta k^3\pi^3 b^3(-C_k(\tau)\cos(k\pi b t_0)+D_k(\tau)\cos(k\pi b t_0))\phi_k,\end{aligned}\tag{57}$$

where

$$\begin{aligned}\gamma_k = & \int_0^d(-d(d^2-3d+2)x+(1-d)x^3)\sin(k\pi x)\,\mathrm{d}x\\ & +\int_d^1(d^3-d(d^2+2)x+3dx^2-dx^3)\sin(k\pi x)\,\mathrm{d}x,\end{aligned}$$

$$\begin{aligned}\sigma_k = & \int_0^d(-(3d^2-6d+2)x-x^3)\sin(k\pi x)\,\mathrm{d}x\\ & +\int_d^1(3d^2-(3d^2+2)x+3x^2-x^3)\sin(k\pi x)\,\mathrm{d}x,\end{aligned}$$

$$\phi_k = -\int_0^d(d-1)x\sin(k\pi x)\,\mathrm{d}x+\int_d^1 d(1-x)\sin(k\pi x)\,\mathrm{d}x.$$

Now it should be observed that in the right-hand side of Eqs. (56) and (57) terms like $\cos(k^2\pi^2 t_0)$, $\sin(k^2\pi^2 t_0)$, $\cos(k\pi b t_0)$, and $\sin(k\pi b t_0)$ occur. These terms are solutions of the corresponding homogeneous equations (56) and (57). So, secular terms will occur in the solutions for $f_k(t_0,\tau)$ and $g_k(t_0,\tau)$. To avoid these secular terms the coefficients before the terms $\cos(k^2\pi^2 t_0)$, $\sin(k^2\pi^2 t_0)$, $\cos(k\pi b t_0)$, and $\sin(k\pi b t_0)$ have to be set equal to zero. Now two cases have to be distinguished (a) $b\neq k\pi$ for all k and (b) $b=k\pi$ for a certain value of k. This first case will be referred to as the nonresonant case and the other one will be called the resonant case.

4.1. The nonresonant case ($b\neq k\pi$ for all k)

In this case it follows from Eqs. (56)–(57) that $A_k(\tau)$, $B_k(\tau)$, $C_k(\tau)$, $D_k(\tau)$ have to satisfy

$$\dot{A}_k = -\left(\alpha\sin^2(k\pi d)+\beta k^2\pi^2\cos^2(k\pi d)-\frac{a_1}{2}\right)A_k,$$

$$\dot{B}_k = -\left(\alpha\sin^2(k\pi d)+\beta k^2\pi^2\cos^2(k\pi d)-\frac{a_1}{2}\right)B_k,$$

$$\dot{C}_k = -\delta b^2 \sin^2(k\pi d) C_k - \frac{b_1}{2k\pi b} D_k,$$

$$\dot{D}_k = -\delta b^2 \sin^2(k\pi d) D_k + \frac{b_1}{2k\pi b} C_k, \tag{58}$$

where the dot represents differentiation with respect to τ. The eigenvalues of system (58) can easily be determined, yielding

$$\lambda_{1,2} = -\left(\alpha \sin^2(k\pi d) + \beta k^2\pi^2 \cos^2(k\pi d) - \frac{a_1}{2}\right), \quad \lambda_{3,4} = -\delta b^2 \sin^2(k\pi d) \pm \mathrm{i}\frac{b_1}{2k\pi b}. \tag{59}$$

It is clear that for all sufficiently large values of the damping parameters α and β the real parts of the eigenvalues λ_1 and λ_2 will be negative. Moreover, it follows from Eq. (59) that also the real part of λ_3 and λ_4 are negative when $\sin(k\pi d) \neq 0$. So by choosing α, β, and d appropriately it follows that damping in the beam system can always be obtained.

4.2. The resonant case $b = k\pi$ for a certain fixed k

In this case for $b = k\pi$ the frequency of a vertical and the torsional oscillation models will coincide. Then, it follows from Eqs. (56)–(57) that $A_k(\tau)$, $B_k(\tau)$, $C_k(\tau)$, $D_k(\tau)$ have to satisfy

$$\dot{A}_k = -\left(\alpha \sin^2(bd) + \beta k^2\pi^2 \cos^2(bd) - \frac{a_1}{2}\right) A_k + \frac{a_1}{2k^2\pi^2} D_k, \tag{60}$$

$$\dot{B}_k = -\left(\alpha \sin^2(bd) + \beta k^2\pi^2 \cos^2(bd) - \frac{a_1}{2}\right) B_k - \frac{a_1}{2k^2\pi^2} C_k, \tag{61}$$

$$\dot{C}_k = -\delta b^2 \sin^2(bd) C_k - \frac{b_1}{2b^2} D_k - \frac{b_1 k\pi}{2b} A_k, \tag{62}$$

$$\dot{D}_k = -\delta b^2 \sin^2(bd) D_k + \frac{b_1}{2b^2} C_k - \frac{b_1 k\pi}{2b} B_k. \tag{63}$$

The eigenvalues of system (60)–(63) are given by

$$\lambda = -\frac{a}{2} + \sqrt{|c|}\big(\cos(Arg(c)/2 + m\pi) + \mathrm{i}\,\sin(Arg(c)/2 + m\pi)\big), \tag{64}$$

for $m = 0, 1$, where

$$a = \alpha \sin^2(bd) + \beta k^2\pi^2 \cos^2(bd) - \frac{a_1}{2} + \delta b^2 \sin^2(bd) \mp \mathrm{i}\frac{b_1}{2b^2},$$

$$c = \frac{1}{4}\left(\alpha \sin^2(bd) + \beta k^2\pi^2 \cos^2(bd) - \frac{a_1}{2} - \delta b^2 \sin^2(bd)\right)^2 \pm \frac{b_1^2}{16b^4}$$
$$\pm \mathrm{i}\frac{b_1}{4b^2}\left(\alpha \sin^2(bd) + \beta k^2\pi^2 \cos^2(bd) - \delta b^2 \sin^2(bd)\right).$$

In Fig. 4 the maximum values of the real parts of the eigenvalues are given (on a gray scale) for different values of the damping parameters. The parameters a_1 and b_1 describing the wind velocity are taken from experiments [17] and are in this case taken to be equal to 0.42 and 0.75, respectively. Positive values of the real part of the eigenvalues correspond to a white coloring in Fig. 4. It can be seen that no damping in the system will occur only for quite a small number of combinations of the parameter values α, β, and δ. To construct these plots one parameter, in this case the parameter α, and the position of dampers on the beam have been fixed. From the plots it can be seen for which combinations of the other two parameters β and δ damping will occur or not. By taking the parameter α sufficiently large (larger than 0.01) damping will occur for all combinations of the parameters β and δ. We can always fix an other parameter, for example δ. So for $\delta = 0.001$ for $\alpha < 0.5$ and $\beta < 0.35$ damping will always occur.

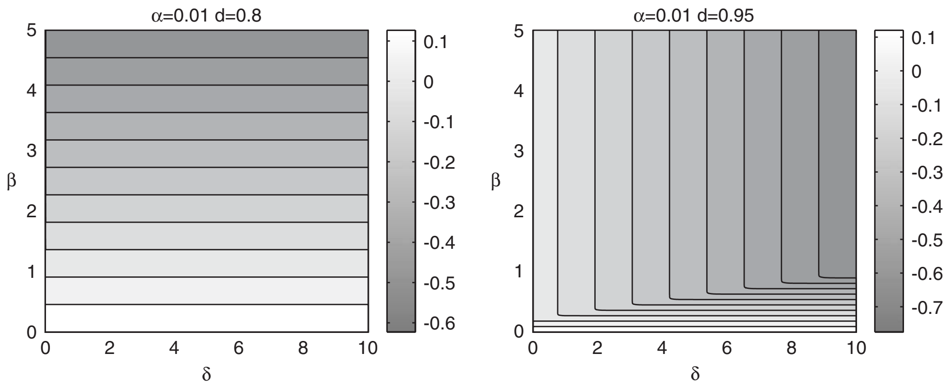

Fig. 4. Plots of the real parts of the eigenvalues for different values of the damping parameters.

5. Conclusions

In this paper the oscillations of a suspension bridge have been modeled by coupled vertical and torsional oscillations of a simply supported beam. This model has been described by coupled, linear partial differential equations. Several types of dampers have been added to the beam to diminish undesirable oscillations. It has been shown that a combination of three types of dampers (damping proportional to the lateral velocity, damping proportional to the rotational velocity, and damping proportional to the torsional velocity) guarantees the presence of damping in the system when the damping parameters and positions of the dampers are chosen appropriately. The use of one or two types of dampers will not always generate damping in the system. As, for instance, can be seen from Eq. (59) if parameter d is such that $\sin(k\pi d) = 0$ and parameter $\beta = 0$ or $\cos(k\pi d) = 0$ and parameter $\alpha = 0$. Because then two of the eigenvalues have a positive real part. Since it is in all cases possible to calculate the eigenvalues explicitly, it is not so difficult to generate stability diagrams for given parameter values $\alpha, \beta, \delta, d, b, a_1$ and b_1.

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