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Vibration analysis of non-homogeneous circular plate of nonlinear thickness variation by differential quadrature method

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Abstract

Free axisymmetric vibrations of non-homogeneous isotropic circular plates of nonlinear thickness variation have been analyzed on the basis of classical plate theory employing the differential quadrature (DQ) method. The non-homogeneity is assumed to arise due to the variation in Young's modulus and also the density of the plate material. The first three natural frequencies have been obtained for clamped, simply supported and free edge conditions, taking grid points as zeros of Chebyshev polynomials. The effect of non-homogeneity and thickness variation on natural frequencies of vibration has been investigated for the first three modes of vibration. The results for linear as well as parabolic thickness variations have been obtained as special cases. Comparison studies have been carried out to establish the accuracy and versatility of the present DQ method.

1. Introduction

Plates of uniform/non-uniform thickness are widely used as structural components in various engineering fields such as aerospace industry, missile technology, naval ship design and telephone industry, etc. Various numerical techniques such as Frobenius method [1], finite-difference method [2], simple polynomial approximation [3], Galerkin's method [4,5], Rayleigh–Ritz method [6–8], characteristic orthogonal polynomials [9], quintic splines method [10], finite element method [11,12] and Chebyshev collocation method [13,14], etc. have been employed to study the vibrational characteristics of plates of various geometries. The above numerical methods such as finite difference and finite element require fine mesh size to obtain accurate results but are computationally expensive. The method of quintic splines, the characteristic orthogonal polynomials and Frobenius method require an appreciable number of terms for plates of variable thickness. Recently, differential quadrature method (DQM), introduced by Bellman et al. [15,16] has emerged as a distinct numerical technique which has capability of producing highly accurate results with minimum computational efforts for initial and boundary value problems. This led to the study of the vibrational behavior of plates of various geometries using DQ method by a number of researchers [17–22], to mention a few.

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In this paper, a DQ procedure is developed for obtaining the natural frequencies of non-homogeneous circular plates of quadratically varying thickness. The consideration of present type of thickness variation was taken earlier by Singh and Saxena [7] and has the advantage of approximating general thickness variations by proper choice of taper constants besides dealing with linear and parabolic variations which are of practical importance. The plate type structural components in aircraft and rockets have to operate under elevated temperatures which cause non-homogeneity in the plate material i.e. elastic constants of the material become functions of the space variables. In an up-to-date survey of literature, authors have come across various models to account for non-homogeneity of plate material proposed by researchers dealing with vibration. Bose [23] analyzed the vibrations of thin non-homogeneous circular plates with a central hole assuming the variation in Young's modulus and density in radial direction as $E = E_0 r$ and $\rho = \rho_0 r$ where E_0 and ρ_0 are constants. Biswas [24] considered a non-homogeneous material for which rigidity $\mu = \mu_0 e^{-\mu z}$ and density $\rho = \rho_0 e^{-\mu z}$ both were assumed to vary exponentially where μ_0 and ρ_0 are constants. Rao et al. [25] dealing with vibration of non-homogeneous isotropic thin plates have assumed linear variations for Young's modulus and density given by $E = E_0(1+\alpha x)$ and $\rho = \rho_0(1+\beta x)$. In a series of papers, Tomar et al. [26–29] have assumed exponential variations i.e. $E = E_0 e^{\mu x}$ and $\rho = \rho_0 e^{\mu x}$ in the study of vibrational behavior of non-homogeneous isotropic plates. The Poisson ratio is assumed to remain constant. The assumption of variation in which the parameter μ is same for Young's modulus as well as density does not seem to have any justification. In the present study, a more general model has been proposed in which, the Young's modulus and density are assumed to vary exponentially in radial direction in distinct manner i.e. $E = E_0 e^{\mu x}$ and $\rho = \rho_0 e^{\eta x}$.

2. Mathematical formulation

The small deflection axisymmetric motion of an isotropic non-homogeneous circular plate of radius a, thickness $h(r)$ and density $\rho(r)$, referred to cylindrical polar coordinates system (r, θ, z) is governed by the equation

$$D w_{,rrrr} + \frac{2(D + rD_{,r})}{r} w_{,rrr} + \frac{\{-D + (2+\upsilon) r D_{,r} + r^2 D_{,rr}\}}{r^2} w_{,rr} + \frac{(D - rD_{,r} + r^2 \upsilon D_{,rr})}{r^3} w_{,r} + \rho h w_{,tt} = 0, \tag{1}$$

where a comma followed by a suffix represents the partial differentiation with respect to that variable and $D(r) = E(r)h^3/12(1-\upsilon^2)$ is the flexural rigidity, w the transverse deflection, t the time and $E(r)$ Young's modulus.

Introducing the non-dimensional variables $x = r/a$, $\bar{w} = w/a$, $\bar{h} = h/a$ together with the quadratic variation in thickness

$$\bar{h} = h_0(1 + \alpha x + \beta x^2) \quad \text{such that} \quad |\alpha| \leqslant 1, \quad |\beta| \leqslant 1 \text{ and } \alpha + \beta > -1 \tag{2}$$

and assuming the exponential variation for the non-homogeneity of material as

$$E = E_0 e^{\mu x}, \qquad \rho = \rho_0 e^{\eta x}, \tag{3}$$

Eq. (1) reduces to

$$P_0 \frac{d^4 W}{dx^4} + P_1 \frac{d^3 W}{dx^3} + P_2 \frac{d^2 W}{dx^2} + P_3 \frac{dW}{dx} + P_4 W = 0, \tag{4}$$

where $\bar{w}(x,t) = W(x)e^{i\omega t}$ (for harmonic vibrations), ω is the radian frequency, μ and η are non-homogeneity parameters, α and β are the taper parameters, h_0, ρ_0 and E_0 are the thickness, density and Young's modulus respectively, at the center of the plate and

$$P_0 = 1, \quad P_1 = 2\{1 + \beta x\}/x, \quad P_2 = B^2 + C + (2 + \nu)B/x - 1/x^2,$$
$$P_3 = (1 - Bx)/x^3 + \nu(B^2 + C)/x, \quad P_4 = -\Omega^2 e^{(\eta-\mu)x}/A^2,$$
$$A = 1 + \alpha x + \beta x^2, \quad B = \mu + 3(\alpha + 2\beta x)/A, \quad C = 3(2\beta - \alpha^2 - 2\beta^2 x^2 - 2\alpha\beta x)/A^2,$$
$$\Omega^2 = 12\rho_0 a^2 \omega^2 (1 - \nu^2)/(E_0 h_0^2).$$

Eq. (4) which is a fourth-order linear differential equation with variable coefficients involving several plate parameters becomes quite complex and so its exact solution is not possible. An approximate solution of Eq. (4) together with the boundary conditions at the edge $x = 1$ and regularity condition at the center $x = 0$, has been obtained by DQ method.

3. Method of solution

Let $x_1, x_2, \ldots, x_m$ be the m grid points in the applicability range [0,1] of the plate. The DQ method approximates the nth order derivative of $W(x)$ w.r.t. x at discrete point x_i as

$$W_x^{(n)}(x_i) = \sum_{j=1}^{m} c_{ij}^{(n)} W(x_j), \qquad i = 1, 2, \ldots, m. \tag{5}$$

Following Shu [30, pp. 31,35], the weighting coefficients $c_{ij}^{(n)}$ in Eq. (5) are given by

$$c_{ij}^{(1)} = \frac{M^{(1)}(x_i)}{(x_i - x_j)M^{(1)}(x_j)}, \qquad i,j = 1, 2, \ldots, m, \quad \text{but } j \neq i, \tag{6}$$

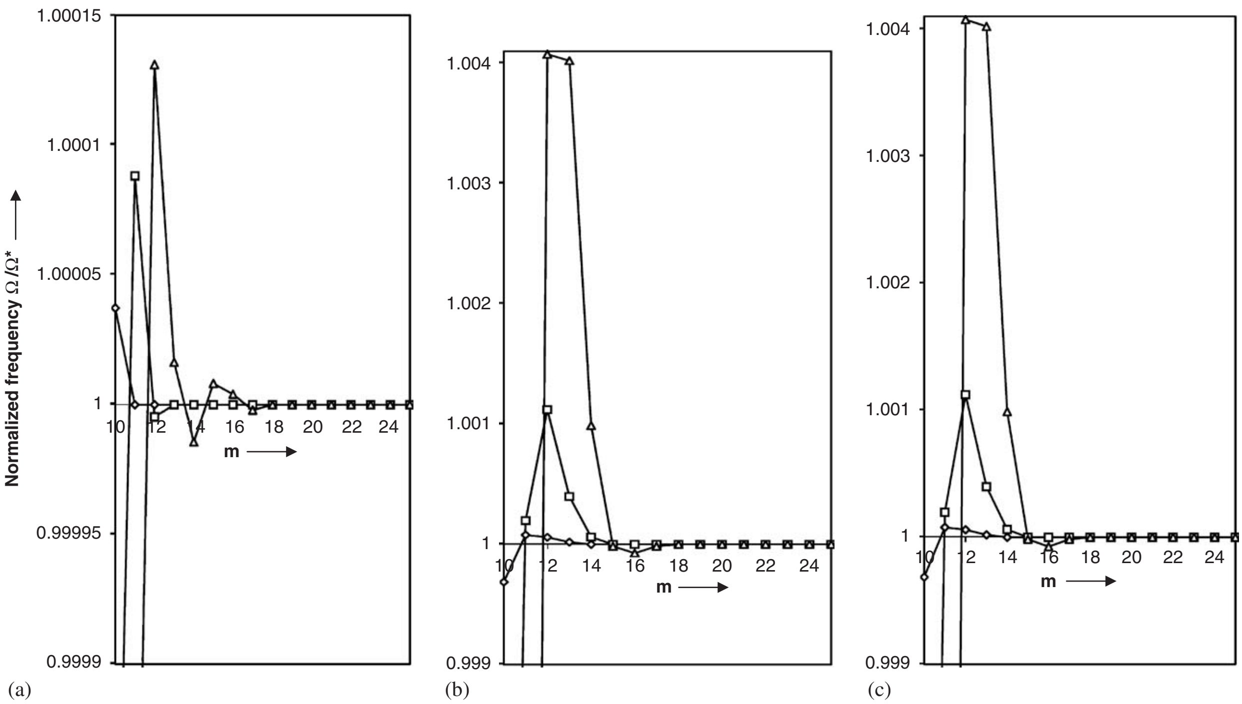

Fig. 1. Convergence of the normalized frequency parameter, Ω/Ω^*, for the first three modes of vibration with grid refinement for $\eta = 1.0$, $\mu = -0.5$, $\alpha = -0.4$, $\beta = 0.1$ for (a) clamped (b) simply supported and (c) free plate. Ω^*—the DQ results using 25 grid points.

Table 1
Values of frequency parameter Ω for clamped plate for $\nu = 0.3$

Mode	α	β	η = −0.5, μ = −0.5	0	1	η = 0, μ = −0.5	0	1	η = 1, μ = −0.5	0	1
I	−0.5	0	5.7823	6.7376	9.1707	5.2676	6.1504	8.4058	4.3141	5.0589	6.9743
		0.5	8.9842	10.5316	14.4740	8.2329	9.6732	13.3572	6.8269	8.0606	11.2424
		−0.5	6.2569	7.2797	9.8979	5.6889	6.6320	9.0529	4.6407	5.4328	7.4779
	0	0	9.5005	11.1464	15.3791	8.6879	10.2158	14.1597	7.1733	8.4746	11.8597
		0.5	12.6893	14.9226	20.6014	11.6473	13.7310	19.0532	9.6912	11.4843	16.1078
		−0.5	9.9883	11.7224	16.2153	9.1180	10.7241	14.8998	7.5016	8.8627	12.4275
	0.5	0	13.2232	15.5730	21.6005	12.1153	14.3021	19.9359	10.0431	11.9150	16.7809
		0.5	16.4301	19.3602	26.7851	15.0951	17.8331	24.8054	12.5840	14.9471	21.0272
II	−0.5	0	26.9194	30.8689	40.4535	23.7356	27.3002	35.9954	18.3158	21.1908	28.2756
		0.5	36.0125	41.2360	53.8464	32.0322	36.7861	48.3178	25.1528	29.0528	38.6035
		−0.5	29.5624	33.9258	44.5205	26.0750	30.0152	39.6350	20.1283	23.3085	31.1577
	0	0	38.9153	44.5753	58.2294	34.6170	39.7711	52.2669	27.1791	31.4115	41.7754
		0.5	46.9682	53.7238	69.9704	42.0016	48.1833	63.1185	33.3287	38.4559	50.9585
		−0.5	41.7988	47.9016	62.6145	37.1793	42.7391	56.2127	29.1787	33.7466	44.9341
	0.5	0	50.0417	57.2557	74.5872	44.7438	51.3480	67.2899	35.4855	40.9669	54.3272
		0.5	57.5710	65.7867	85.4815	51.6715	59.2181	77.3894	41.2949	47.6025	62.9284
III	−0.5	0	62.6436	71.6301	93.0924	54.9910	63.0611	82.4331	42.0591	48.5050	64.1358
		0.5	81.0526	92.2944	118.9703	71.7238	81.8995	106.1599	55.7456	64.0058	83.8865
		−0.5	69.3735	79.3069	102.9940	60.9308	69.8624	91.2716	46.6401	53.7902	71.1093
	0	0	88.1704	100.3867	129.3342	78.0372	89.1041	115.4552	60.6645	69.6613	91.2929
		0.5	103.8901	117.9711	151.2091	92.3972	105.2133	135.6044	72.5320	83.0515	108.2282
		−0.5	95.2048	108.3867	139.5838	84.2756	96.2259	124.6483	65.5218	75.2493	98.6172
	0.5	0	111.2158	126.2833	161.8083	98.9144	112.6360	145.1395	77.6391	88.9129	115.8719
		0.5	125.5564	142.2857	181.6230	112.0582	127.3436	163.4471	88.5764	101.2208	131.3580

where

$$M^{(1)}(x_i) = \prod_{\substack{j=1 \\ j \neq i}}^{m} (x_i - x_j) \tag{7}$$

and

$$c_{ij}^{(n)} = n\left(c_{ii}^{(n-1)} c_{ij}^{(1)} - \frac{c_{ij}^{(n-1)}}{x_i - x_j}\right), \qquad i,j = 1,2,\ldots,m, \quad \text{but } j \neq i \text{ and } n = 2,3,\ldots, \tag{8}$$

$$c_{ii}^{(n)} = -\sum_{\substack{j=1 \\ j \neq i}}^{m} c_{ij}^{(n)}, \qquad i = 1,2,\ldots,m. \tag{9}$$

Now, discretizing Eq. (4) at the grid point $x = x_i$ and substituting the values of first four derivatives of W from Eq. (5), we get

$$\sum_{j=1}^{m}\left(P_0 c_{ij}^{(4)} + P_{1,i} c_{ij}^{(3)} + P_{2,i} c_{ij}^{(2)} + P_{3,i} c_{ij}^{(1)}\right) W(x_j) + P_{4,i} W(x_i) = 0 \quad \text{for } i = 2,3,\ldots,(m-2). \tag{10}$$

The satisfaction of Eq. (10) at $(m-3)$ grid points x_i, $i = 2, 3, \ldots, (m-2)$ together with the regularity condition at the center provides a set of $(m-2)$ equations in terms of unknowns $W_j (\equiv W(x_j))$, $j = 1, 2, \ldots, m$. The

Table 2
Values of frequency parameter Ω for simply-supported plate for $\nu = 0.3$

Mode	α	β	η = −0.5			0			1		
			μ = −0.5	0	1	−0.5	0	1	−0.5	0	1
I	−0.5	0	3.4712	3.9408	5.0531	3.1222	3.5498	4.5644	2.4868	2.8360	3.6678
		0.5	4.3703	4.9682	6.4540	3.9284	4.4716	5.8231	3.1239	3.5653	4.6660
		−0.5	3.9578	4.4870	5.7368	3.5575	4.0392	5.1786	2.8303	3.2236	4.1571
	0	0	4.8363	5.4854	7.0874	4.3455	4.9351	6.3917	3.4540	3.9330	5.1187
		0.5	5.7008	6.5136	8.5969	5.1172	5.8537	7.7432	4.0582	4.6532	6.1828
		−0.5	5.3019	6.0043	7.7289	4.7624	5.4002	6.9676	3.7838	4.3019	5.5774
	0.5	0	6.1453	7.0035	9.1877	5.5152	6.2927	8.2732	4.3736	5.0018	6.6047
		0.5	7.0302	8.0765	10.8085	6.3033	7.2492	9.7218	4.9877	5.7486	7.7411
II	−0.5	0	21.1646	24.1560	31.3380	18.5522	21.2386	27.7240	14.1696	16.3177	21.5609
		0.5	26.5982	30.2135	38.7592	23.5495	26.8323	34.6339	18.3632	21.0515	27.5109
		−0.5	23.7449	27.1551	35.3799	20.8236	23.8870	31.3192	15.9085	18.3578	24.3697
	0	0	29.4088	33.4691	43.0951	26.0326	29.7200	38.5132	20.2762	23.2937	30.5755
		0.5	34.2642	38.8520	49.6169	30.5313	34.7290	44.6324	24.1079	27.5988	35.9295
		−0.5	32.1785	36.6866	47.4039	28.4755	32.5691	42.3620	22.1519	25.4982	33.6053
	0.5	0	37.1930	42.2449	54.1239	33.1209	37.7423	48.6734	26.1046	29.9426	39.1333
		0.5	41.7869	47.3208	60.2327	37.3972	42.4881	54.4319	29.7804	34.0603	44.2205
III	−0.5	0	53.3298	60.8744	78.8013	46.6827	53.4404	69.5828	35.5298	40.9040	53.8779
		0.5	67.0534	76.0816	97.2716	59.2161	67.3808	86.6427	45.8806	52.5035	68.2906
		−0.5	59.7779	68.2611	88.4161	52.3468	59.9533	78.1234	39.8575	45.9172	60.5504
	0	0	73.7511	83.7329	107.1722	65.1256	74.1561	95.4730	50.4382	57.7665	75.2496
		0.5	85.5072	96.7042	122.7873	75.9212	86.1120	109.9742	59.4550	67.8279	87.6393
		−0.5	80.3894	91.3164	116.9831	70.9833	80.8729	104.2267	54.9546	62.9843	82.1519
	0.5	0	92.3434	104.4969	132.8232	81.9714	93.0342	118.9535	64.1490	73.2383	94.7611
		0.5	103.0955	116.3255	146.9773	91.8815	103.9770	132.1453	72.4891	82.5168	106.1166

resulting system of equations can be written in the matrix form as

$$[B][W^*] = [0], \tag{11}$$

where B and W^* are matrices of order $(m-2) \times$ m and $m \times 1$, respectively.

The $(m-2)$ internal grid points chosen for collocation are the zeros of shifted Chebyshev polynomial of order $(m-2)$ with orthogonality range $(0, 1)$ given by

$$x_{k+1} = \frac{1}{2}\left[1 + \cos\left(\frac{2k-1}{m-2}\frac{\pi}{2}\right)\right], \quad k = 1, 2, \ldots, (m-2). \tag{12}$$

However, for a specified plate, the following three different sets of grid points have also been considered for a comparative study:

(i) Zeros of shifted Legendre polynomial $P_n^*(x)$ (Bellman et al. [16]), satisfying the differential equation

$$x(1-x)P''^*_n(x) + (1-2x)P'^*_n + n(n+1)P_n^*(x) = 0,$$

(ii) grid points taken by Liew et al. [21]

$$x_k = \frac{1}{2}\left[1 - \cos\frac{(k-1)\pi}{m-1}\right], \qquad k = 1, 2, \ldots, m, \tag{13}$$

(iii) equally spaced grid points (Bert et al. [17]).

Table 3
Values of frequency parameter Ω for free plate for $v = 0.3$

Mode	α	β	η = −0.5			0			1		
			μ = −0.5	0	1	−0.5	0	1	−0.5	0	1
I	−0.5	0	7.6512	8.5256	10.5669	6.5544	7.3210	9.1148	4.8080	5.3956	6.7773
		0.5	8.2370	9.2473	11.7358	7.1209	8.0104	10.2059	5.3145	6.0017	7.7045
		−0.5	8.8436	9.8303	12.1335	7.5871	8.4538	10.4805	5.5760	6.2423	7.8063
	0	0	9.2861	10.3962	13.1162	8.0258	9.0031	11.4019	5.9841	6.7388	8.5971
		0.5	10.1677	11.4956	14.8727	8.8302	10.0019	12.9879	6.6397	7.5474	9.8686
		−0.5	10.3456	11.5593	14.5223	8.9408	10.0093	12.6218	6.6624	7.4873	9.5095
	0.5	0	11.1358	12.5551	16.1444	9.6649	10.9165	14.0871	7.2577	8.2260	10.6863
		0.5	12.1778	13.8588	18.2126	10.6010	12.0854	15.9387	8.0002	9.1506	12.1479
II	−0.5	0	28.9596	32.9017	42.1928	25.1820	28.6948	37.0232	18.9596	21.7315	28.3819
		0.5	34.3938	38.8313	49.1903	30.2380	34.2393	43.6261	23.2871	26.5254	34.2015
		−0.5	33.2679	37.8344	48.5867	28.9266	33.0007	42.6575	21.7646	24.9838	32.7162
	0	0	38.5853	43.6084	55.3258	33.9100	38.4432	49.0768	26.0797	29.7511	38.4637
		0.5	43.5454	49.0296	61.7573	38.5366	43.5210	55.1422	30.0688	34.1688	43.8251
		−0.5	42.8138	48.4339	61.5353	37.6092	42.6843	54.5883	28.8839	32.9955	42.7632
	0.5	0	47.7623	53.8273	67.8908	42.2438	47.7593	60.6154	32.9048	37.4425	48.1396
		0.5	52.4791	58.9816	74.0091	46.6550	52.5980	66.3917	36.7318	41.6763	53.2666
III	−0.5	0	64.7299	73.7404	94.9875	56.4716	64.5147	83.5836	42.7249	49.0844	64.3256
		0.5	79.1946	89.5658	113.7101	69.7483	79.1053	100.9993	53.7865	61.3479	79.2246
		−0.5	73.4335	83.6969	107.8786	64.0616	73.2293	94.9522	48.4505	55.7064	73.0949
	0	0	87.8164	99.3786	126.2877	77.3162	87.7502	112.1630	59.5707	68.0039	87.9488
		0.5	100.2487	112.9478	142.2784	88.7759	100.3099	127.0838	69.2079	78.6545	100.8112
		−0.5	96.4352	109.1903	138.8659	84.8794	96.3928	123.3271	65.3466	74.6541	96.6718
	0.5	0	108.9059	122.7715	154.7914	96.4035	108.9980	138.2345	75.0806	85.3944	109.5943
		0.5	120.2682	135.1474	169.3227	106.9103	120.4877	151.8287	83.9766	95.2018	121.3824

4. Boundary conditions and frequency equations

By satisfying the relations:

(i) $W = \frac{dW}{dx} = 0$: for clamped edge,

(ii) $W = \frac{d^2W}{dx^2} + \frac{\upsilon}{x}\frac{dW}{dx} = 0$: for simply supported edge,

(iii) $\frac{d^2W}{dx^2} + \frac{\upsilon}{x}\frac{dW}{dx} = \frac{d^3W}{dx^3} + \frac{1}{x}\frac{d^2W}{dx^2} - \frac{1}{x^2}\frac{dW}{dx} = 0$: for free edge,

a set of two homogeneous equations in terms of W_j is obtained. These equations together with field Eq. (11) give a complete set of m equations in m unknowns.

For a clamped plate, the above set of homogeneous equations can be written as

$$\left[\frac{B}{B^c}\right][W^*] = [0], \tag{14}$$

where B^c is a matrix of order $2 \times m$.

For a non-trivial solution of Eq. (14), the frequency determinant must vanish and hence

$$\left|\frac{B}{B^c}\right| = 0. \tag{15}$$

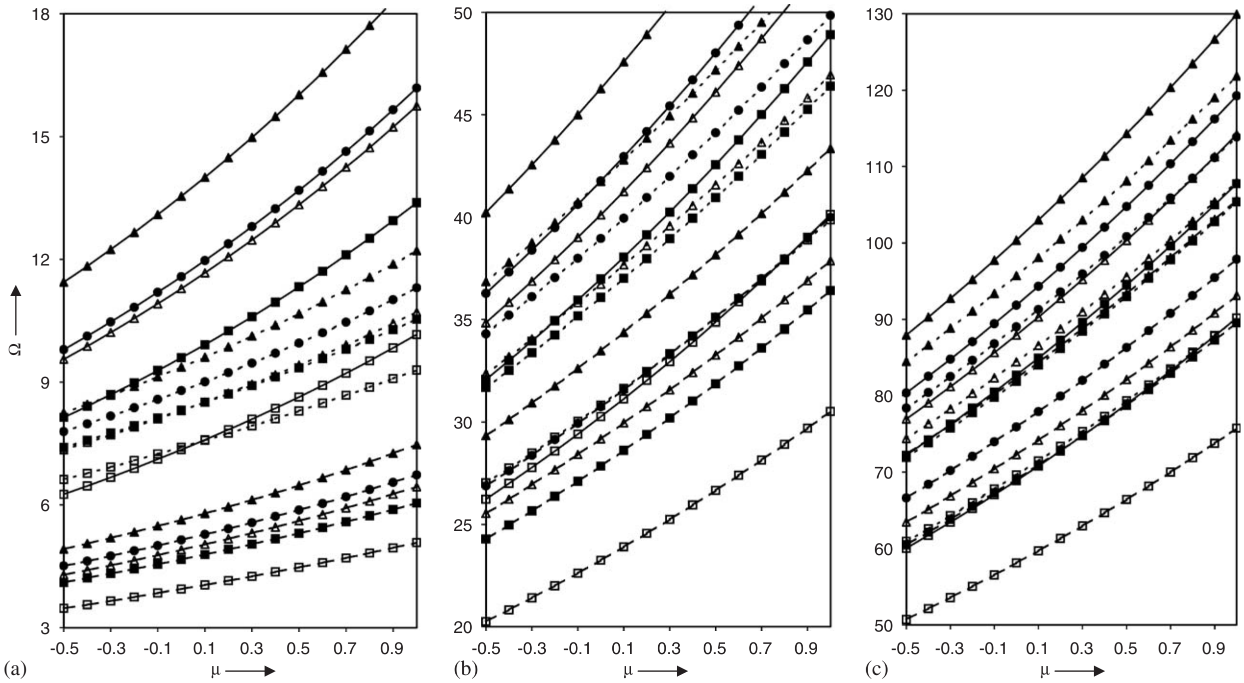

Fig. 2. Frequency parameter for clamped, simply supported and free plates vibrating in (a) fundamental mode (b) second mode (c) third mode for $\eta = 0.5$. ——, clamped; – – – – – –, simply supported; - - - - - - - - - - - -, free; □, $\alpha = 0$, $\beta = -0.3$; △, $\alpha = 0$, $\beta = 0.3$; ■, $\alpha = 0.3$, $\beta = -0.3$; ●, $\alpha = 0.3$, $\beta = 0$; ▲, $\alpha = 0.3$, $\beta = 0.3$.

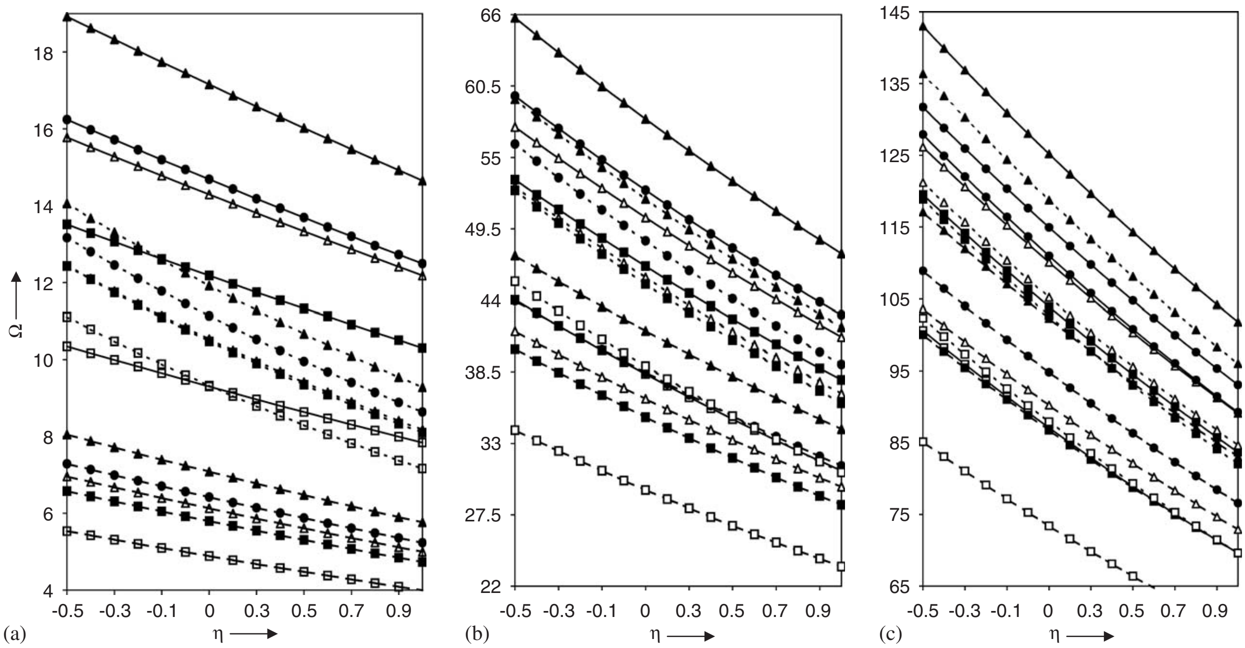

Fig. 3. Frequency parameter for clamped, simply supported and free plates vibrating in (a) fundamental mode (b) second mode (c) third mode for $\mu = 0.5$. ——, clamped; – – – – – –, simply supported; - - - - - - - - - - - -, free; □, $\alpha = 0$, $\beta = -0.3$; △, $\alpha = 0$, $\beta = 0.3$; ■, $\alpha = 0.3$, $\beta = -0.3$; ●, $\alpha = 0.3$, $\beta = 0$; ▲, $\alpha = 0.3$, $\beta = 0.3$.

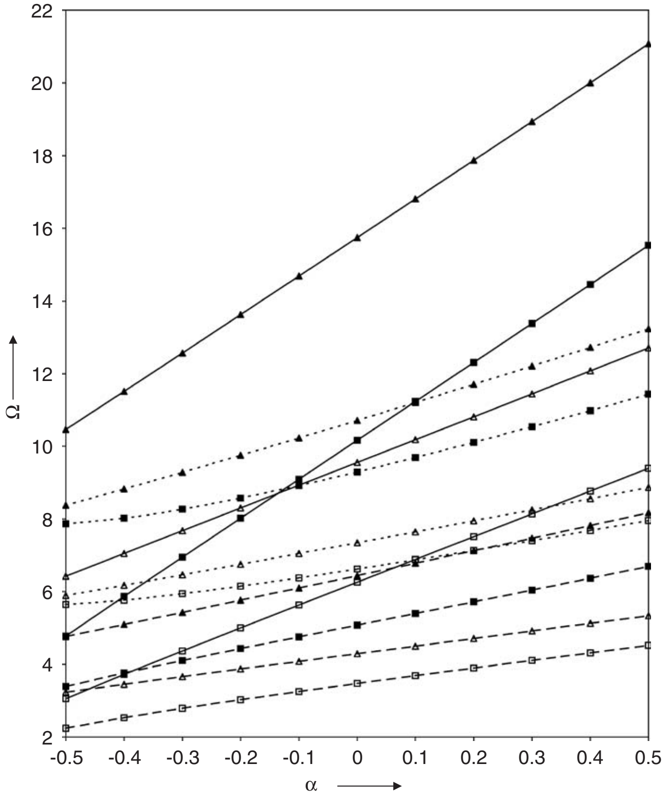

Fig. 4. Frequency parameter for clamped, simply supported and free plates vibrating in fundamental mode for $\eta = 0.5$. ——, clamped; – – – – – –, simply supported; - - - - - - - - - - - -, free; □, $\mu = -0.5$, $\beta = -0.3$; △, $\mu = -0.5$, $\beta = 0.3$; ■, $\mu = 1.0$, $\beta = -0.3$; ▲, $\mu = 1.0$, $\beta = 0.3$.

Similarly, for simply supported and free edge boundary conditions, the frequency determinants can be written as

$$\left|\frac{B}{B^s}\right| = 0, \qquad \left|\frac{B}{B^F}\right| = 0, \tag{16,17}$$

respectively.

5. Numerical results and discussion

The frequency Eqs. (15)–(17) provide the values of the frequency parameter Ω for various values of plate parameters. In the present work, the first three natural frequencies of vibration have been computed for all the three boundary conditions for non-homogeneity parameter $\mu = -0.5(0.1)1.0$; density parameter $\eta = -0.5(0.1)1.0$ and taper constants $\alpha = -0.5(0.1)0.5$; $\beta = -0.5(0.1)0.5$ (such that $\alpha + \beta > -1$) for $v = 0.3$.

To choose the appropriate number of grid points m, a computer program was developed and run for $m = 10(1)25$ for different sets of plate parameters for all the three boundary conditions. The numerical values showed a consistent improvement with the increase of the number of grid points. In all the computations, the number of grid points has been taken as $m = 18$, since further increase in m does not improve the results even in the fourth place of decimal (Fig. 1).

The numerical results are given in Tables 1–3 and Figs. 2–8. From the results, it is found that for $\alpha > 0$, $\beta > 0$, the frequency parameter for free plate is smaller than that of clamped plate and greater than that for simply supported plate whatever are the values of other plate parameters.

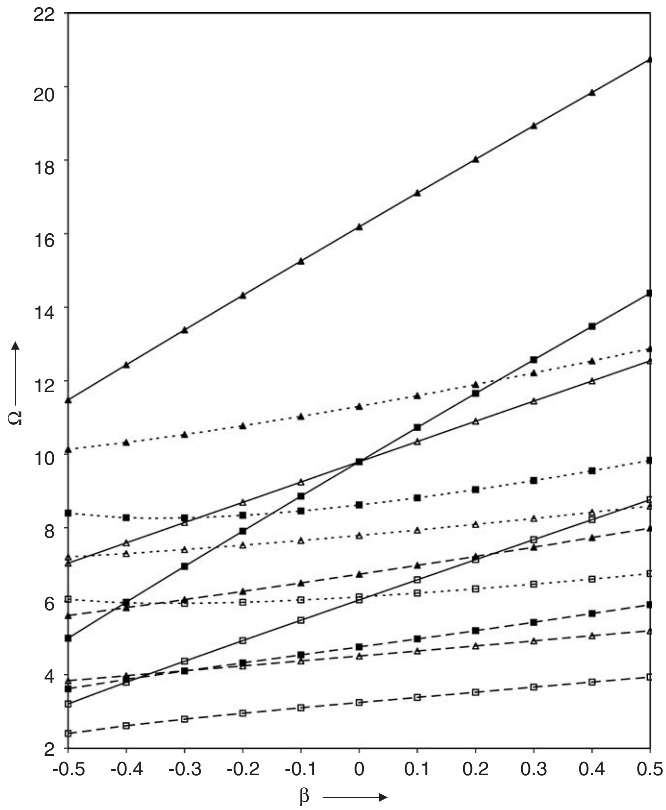

Fig. 5. Frequency parameter for clamped, simply supported and free plates vibrating in fundamental mode for $\eta = 0.5$. ——, clamped; – – – – – –, simply supported; - - - - - - - - - - - -, free; □, $\mu = -0.5$, $\alpha = -0.3$; △, $\mu = -0.5$, $\alpha = 0.3$; ■, $\mu = 1.0$, $\alpha = -0.3$; ▲, $\mu = 1.0$, $\alpha = 0.3$.

Fig. 2(a) shows the effect of non-homogeneity parameter μ on the frequency parameter Ω for $\eta = 0.5$, $\alpha = 0.0$, 0.3 and $\beta = 0.0$, ± 0.3 for all the three plates vibrating in fundamental mode. It is observed that frequency parameter increases with increasing value of non-homogeneity parameter μ for all the three cases. Also, the frequency parameter increases with increasing values of α or β or both for all the three plates. The increase is more pronounced in case of clamped plate as compared to simply supported and free plates. Fig. 2(b) shows the plots for Ω versus μ for the second mode of vibration. It is observed that the rate of increase of Ω in all the three cases is higher than that of the fundamental mode. A similar behavior can be seen from Fig. 2(c) when the plate is vibrating in third mode.

Fig. 3(a) depicts the variation of frequency parameter Ω with density parameter η for $\mu = 0.5$, $\alpha = 0.0$, 0.3 and $\beta = 0.0$, ± 0.3 for all the three plates vibrating in fundamental mode. It is observed that frequency decreases with the increasing value of density parameter η. The rate of decrease with increasing value of η is more pronounced in case of free plate as compared to clamped or simply supported plate, whatever are the values of other plate parameters. A similar inference was observed when the plate is vibrating in second and third modes (Figs. 3(b) and (c)).

Fig. 4 shows the effect of taper parameter α on frequency parameter Ω for $\mu = -0.5$, 1.0, $\eta = 0.5$ and $\beta = -0.3, 0.3$ for plates vibrating in fundamental mode. It is observed that frequency parameter increases with increasing value of taper parameter α. The rate of increase of Ω is higher for clamped plate as compared to those of simply supported and free plates. Further the frequency parameter can be increased/decreased by increasing/decreasing the value of β as well as μ. For second and third modes the behavior of Ω with α is the same as that for the fundamental mode except for the fact that the rate of increase gets more pronounced with increase in number of modes.

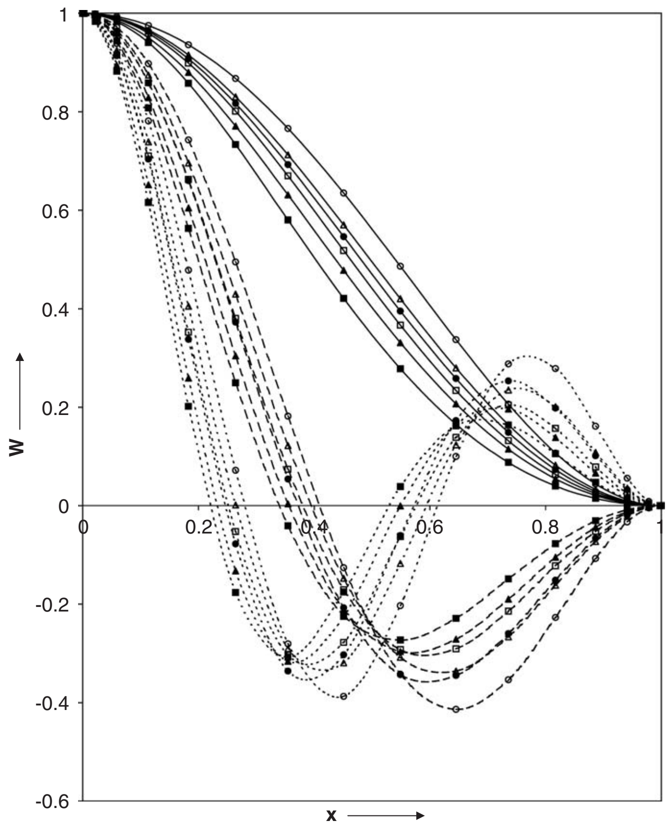

Fig. 6. Normalized displacements for the first three modes of vibration for clamped plate for $\eta = 0.5$. ——, fundamental mode; – – – – – –, second mode; - - - - - - - - - - - -, third mode. $\mu = -0.5$: □, $\alpha = 0.5$, $\beta = 0.5$; △, $\alpha = 0.5$, $\beta = 0$; ○, $\alpha = 0$, $\beta = 0$. $\mu = 1.0$: ■, $\alpha = 0.5$, $\beta = 0.5$; ▲, $\alpha = 0.5$, $\beta = 0$; ●, $\alpha = 0$, $\beta = 0$.

Fig. 5 shows the plot of frequency parameter Ω versus taper parameter β for $\mu = -0.5$, 1.0, $\eta = 0.5$ and taper parameter $\alpha = -0.3$, 0.3 for plates vibrating in fundamental mode. It is found that frequency parameter increases with increasing value of taper parameter β except in case of free plate for $\alpha = -0.3$. In this case, there appears a local minima in the vicinity of $\beta = -0.3$. This may be attributed to the increased mass of the plate towards the center. However, from the results for the second and third modes, the frequency parameter Ω is found to increase continuously with the increasing value of β. The rate of increase of Ω with increasing values of β is higher for clamped plate as compared to simply supported and free plates.

Figs. 6–8 show the plots for normalized transverse displacements for $\mu = -0.5$, 1.0, $\eta = 0.5$, $\alpha = 0.0$, $\beta = 0.0$; $\alpha = 0.5$, $\beta = 0.0$ and $\alpha = 0.5$, $\beta = 0.5$ for the first three modes of vibration for clamped, simply supported and free plates, respectively. The radii of nodal circles decrease as the outer edge becomes thicker and thicker for all three boundary conditions. The effect of non-homogeneity μ also decreases the radii of nodal circles.

Table 4 shows a comparison of results for homogeneous ($\mu = 0.0$, $\eta = 0.0$) circular plate of uniform thickness ($\alpha = 0.0$, $\beta = 0.0$) with those of exact solutions given by Leissa [31] and approximate solutions obtained by Ritz method [32] and receptence method [33]. Table 5 shows a comparison of results for homogeneous circular plate of linearly varying thickness with results obtained by Frobenius method [1] and by Rayleigh–Ritz method [6,34] for clamped and simply supported plate. A comparison of results for homogeneous circular plate of parabolically varying thickness with those obtained by Frobenius method [1], Rayleigh–Ritz method [6] and Ritz method [32] is presented in Table 6.

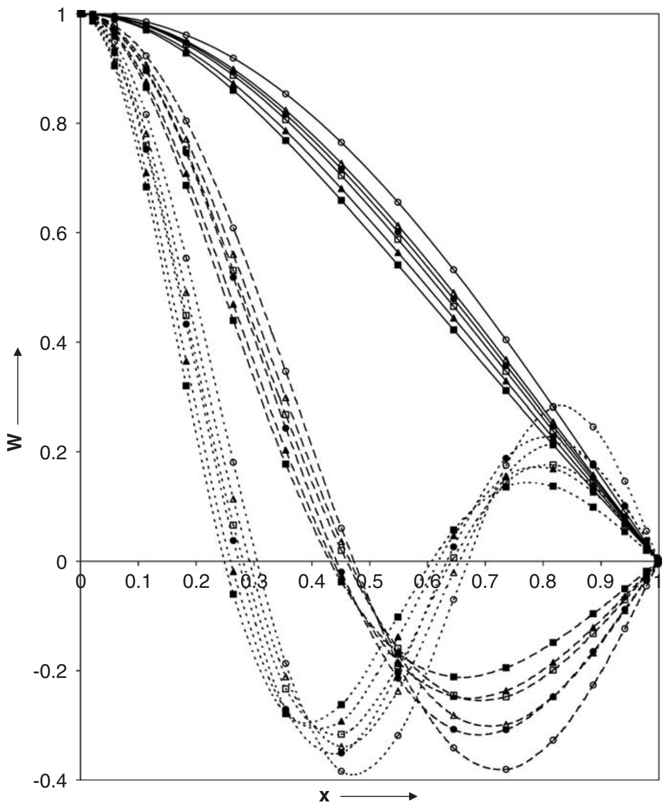

Fig. 7. Normalized displacements for the first three modes of vibration for simply supported plate for $\eta = 0.5$. ——, fundamental mode; – – – – – –, second mode; - - - - - - - - - - - -, third mode. $\mu = -0.5$: □, $\alpha = 0.5$, $\beta = 0.5$; △, $\alpha = 0.5$, $\beta = 0$; ○, $\alpha = 0$, $\beta = 0$. $\mu = 1.0$: ■, $\alpha = 0.5$, $\beta = 0.5$; ▲, $\alpha = 0.5$, $\beta = 0$; ●, $\alpha = 0$, $\beta = 0$.

A comparative study for evaluation of frequency parameter Ω for a specified plate for the first three modes of vibration has been presented in Table 7 by taking equally spaced and three unequally spaced grid points i.e. zeros of shifted Chebyshev polynomials obtained from Eqs. (12) and (13) and also that of shifted Legendre polynomials. During numerical computation, it is found that for uniform grid spacing the number of grid points is considerably greater as compared to non-uniform grid spacing. It is worth noting that in case of uniform grid points the results converge with the increasing value of m up to a certain extent and after that results become unstable which is due to round-off errors.

It is observed that the number of grid points taken as zeros of shifted Chebyshev polynomials (used in the present investigation) are found not to exceed the number of grid points as taken by Liew et al. [21] and Bellman et al. [16]. Thus, the present choice of grid points not only provides a comparatively faster rate of convergence but also leads to reliable results. After verifying the convergence trends and the accuracy of results available in the literature by the present DQ method (Tables 4, 5 and 6) the new results for non-homogeneous circular plates of linear, parabolic and quadratic variation in thickness can be used as benchmark for future researches.

6. Conclusion

The DQM has been applied to study the effect of the non-homogeneity of the material on the natural frequencies of circular plates of quadratically varying thickness on the basis of classical plate theory. It is observed that in case of plate for which thickness increases towards the outer edge, the frequency parameter

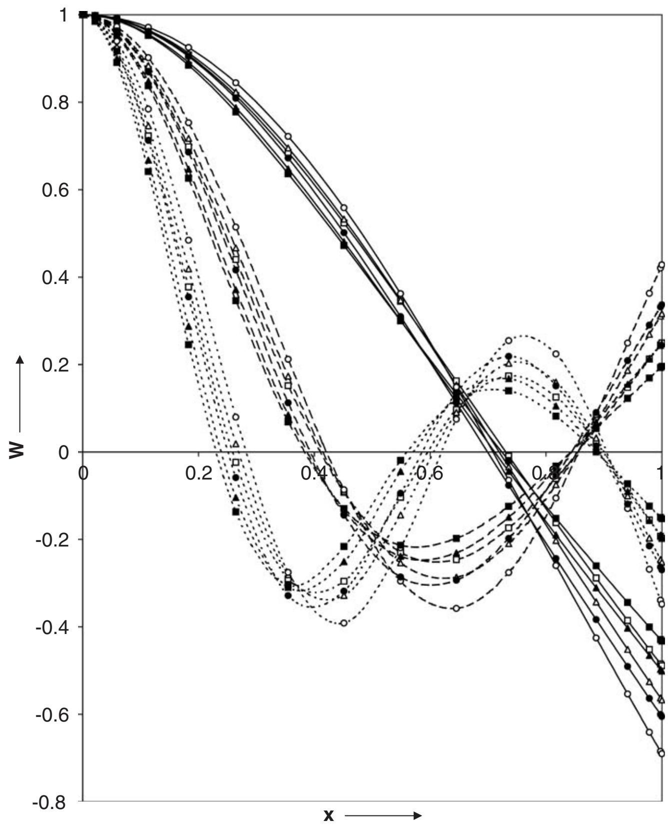

Fig. 8. Normalized displacements for the first three modes of vibration for free plate for $\eta = 0.5$. ——, fundamental mode; – – – – – –, second mode; - - - - - - - - - - - -, third mode. $\mu = -0.5$: □, $\alpha = 0.5$, $\beta = 0.5$; △, $\alpha = 0.5$, $\beta = 0$; ○, $\alpha = 0$, $\beta = 0$. $\mu = 1.0$: ■, $\alpha = 0.5$, $\beta = 0.5$; ▲, $\alpha = 0.5$, $\beta = 0$; ●, $\alpha = 0$, $\beta = 0$.

Table 4
Comparison of frequency parameter Ω for homogeneous ($\mu = 0.0$, $\eta = 0.0$) circular plate of uniform thickness ($\alpha = 0.0$, $\beta = 0.0$)

Mode	$v = 0.3$				$v = 0.33$	
	Clamped plate		S-S plate		Free plate	
I	10.2158	10.2158[a]	4.9351	4.977[a]	9.0689	9.084[a]
	10.2158[b]	10.216[c]	4.9352[b]	4.935[c]		
II	39.7711	39.771[a]	29.7200	29.76[a]	38.507	38.55[a]
	39.7711[b]	39.771[c]	29.7200[b]	29.720[c]		
III	89.1041	89.104[a]	74.1561	74.20[a]	87.8127	87.80[a]
	89.1041[b]	89.103[c]	74.1961[b]	74.156[c]		

[a]Values taken from Ref. [31].
[b]Values taken from Ref. [32] by Ritz method.
[c]Values taken from Ref. [33] by receptence method.

for linear thickness variation (LTV) is higher than that for parabolic thickness variation (PTV) and smaller than that for quadratic thickness variation (QTV). However, in case of plate for which thickness decreases towards the outer edge, the frequency parameter for LTV is smaller than that for PTV and higher than that for

Table 5
Comparison of frequency parameter Ω for homogeneous ($\mu = 0.0$, $\eta = 0.0$) circular plate of linear thickness variation ($\beta = 0.0$)

α	Clamped						S-S					
	I		II		III		I		II		III	
−0.5	6.1504 6.1504[b]	6.1522[a]	27.3002 27.300[b]	27.3006[a]	63.0611 63.062[b]	63.0605[a]	3.5498 3.5498[b]	3.5507[a]	21.2386 21.239[b]	21.2419[a]	53.4404 53.441[b]	53.4095[a]
−0.3	7.7783 7.7783[b]	7.7769[a] 7.778[c]	32.4610 32.461[b]	32.4586[a] 32.463[c]	73.9467 73.947[b]	73.9586[a]	4.1158 4.1158[b]	4.1154[a] 4.116[c]	24.7265 24.727[b]	24.7268[a] 24.728[c]	62.0704 62.071[b]	62.0732[a]
−0.1	9.4027 9.4027[b]	9.4016[a] 9.402[c]	37.3763 37.376[b]	37.3742[a] 37.376[c]	84.1680 84.168[b]	84.1188[a]	4.6637 4.6637[b]	4.6627[a] 4.664[c]	28.0774 28.077[b]	28.0765[a] 28.078[c]	70.2127 70.213[b]	70.2104[a]
0.1	11.0301 11.030[b]	11.0297[a] 11.03[c]	42.1337 42.134[b]	42.1408[a] 42.133[c]	93.9486 93.949[b]	93.9014[a]	5.2061 5.2061[b]	5.2065[a] 5.206[c]	31.3465 31.346[b]	31.3467[a] 31.346[c]	78.0323 78.032[b]	78.0254[a]
0.3	12.6631 12.663[b]	12.6648[a] 12.663[c]	46.7813 46.782[b]	46.7965[a] 46.784[c]	103.4123 103.41[b]	103.8434[a]	5.7483 5.7483[b]	5.7469[a] 5.748[c]	34.5625 34.563[b]	34.5613[a] 34.564[c]	85.6205 85.623[b]	85.5148[a]
0.5	14.3021 14.302[b]	14.3033[a]	51.3480 51.349[b]	51.3588[a]	112.6360 112.64[b]	112.4586[a]	6.2927 6.2928[b]	6.2908[a]	37.7423 37.743[b]	37.7414[a]	93.0342 93.042[b]	92.7375[a]

[a]Values taken from Ref. [1] by Frobenius method.
[b]Values taken from Ref. [34] by Ritz method.
[c]Values taken from Ref. [6] by Rayleigh–Ritz method.

Table 6
Comparison of frequency parameter Ω for homogeneous ($\mu = 0.0$, $\eta = 0.0$) circular plate of parabolic thickness variation ($\alpha = 0.0$)

β	Clamped						S-S					
	I		II		III		I		II		III	
−0.5	6.6320	6.6303[a]	30.0152	30.0130[a]	69.8624	69.8709[a]	4.0392	4.0391[a]	23.887	23.8884[a]	59.9533	59.9567[a]
−0.3	8.0759 8.0759[b]	8.0748[a] 8.076[c]	34.161 34.1610[b]	34.1768[a] 34.161[c]	78.1241	78.1086[a]	4.4034 4.4034[b]	4.4029[a] 4.403[c]	26.3765 26.3765[b]	26.3757[a] 26.376[c]	66.0394	66.0258[a]
−0.1	9.5055 9.5055[b]	9.5055[a] 9.505[c]	37.9627 37.9627[b]	37.9631[a] 37.963[c]	85.5877	85.5598[a]	4.7576 4.7576[b]	4.7562[a] 4.758[c]	28.6437 28.6437[b]	28.6447[a] 28.644[c]	71.5534	71.5579[a]
0.1	10.9235 10.9235[b]	10.9223[a] 10.924[c]	41.5301 41.5301[b]	41.5380[a] 41.529[c]	92.505	92.5123[a]	5.1142 5.1142[b]	5.1130[a] 5.114[c]	30.7664 30.7664[b]	30.7682[a] 30.768[c]	76.6758	76.6675[a]
0.3	12.3317 12.3317[b]	12.3287[a] 12.332[c]	44.9242 44.9242[b]	44.9329[a] 44.921[c]	99.0172	99.2534[a]	5.4787 5.4787[b]	5.4802[a] 5.479[c]	32.7863 32.7863[b]	32.7877[a] 32.786[c]	81.5074	81.5172[a]
0.5	13.731	13.7317[a]	48.1833	48.1822[a]	105.2133	—	5.8537	5.8509[a]	34.729	34.7138[a]	86.112	—

[a]Values taken from Ref. [1] by Frobenius method.
[b]Values taken from Ref. [32] by Ritz method.
[c]Values taken from Ref. [6] by Rayleigh–Ritz method.

QTV. The frequency parameter increases with increasing values of non-homogeneity parameter μ, while it decreases with increasing values of density parameter η. Thus, a desired frequency can be achieved by altering one or more plate parameters: non-homogeneity parameter μ, density parameter η and taper parameters α, β. The accuracy of the approach has been verified by demonstrating a close agreement of our results with those of exact solutions and obtained by various techniques: Frobenius method, Ritz method, Rayleigh–Ritz method, receptence method.

Table 7
Number of grid points for convergence of frequency parameter Ω by using zeros of Chebyshev polynomial, Legendre polynomial and equidistant collocation points for clamped plate for $\eta = 0.5$, $\mu = -0.5$

Nature of grid points	$\alpha = -0.5$, $\beta = 0.5$			$\alpha = 0.0$, $\beta = 0.5$			$\alpha = 0.5$, $\beta = -0.5$			$\alpha = 0.5$, $\beta = 0.5$		
	Mode											
	I	II	III	I	II	III	I	II	III	I	II	III
	Ω											
	7.5131	28.4203	63.3102	10.6470	37.4630	81.9673	8.2884	32.9813	74.4059	13.8119	46.2540	99.7549
Zeros of Chebyshev polynomial	13	15	17	11	13	16	11	12	14	11	12	15
Zeros of Legendre polynomial	13	15	17	11	14	17	11	14	16	11	13	16
Equidistant	17	22	24	16	20	23	16	19	22	13	18	21
Ref. [21]	13	14	18	12	15	18	12	15	16	12	15	16

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