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Short Communication

Solutions of a quadratic nonlinear oscillator: Iteration procedure

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Abstract

A modified iteration procedure is applied to a quadratic nonlinear oscillator (QNO). When the solutions of the two auxiliary equations are available, we obtain the first and second analytical approximate solutions to the QNO. Our second approximation result significantly improves on the accuracy of the approximate solution obtained by using the first-order harmonic balance method.

1. Introduction

Consider a quadratic nonlinear oscillator (QNO) modeled by

$$\ddot{x} + x + \varepsilon x^2 = 0, \quad x(0) = A > 0, \quad \dot{x}(0) = 0, \tag{1}$$

which is used as a mathematical model of the human eardrum oscillation [1]. Unlike cubic nonlinear oscillators, the behavior of QNOs is different for positive and negative directions [2,3]. By using the lowest-order harmonic balance (HB) method [1] and resorting to the two auxiliary equations, Hu [3] obtained the first-order approximate solution to Eq. (1). However, the relative errors of the approximate period and approximate periodic solution are increased when $\varepsilon A \to 0.5$ because Eq. (1) has a homoclinic orbit with period $+\infty$ for $\varepsilon A = 0.5$ [3]. In addition, it is usually rather difficult to use the HB method to produce higher-order analytical approximations because it requires solutions of sets of complicated nonlinear algebraic equations. To improve the accuracy of the approximate solution in Ref. [3], the first and second approximations to the solution of Eq. (1) are presented in this paper using a modified iteration procedure derived by Lim et al. [4] and Mickens [5].

For convenience, an outline of the iteration technique [4,5] is rewritten here.

The nonlinear oscillator equation is assumed to have the form

$$\ddot{x} + g(x) = 0, \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{2}$$

where $g(x)$ is a nonlinear function of x and has the property

$$g(-x) = -g(x). \tag{3}$$

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Let the angular frequency of Eq. (2) be ω, which is unknown to be further determined. Then Eq. (2) can be rewritten as

$$\ddot{x} + \omega^2 x = \omega^2 x - g(x) =: G(x), \quad x(0) = A, \quad \dot{x}(0) = 0. \tag{4}$$

The iteration scheme is [6]

$$\ddot{x}_{k+1} + \omega^2 x_{k+1} = G(x_k), \quad x_k(0) = A, \quad \dot{x}_k(0) = 0, \quad k = 0, 1, 2, \ldots, \tag{5}$$

where the input or starting function is

$$x_0(t) = A \cos\theta = A \cos\omega t. \tag{6}$$

Usually x_1 can easily be obtained from Eq. (5). When $k \geqslant 1$, we have [7]

$$G(x_k) = G[x_{k-1} + (x_k - x_{k-1})] \approx G(x_{k-1}) + G_x(x_{k-1})(x_k - x_{k-1}), \tag{7}$$

where

$$G_x(x) = \frac{\mathrm{d}G}{\mathrm{d}x}. \tag{8}$$

Therefore, Eq. (5) can be rewritten as [4,5]

$$\begin{aligned} \ddot{x}_{k+1} + \omega^2 x_{k+1} &= G(x_{k-1}) + G_x(x_{k-1})(x_k - x_{k-1}), \\ x_k(0) = A, \quad \dot{x}_k(0) &= 0, \quad k = 0, 1, 2, \ldots, \end{aligned} \tag{9}$$

where

$$x_{-1}(t) = x_0(t). \tag{10}$$

The angular frequency ω is calculated anew at each stage of the iteration procedure by demanding that the right-hand side of Eq. (9) contains no terms giving rise to secular terms in the complete solution of Eq. (9) [8].

2. Solutions of the two auxiliary equations

On the basis of the analysis in Ref. [3], we first consider the following auxiliary equation:

$$\ddot{x} + x + \varepsilon|x|x = \ddot{x} + x + \varepsilon x^2 \operatorname{sgn}(x) = 0, \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{11}$$

where sgn(x) is the sign function, equal to $+1$ if $x>0$, 0 if $x=0$, and -1 if $x<0$. Let ω_A be the angular frequency of Eq. (11). For Eq. (11), it follows that

$$G(x) = \omega_A^2 x - x - \varepsilon x^2 \operatorname{sgn}(x) \tag{12}$$

and the equation for $x_1(t)$ is, see Eqs. (9) and (10),

$$\ddot{x}_1 + \omega_A^2 x_1 = G(x_0) = \omega_A^2 x_0 - x_0 - \varepsilon x_0^2 \operatorname{sgn}(x_0), \quad x_1(0) = A, \quad \dot{x}_1(0) = 0. \tag{13}$$

Letting $p = 2$ in Eqs. (15) and (16) of Ref. [7], we have

$$\begin{aligned} \operatorname{sgn}(x_0)x_0^2 &= A^2 \operatorname{sgn}[(A\cos\theta)]\cos^2\theta \\ &= A^2(a_1\cos\theta + a_3\cos 3\theta + a_5\cos 5\theta + \cdots), \end{aligned} \tag{14}$$

where

$$a_1 = 8/(3\pi), \quad a_3 = a_1/5 = 8/(15\pi), \quad a_5 = -a_1/35 = -8/(105\pi). \tag{15a−c}$$

Substituting $x_0 = A\cos\theta = A\cos\omega_A t$ and Eq. (14) into Eq. (13) gives

$$\begin{aligned} \ddot{x}_1 + \omega_A^2 x_1 &= \left(\omega_A^2 - 1 - \frac{8p_A}{3\pi}\right)A\cos\theta - \frac{8p_A A}{15\pi}\cos 3\theta + \frac{8p_A A}{105\pi}\cos 5\theta + \mathrm{HOH}, \\ x_1(0) &= A, \quad \dot{x}_1(0) = 0, \end{aligned} \tag{16}$$

where HOH stands for the higher-order harmonics and

$$p_A = \varepsilon A. \tag{17}$$

The requirement of no secular terms in $x_1(t)$ implies that

$$\omega_A = \omega_{A1} = \sqrt{1 + \frac{8p_A}{3\pi}}. \tag{18}$$

The corresponding first approximate periodic solution becomes

$$x_{A1}(t) = A\cos\omega_A t + b_3(\cos\omega_A t - \cos 3\omega_A t) + b_5(\cos\omega_A t - \cos 5\omega_A t), \tag{19}$$

where ω_A is given by Eq. (18) and

$$b_3 = -\frac{p_A A}{15\pi\omega_{A1}^2}, \quad b_5 = \frac{p_A A}{315\pi\omega_{A1}^2}. \tag{20a,b}$$

The corresponding first approximate period of the oscillation is

$$T_{A1} = \frac{2\pi}{\omega_{A1}}. \tag{21}$$

Continuing to $k = 1$ yields

$$\begin{aligned} x_2^{2} + \omega_A^2 x_2 &= G(x_0) + (\omega_A^2 - 1)(x_1 - x_0) - 2\varepsilon[\operatorname{sgn}(x_0)x_0^2/x_0](x_1 - x_0), \\ x_2(0) &= A, \quad \dot{x}_2(0) = 0, \end{aligned} \tag{22}$$

where use has been made of the relation

$$\frac{\mathrm{d}}{\mathrm{d}x}[\operatorname{sgn}(x)x^2] = 2\operatorname{sgn}(x)x = 2\operatorname{sgn}(x)x^2/x. \tag{23}$$

Letting $p = 2$ in Eq. (22) of Ref. [7] produces

$$2[\operatorname{sgn}(x_0)x_0^2/x_0](x_1 - x_0) = 4A^{-1}\operatorname{sgn}(x_0)x_0^2[b_3(1 - \cos 2\theta) + b_5(\cos 2\theta - \cos 4\theta)]. \tag{24}$$

Now substituting $x_0 = A\cos\theta = A\cos\omega_A t$, Eqs. (19) and (24) into Eq. (22), taking into account the relation given by Eq. (14) and making some arithmetical manipulations results in

$$\begin{aligned} x_2^{2} + \omega_A^2 x_2 = {} & \left[\omega_A^2 - 1 - \frac{8p_A}{3\pi} - \frac{4p_A(\omega_A^2 - 1)}{63\pi\omega_{A1}^2} + \frac{2944p_A^2}{11025\pi^2\omega_{A1}^2}\right] A\cos\theta \\ & + \frac{p_A A}{15\pi}\left(-8 + \frac{\omega_A^2 - 1}{\omega_{A1}^2} - \frac{6704p_A}{2205\pi\omega_{A1}^2}\right)\cos 3\theta \\ & + \frac{p_A A}{105\pi}\left(8 - \frac{\omega_A^2 - 1}{3\omega_{A1}^2} - \frac{368p_A}{45\pi\omega_{A1}^2}\right)\cos 5\theta + \text{HOH}, \quad x_2(0) = A, \quad \dot{x}_2(0) = 0. \end{aligned} \tag{25}$$

Secular terms are eliminated by setting the coefficient of $\cos\theta$ equal to zero; doing this yields

$$\omega_A = \omega_{A2} = \sqrt{1 + \frac{8p_A[1 - 368p_A/(3675\pi\omega_{A1}^2)]}{3\pi[1 - 4p_A/(63\pi\omega_{A1}^2)]}}. \tag{26}$$

The corresponding second approximate periodic solution is

$$x_{A2}(t) = A\cos\omega_A t + c_3(\cos\omega_A t - \cos 3\omega_A t) + c_5(\cos\omega_A t - \cos 5\omega_A t), \tag{27}$$

where ω_A is given by Eq. (26) and

$$c_3 = \frac{p_A A}{120\pi\omega_{A2}^2}\left(-8 + \frac{\omega_{A2}^2 - 1}{\omega_{A1}^2} - \frac{6704p_A}{2205\pi\omega_{A1}^2}\right), \tag{28a}$$

$$c_5 = \frac{p_A A}{2520\pi\omega_{A2}^2}\left(8 - \frac{\omega_{A2}^2 - 1}{3\omega_{A1}^2} - \frac{368p_A}{45\pi\omega_{A1}^2}\right). \tag{28b}$$

The corresponding second approximate period of the oscillation is

$$T_{A2} = \frac{2\pi}{\omega_{A2}}. \tag{29}$$

Now we consider the second auxiliary equation [3]

$$\ddot{x} + x - \varepsilon|x|x = \ddot{x} + x - \varepsilon x^2 \mathrm{sgn}\,(x) = 0, \quad x(0) = B > 0, \quad \dot{x}(0) = 0, \tag{30}$$

where [3,9]

$$B = [3 + 2p_A - \sqrt{3(1 - 2p_A)(3 + 2p_A)}]/(4\varepsilon). \tag{31}$$

If we obtain the approximate solutions to Eq. (11), then we do not need to actually solve Eq. (30). In fact, replacing $p_A = \varepsilon A$ and A in the above calculations by $p_B = -\varepsilon B$ and B respectively gives the first and second approximations to the solution of Eq. (30) immediately. Consequently, the first approximation is

$$x_{B1}(t) = B \cos \omega_B t + b_3'(\cos \omega_B t - \cos 3\omega_B t) + b_5'(\cos \omega_B t - \cos 5\omega_B t), \tag{32}$$

where

$$\omega_B = \omega_{B1} = \sqrt{1 + \frac{8p_B}{3\pi}}, \quad b_3' = -\frac{p_B B}{15\pi\omega_{B1}^2}, \quad b_5' = \frac{p_B B}{315\pi\omega_{B1}^2}. \tag{33a−c}$$

The corresponding approximate period of the oscillation is

$$T_{B1} = \frac{2\pi}{\omega_{B1}}. \tag{34}$$

The second approximation is

$$x_{B2}(t) = B\cos \omega_B t + c_3'(\cos \omega_B t - \cos 3\omega_B t) + c_5'(\cos \omega_B t - \cos 5\omega_B t), \tag{35}$$

where

$$\omega_B = \omega_{B2} = \sqrt{1 + \frac{8p_B[1 - 368p_B/(3675\pi\omega_{B1}^2)]}{3\pi[1 - 4p_B/(63\pi\omega_{B1}^2)]}}, \tag{36a}$$

$$c_3' = \frac{p_B B}{120\pi\omega_{B2}^2}\left(-8 + \frac{\omega_{B2}^2 - 1}{\omega_{B1}^2} - \frac{6704p_B}{2205\pi\omega_{B1}^2}\right), \tag{36b}$$

$$c_5' = \frac{p_B B}{2520\pi\omega_{B2}^2}\left(8 - \frac{\omega_{B2}^2 - 1}{3\omega_{B1}^2} - \frac{368p_B}{45\pi\omega_{B1}^2}\right). \tag{36c}$$

The corresponding second approximate period of the oscillation is

$$T_{B2} = \frac{2\pi}{\omega_{B2}}. \tag{37}$$

3. Results and discussion

When the approximate solutions of the two auxiliary Eqs. (11) and (30) are available, according to the analysis in Ref. [3], the approximations to the solution of Eq. (1) can be easily obtained. Consequently, the first approximate period T_1 and the corresponding periodic solution $x_1(t)$ to Eq. (1) are respectively

$$T_1 = (T_{A1} + T_{B1})/2, \tag{38}$$

$$x_1(t) = x_{A1}(t), \quad 0 \leqslant t \leqslant \frac{T_{A1}}{4}, \tag{39a}$$

$$x_1(t) = x_{B1}\left(t - \frac{T_{A1}}{4} + \frac{T_{B1}}{4}\right), \quad \frac{T_{A1}}{4} \leqslant t \leqslant \frac{T_{A1}}{4} + \frac{T_{B1}}{2}, \tag{39b}$$

$$x_1(t) = x_{A1}\left(t + \frac{T_{A1}}{2} - \frac{T_{B1}}{2}\right), \quad \frac{T_{A1}}{4} + \frac{T_{B1}}{2} \leqslant t \leqslant T_1. \tag{39c}$$

Similarly, the second approximate period T_2 and the corresponding periodic solution $x_2(t)$ to Eq. (1) are respectively

$$T_2 = (T_{A2} + T_{B2})/2, \tag{40}$$

$$x_2(t) = x_{A2}(t), \quad 0 \leqslant t \leqslant \frac{T_{A2}}{4}, \tag{41a}$$

$$x_2(t) = x_{B2}\left(t - \frac{T_{A2}}{4} + \frac{T_{B2}}{4}\right), \quad \frac{T_{A2}}{4} \leqslant t \leqslant \frac{T_{A2}}{4} + \frac{T_{B2}}{2}, \tag{41b}$$

$$x_2(t) = x_{A2}\left(t + \frac{T_{A2}}{2} - \frac{T_{B2}}{2}\right), \quad \frac{T_{A2}}{4} + \frac{T_{B2}}{2} \leqslant t \leqslant T_2. \tag{41c}$$

Eq. (38) is identical to Eq. (35) in Ref. [3]. The first-order HB method solution to Eq. (1) is [3]

$$x_{H1}(t) = A \cos \omega_{A1} t, \quad 0 \leqslant t \leqslant \frac{T_{A1}}{4}, \tag{42a}$$

$$x_{H1}(t) = B \cos \omega_{B1}\left(t - \frac{T_{A1}}{4} + \frac{T_{B1}}{4}\right), \quad \frac{T_{A1}}{4} \leqslant t \leqslant \frac{T_{A1}}{4} + \frac{T_{B1}}{2}, \tag{42b}$$

$$x_{H1}(t) = A \cos \omega_{A1}\left(t + \frac{T_{A1}}{2} - \frac{T_{B1}}{2}\right), \quad \frac{T_{A1}}{4} + \frac{T_{B1}}{2} \leqslant t \leqslant T_1. \tag{42c}$$

For the purpose of comparison, the following exact solution to Eq. (1) is presented [9]

$$x_e(t) = A + a\, sn^2(\Omega t, m), \tag{43}$$

where $sn(\Omega t, m)$ is the Jacobian elliptic sine function and [9]

$$a = \left[\sqrt{3(1-2p_A)(3+2p_A)} - 3(1+2p_A)\right] \Big/ (4\varepsilon), \tag{44a}$$

$$\Omega = \frac{1}{2}\sqrt{\frac{1}{2} + p_A + \frac{1}{6}\sqrt{3(1-2p_A)(3+2p_A)}}, \tag{44b}$$

$$m = \frac{1}{2} + \frac{3(2p_A^2 + 2p_A - 1)}{3 + (1+2p_A)\sqrt{3(1-2p_A)(3+2p_A)}}. \tag{44c}$$

The corresponding exact period of the oscillation is [9]

$$T_e = 2F/\Omega, \tag{45}$$

where F is the complete elliptical integral of the first kind given by the following equation:

$$F = F\left(k, \frac{\pi}{2}\right) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}, \quad k^2 = m. \tag{46}$$

The exact period T_e obtained by using Eq. (45) the approximate periods T_1 and T_2 computed by Eqs. (38) and (40), respectively, are listed in Table 1 for $\varepsilon = 1$. The percentage errors are defined as $100[T_1(T_2) - T_e]/T_e$. Table 1 indicates that T_2 is more accurate than T_1, especially for A close to 0.5.

The exact solution $x_e(t)$ of Eq. (1) obtained from Eq. (43) and the corresponding approximate solutions $x_{H1}(t)$, $x_1(t)$ and $x_2(t)$ computed by Eqs. (42), (39) and (41), respectively, are plotted in Fig. 1 for the time in

Table 1
Comparison of approximate periods with the exact period for $\varepsilon = 1$

A	T_e	T_1 (% error)	T_2 (% error)
0.100	6.311599	6.311242(−0.006)	6.311595(0.000)
0.200	6.411392	6.409514(−0.029)	6.411370(0.000)
0.300	6.629357	6.622552(−0.103)	6.629261(−0.001)
0.400	7.124567	7.096187(−0.398)	7.123948(−0.009)
0.450	7.706476	7.627741(−1.022)	7.703604(−0.037)
0.460	7.905170	7.801416(−1.312)	7.900709(−0.056)
0.470	8.167157	8.023325(−1.761)	8.159562(−0.093)
0.480	8.545168	8.327834(−2.543)	8.530067(−0.177)
0.490	9.207999	8.811815(−4.303)	9.166040(−0.456)
0.495	9.883999	9.234181(−6.574)	9.785314(−0.998)

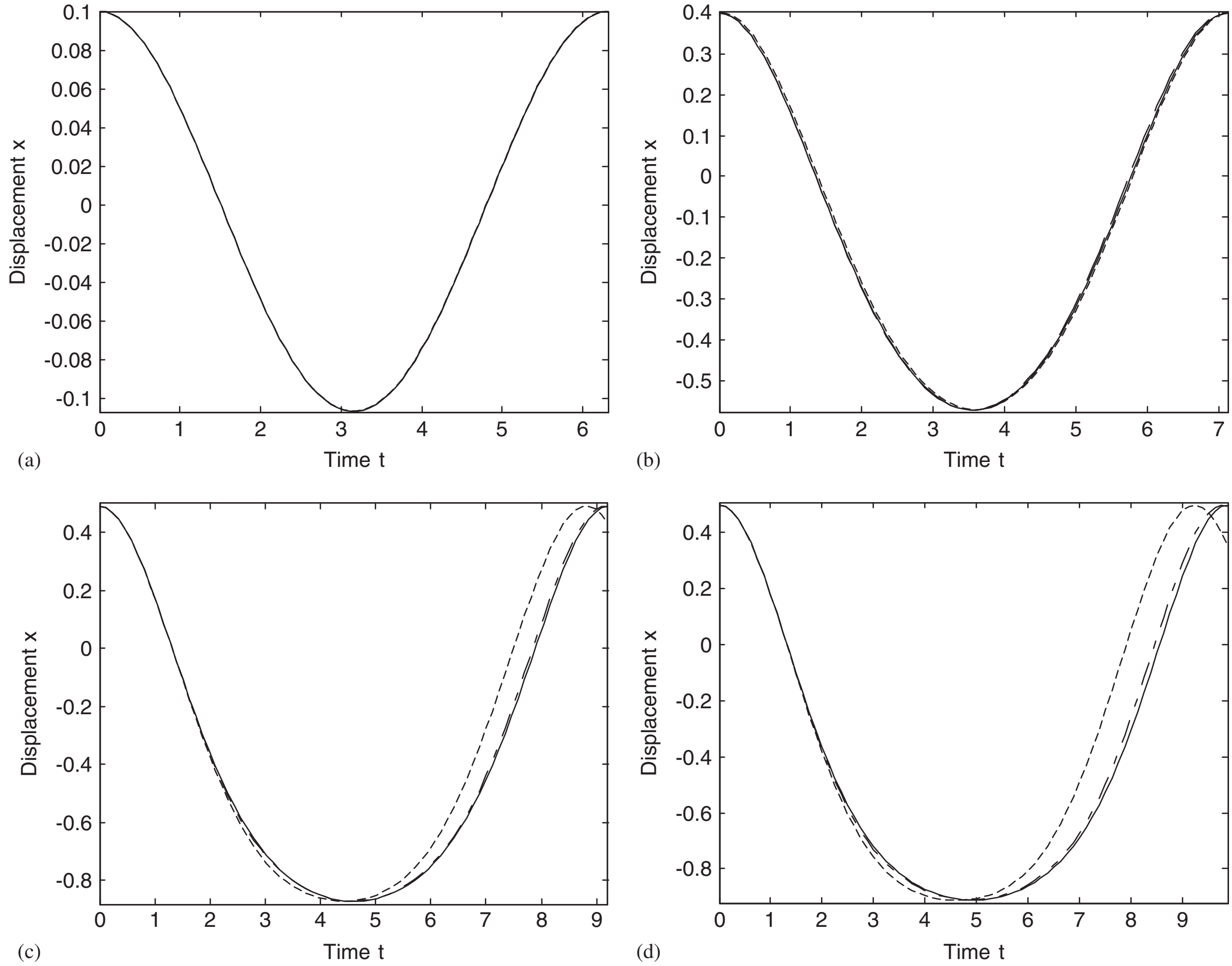

Fig. 1. Comparison of the approximate solutions x_{H1} (dotted curve), x_1 (dashed curve) and x_2 (dash-dot curve) with the exact solution x_e (solid curve) for $\varepsilon = 1$: (a) $A = 0.100$; (b) $A = 0.400$; (c) $A = 0.490$; (d) $A = 0.495$.

one exact period ($\varepsilon = 1$). Fig. 1 shows that $x_{H1}(t)$, $x_1(t)$ and $x_2(t)$ are nearly identical to the exact solution for $A = 0.100$. But for A close to 0.5, $x_1(t)$ is more accurate than $x_{H1}(t)$ and $x_2(t)$ is more accurate than $x_1(t)$, as shown in Fig. 1.

4. Conclusions

A QNO modeled by Eq. (1) has been attacked by a modified iteration technique. First, the technique is applied to the two auxiliary Eqs. (11) and (30), where the force functions are odd. Once the approximate solutions of the two equations are available, we obtain the analytical approximate periodic solutions to the QNO immediately. The second analytical approximation result derived here greatly improves the accuracy of the first-order approximation obtained by using the HB method in Ref [3]. Although the solutions of the two auxiliary equations are needed, actually we only solved one of these equations in great detail. Therefore, the procedure used in this paper is not complicated.

The present method can also be used to deal with the mixed parity differential equation:

$$\ddot{x} + x + \alpha x^2 + \beta x^3 = 0. \tag{47}$$

The two auxiliary equations are respectively [3]

$$\ddot{x} + x + \alpha|x|x + \beta x^3 = \ddot{x} + x + \alpha x^2 \mathrm{sgn}\,(x) + \beta x^3 = 0 \quad \text{for} \quad x \geqslant 0, \tag{48a}$$

$$\ddot{x} + x - \alpha|x|x + \beta x^3 = \ddot{x} + x - \alpha x^2 \mathrm{sgn}\,(x) + \beta x^3 = 0 \quad \text{for} \quad x < 0. \tag{48b}$$

The corresponding results will be presented in another paper.

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