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Energy flow relations from quadratic quantities in three-dimensional isotropic medium and exact formulation for one-dimensional waves

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Abstract

From the basic equations of continuum mechanics in a three-dimensional (3D) isotropic and damped medium excited by harmonic body forces, exact expressions are obtained from quadratic variables for time-averaged energy quantities: kinetic- and strain-energy densities, structural intensity, structural intensity divergence and curl. These energy quantities are split into four components: longitudinal, shear and two mixed ones. Each component is governed by similar relations of different quadratic variables. For 1D wave fields, an exact formulation based on quadratic variables is derived. The fundamental solutions of this formulation are analyzed for unloaded, and for concentrated loaded systems. The energy-models reported in the literature consider only some components of these solutions. The energy density and structural intensity components obtained from the quadratic formulation and from the usual displacement formulation are equivalent; this is illustrated for the energy transfers modeled by the quadratic formulation, in comparison with the displacement formulation, for a one-dimensional, longitudinal- and shear-wave field with wave conversion at one end.

1. Introduction

It is well known that the prediction of vibration behavior of structures is generally difficult, due to the complexity of a structure consisting of numerous connected elements and the large frequency range of interest. The deterministic methods based on displacement, like Finite Element Modeling or modal formulations, are efficient approaches in the low frequency range, but have severe limitations in the high frequency range. Short wavelengths lead to too fine a mesh or to too many modes, only some of which can be realistically determined due to imprecise boundary conditions. It quickly appeared that a deterministic description based on the analytical solution of the displacement of the structure elements and their interconnection would be a very difficult problem to solve. Skudrzyk [1,2] was one of the first to present an approximate description of the dynamic response of complex vibrators. At high frequencies, obtaining accurate results by large computational models of complex systems is unrealistic when too many modes participate in the response.

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Nomenclature

c_0 adiabatic speed of sound
c_{im} constants (quadratic formulation)
$\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ unit vectors of the Cartesian coordinate system
$\mathbf{f}$ external force
i, m, n integers
$\mathrm{j} = \sqrt{-1}$
k wavenumber (waves propagating along $\pm x$)
s external load quadratic term
t time
$\mathbf{u}$ displacement
$\mathbf{v}$ velocity
x, y, z Cartesian coordinate system
C_i constants (displacement formulation)
E Young modulus
$\mathbf{I}$ structural intensity
L Lagrangian density
L_1, L_2 length of 1D system
Q quadratic variable
T kinetic energy density
U strain energy density
W total energy density
γ_l, $\boldsymbol{\gamma}_t$ potentials of external force (respectively, scalar- and vector-)
δ_{mn} Kronecker symbol
$\delta(x)$ Dirac
$\boldsymbol{\varepsilon}$ strain tensor
η damping ratio
θ angle
λ, μ Lamé constants
ν Poisson coefficient
ξ general variable
ρ density
$\boldsymbol{\sigma}$ stress tensor
ϕ longitudinal scalar potential
$\boldsymbol{\Psi}$ shear vector potential
ω angular frequency

Functions, operators, and general symbols

$H(x) = \begin{cases} 0 & x<0 \\ 1/2 & x=0 \\ 1 & x>0 \end{cases}$ Heaviside step function

$\delta_{mn} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$ Kronecker symbol

ε_{imn} permutation symbol, $\varepsilon_{imn} = (i-m)(m-n)(n-i)/2$ (without subscript summation)
$\mathbf{1}$ unit tensor, $(1)_{mn} = \delta_{mn}$
ξ scalar
$\boldsymbol{\xi}$ vector (components ξ_i), tensor (components ξ_{mn})
$\mathbf{a} = \boldsymbol{\xi}\mathbf{q}$ tensor-vector product, $a_i = \xi_{im}q_m$
$\mathbf{a} = \boldsymbol{\xi}\boldsymbol{\zeta}$ tensor product, $a_{mn} = \xi_{mi}\zeta_{in}$
$\mathrm{a} = \boldsymbol{\xi} \cdot \mathbf{q}$ scalar product, $a = \xi_i q_i$
$\mathbf{a} = \boldsymbol{\xi} \times \mathbf{q}$ vector product, $\boldsymbol{a}_i = \varepsilon_{imn}\xi_m q_n$
$a = \underline{\underline{\boldsymbol{\xi}}} : \underline{\underline{\boldsymbol{\zeta}}}$ contracted tensor product, $a = \xi_{im}\zeta_{mi}$
$u_{m,n} = \partial u_m/\partial x_n$, $\xi_{,x} = \partial \xi/\partial x$ partial derivative
$\nabla^2 \xi$ scalar Laplacian, $\nabla^2 \xi = \operatorname{div} \operatorname{grad} \xi$
$\nabla^2 \boldsymbol{\xi}$ vector Laplacian, $\nabla^2 \boldsymbol{\xi} = \operatorname{div} \operatorname{grad} \boldsymbol{\xi} = \operatorname{grad} \operatorname{div} \boldsymbol{\xi} - \operatorname{curl} \operatorname{curl} \boldsymbol{\xi}$
$\operatorname{div} \boldsymbol{\xi}$ tensor divergence, $(\operatorname{div} \boldsymbol{\xi})_i = \partial \xi_{im}/\partial x_m$
$\operatorname{grad} \mathbf{q}$ vector gradient, $(\operatorname{grad} \mathbf{q})_{mn} = \partial q_m/\partial x_n$
$\arg(\xi)$ complex argument
$\mathrm{Re}(\xi)$ real part
$\mathrm{Im}(\xi)$ imaginary part
ξ^* complex conjugate

Subscript symbols

l longitudinal component
t shear (transverse) component
$\xi|_{x_0}$, $\xi|_{0-}$, $\xi|_{0+}$ value of ξ at $x = x_0$, left-, right-limit at $x = 0$

Statistical Energy Analysis (SEA) can give some responses "in the statistical sense" to the dynamic problem at high frequencies for complex systems. But SEA is not a reliable predictive approach (a lot of complementary information has to be obtained by experiment or additional modeling) and the resulting information consists only in averaged quantities describing the behavior on a population of modes of the sub-systems. With increasing interest in the measurement of structural intensity and the examination of experimental results, structural intensity began to be considered in the 1980s as a tool to represent the dynamic behavior of a structure. Other attempts have since been made to derive expressions for energy conservation and power transfer in a local form, where energy variables are considered (as in SEA) and a local formulation is used (as in the Finite Element Method based on the displacement formulation). After the early papers of Belov, Rybak, Tartakovski [3], and Buvailo and Ionov [4], the good results in one-dimensional (1D) systems obtained by Nefske and Sung [5] and by Wohlever and Bernhard [6] motivated Ichchou, Le Bot and Jezequel [7] to work out a more general formulation. Different energy models have been developed, dedicated to the medium frequency range and known as the "conductivity approach", the "thermal analogy" or the "heat transfer model", based on different assumptions like light damping or plane wave superposition with interference neglected. These models assume that there exists a relationship between mechanical energy exchange and vibrational dynamics, in the form of the structural intensity expressed as the gradient of energy density variables. The common point of the previous different works is the approximate energy conservation equation determined from various procedures. However, the extension of these methods to 2D systems fails [8–11]. The main reason for this failure is probably the assumption that the interference between elementary wave components can be neglected. However, this assumption is particularly important because it enables the use of a simple relation, which exists for an elementary wave, between its energy flow and the amplitude of the vibration field (represented by the kinetic energy, or sometimes by the total energy).

Based on the observation that the heat equation (analogue with an equation of thermal conductivity) is not as yet capable of providing the basic model to predict the dynamics of the complex structures, the present work proposes to present exact relationships linking the time-averaged energy variables (Section 2), and more generally quadratic variables, for the following general assumptions:

(a1) small displacement and small strain without stress stiffening,
(a2) steady-state harmonic waves,
(a3) hysteretic damping material,
(a4) homogeneous and isotropic medium.

After the wave field is decomposed into longitudinal and shear components, the number of variables and equations is discussed, with the objective of deriving closed equation sets, i.e. an exact energy- or quadratic-formulation dedicated to a wide frequency range (Section 3). As a consequence of the simplifications occurring in this configuration, an exact quadratic formulation is only derived for 1D systems excited by distributed loads (Section 4). The solutions of this formulation, for free waves produced by exciting sources, are analyzed in terms of wavenumbers, and compared to the solutions used in the energy models reported in the literature. The equivalence of the 1D quadratic formulation and the displacement formulation is illustrated in the case of waves excited by concentrated loads with wave conversion at one end (Section 5).

2. Fundamental equations

2.1. Linear and quadratic variables

Time harmonic wave fields of angular frequency ω are considered in a medium of density ρ. The standard complex notations are used for the displacement ($\mathbf{u}$) and the external force ($\mathbf{f}$) vectors, the strain ($\boldsymbol{\varepsilon}$) and stress ($\boldsymbol{\sigma}$) tensors, where the time dependence $e^{j\omega t}$ and the real part are omitted. From assumptions (a1–a2), the time derivative is $d/dt \approx \partial/\partial t = j\omega$ where $j = \sqrt{-1}$, and the complex velocity $\mathbf{v}$ and the strain tensor components are, respectively

$$\mathbf{v} = j\omega\mathbf{u}, \quad \varepsilon_{kl} = (u_{k,l} + u_{l,k})/2. \tag{1}$$

The equation of momentum conservation can be written

$$-\rho\omega^2\mathbf{u} = \mathbf{f} + \operatorname{div}\boldsymbol{\sigma}. \tag{2}$$

Energy quantities are proportional to the product of two of the above harmonic variables. In this sense they are quadratic variables, and present a harmonic component at angular frequency 2ω, and a steady component. Energy and quadratic quantities are hereafter denoted by complex variables, expressed as the products of linear complex and a linear complex conjugate terms [12]:

$$\mathbf{I} = \mathrm{j}\omega\boldsymbol{\sigma}\mathbf{u}^*/2, \tag{3a}$$

$$T = \rho\omega^2\mathbf{u}\cdot\mathbf{u}^*/4, \tag{3b}$$

$$U = \boldsymbol{\sigma} : \boldsymbol{\varepsilon}^*/4, \tag{3c}$$

where the asterisk implies the complex conjugate. The real part of each of the complex quadratic variables (3) is the time-averaged value of the instantaneous energy variable. The real and imaginary parts of structural intensity $\mathbf{I}$ represent the active and reactive time-averaged power flux density; kinetic energy density is real and positive, and the strain energy density is real and positive for non dissipative materials. The divergence of time-averaged structural intensity can be expressed using the Lagrangian density $L = T - U$ and the power of external loads [13], in a similar manner to that used in acoustics [14]

$$\operatorname{div}\mathbf{I} = -2\mathrm{j}\omega(T-U) - \frac{\mathrm{j}\omega}{2}\,\mathbf{f}\cdot\mathbf{u}^*. \tag{4}$$

This relationship for the divergence of structural intensity expresses the local energy conservation: for non dissipative materials, the real part is the active power developed by external loads (real part of the last term), and both the reactive power of external loads and the Lagrangian density contribute to the local reactive transfers.

2.2. Fundamental energy equations in isotropic and hysteretic damping material

According to the linear theory of elasticity, dissipative materials (a3) can be considered by defining complex Lamé coefficients (a4), expressed as

$$\lambda = \lambda_R(1+\mathrm{j}\eta_\lambda), \quad \mu = \mu_R(1+\mathrm{j}\eta_\mu), \tag{5}$$

where λ_R and μ_R are the elastic Lamé constants and η_λ and η_μ their loss factor, all of them real. Damping ratios for both coefficients are identical in materials presenting real Poisson coefficients $v = \lambda/(2(\lambda+\mu))$. This equality for loss factors $\eta_\lambda \approx \eta_\mu$ is a good approximation for isotropic materials [15]. In isotropic materials, the stress tensor and its divergence are expressed as

$$\boldsymbol{\sigma} = \lambda \operatorname{div}\mathbf{u}\mathbf{1} + 2\mu\boldsymbol{\varepsilon}, \quad \operatorname{div}\boldsymbol{\sigma} = (\lambda+2\mu)\operatorname{grad}\operatorname{div}\mathbf{u} - \mu\operatorname{curl}\operatorname{curl}\mathbf{u}.$$

This expression for the stress tensor divergence gives the displacement formulation for Eq. (2)

$$\rho\omega^2\mathbf{u} + (\lambda+2\mu)\operatorname{grad}\operatorname{div}\mathbf{u} - \mu\operatorname{curl}\operatorname{curl}\mathbf{u} + \mathbf{f} = \mathbf{0}, \tag{6}$$

and the following form for energy variables (3)

$$\mathbf{I} = \mathrm{j}\omega(\lambda\operatorname{div}\mathbf{u}\mathbf{u}^* + 2\mu\boldsymbol{\varepsilon}\mathbf{u}^*)/2, \tag{7a}$$

$$T = \rho\omega^2\mathbf{u}\cdot\mathbf{u}^*/4, \tag{7b}$$

$$U = (\lambda\operatorname{div}\mathbf{u}\operatorname{div}\mathbf{u}^* + 2\mu\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon}^*)/4. \tag{7c}$$

The goal of this work is to obtain, if possible for general assumptions, an energy- or quadratic-formulation for wave fields, i.e. a closed equation set for energy variables (this is reached only in the case of 1D systems, in Section 4). The structural intensity divergence (4) can be written from energy densities and injected power. A self-contained equation set would be obtained if structural intensity could be derived from energy densities. But such an expression cannot be obtained, unless a very special case is considered [6]. In order to obtain a linear closed equation set, we use not only energy variables, but also other quadratic variables. From these extended variables, energy quantities can be linearly expressed: using identity (32a)–(32b) of Appendix A, structural intensity and energy densities are expressed from quadratic variables as

$$\mathbf{I} = \frac{\mathrm{j}\omega}{2}((\lambda+\mu)\mathrm{div}\,\mathbf{u}\,\mathbf{u}^* - \mu\,\mathrm{div}\,\mathbf{u}^*\mathbf{u} + \mu(\mathrm{curl}\,\mathbf{u}^* \times \mathbf{u}) + \mu\,\mathrm{grad}(\mathbf{u}\cdot\mathbf{u}^*) + \mu\,\mathrm{curl}(\mathbf{u}\times\mathbf{u}^*)), \tag{8a}$$

$$T = \rho\omega^2\mathbf{u}\cdot\mathbf{u}^*/4, \tag{8b}$$

$$U = \frac{1}{4}(\lambda\,\mathrm{div}\,\mathbf{u}\,\mathrm{div}\,\mathbf{u}^* + \mu(\nabla^2(\mathbf{u}\cdot\mathbf{u}^*) - \mathbf{u}\cdot\nabla^2\mathbf{u}^* - \nabla^2\mathbf{u}\cdot\mathbf{u}^* - \mathrm{curl}\,\mathbf{u}\cdot\mathrm{curl}\,\mathbf{u}^*)). \tag{8c}$$

The disadvantage in the use of these general quadratic variables is that some of them have no clear physical meaning for energy transfers. From a mathematical point of view, structural intensity $\mathbf{I}$ (8a) cannot be expressed as a simple gradient of energy densities: when the fourth term $(\mathrm{j}\omega/2)\mu\,\mathrm{grad}(\mathbf{u}\cdot\mathbf{u}^*)$ is proportional to the kinetic energy density gradient (but contributes to the reactive intensity for real μ), the last term is derived from a vector potential, and the first three ones cannot be easily expressed from scalar and vector potentials.

The intensity field vector is mathematically characterized by its divergence and curl:

$$\begin{aligned}\mathrm{div}\,\mathbf{I} = \frac{\mathrm{j}\omega}{2}&(\lambda\,\mathrm{div}\,\mathbf{u}\,\mathrm{div}\,\mathbf{u}^* - \mu\,\mathrm{curl}\,\mathbf{u}\cdot\mathrm{curl}\,\mathbf{u}^* + \mu\nabla^2(\mathbf{u}\cdot\mathbf{u}^*) + (\lambda+\mu)(\mathrm{grad}\,\mathrm{div}\,\mathbf{u})\cdot\mathbf{u}^*\\ &- \mu(\mathrm{grad}\,\mathrm{div}\,\mathbf{u}^*)\cdot\mathbf{u} + \mu\,\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}^*\cdot\mathbf{u}),\end{aligned} \tag{9a}$$

$$\begin{aligned}\mathrm{curl}\,\mathbf{I} = \frac{\mathrm{j}\omega}{2}&(\lambda\,\mathrm{div}\,\mathbf{u}\,\mathrm{curl}\,\mathbf{u}^* + \mu(\mathrm{div}\,\mathbf{u}\,\mathrm{curl}\,\mathbf{u}^* - \mathrm{div}\,\mathbf{u}^*\,\mathrm{curl}\,\mathbf{u}) + \mu\,\mathrm{curl}\,\mathrm{curl}(\mathbf{u}\times\mathbf{u}^*)\\ &+ (\lambda+\mu)\mathrm{grad}\,\mathrm{div}\,\mathbf{u}\times\mathbf{u}^* - \mu\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}^*\times\mathbf{u} + \mu\,\mathrm{curl}(\mathrm{curl}\,\mathbf{u}^*\times\mathbf{u})).\end{aligned} \tag{9b}$$

The divergence (9a) expresses, as mentioned for Eq. (4), the local power conservation for longitudinal components, and the intensity curl (9b) accounts for the local vorticity of the power density field. These derivatives of structural intensity are used in the next sections as they present simpler expressions and less variables than the intensity vector itself. Since Eqs. (8)–(9) involve many different quadratic variables, it is convenient to introduce the longitudinal and shear decomposition for the displacement field, giving rise to simpler expressions for the corresponding energy variables.

3. Longitudinal, shear and mixed components

3.1. Longitudinal and shear displacement decomposition

The linear expressions (8)–(9) for energy variables involve many different variables for only a few relationships. In the present section, quadratic variables are split into different components. The number of variables considered is thus increased, but the mathematical properties satisfied by each of these components enables the production of a larger number of simpler relationships.

Simple scalar variables, such as energy densities, consist in condensed data, and are not suited to describe in detail the complexity of the vectorial energy flow field [16]. Working with different components of structural intensity has been suggested [17]. The present method starts with the key longitudinal- and shear-wave decomposition: from the theorem of Helmholtz, the displacement and external load vectors can be expressed as the summation of a curl-free and a divergence-free components [12,15,18], $\mathbf{u} = \mathbf{u}_l + \mathbf{u}_t$, $\mathbf{f} = \mathbf{f}_l + \mathbf{f}_t$, such that the curl of longitudinal components and the divergence of shear component vanish. Each of this vector can be introduced by a scalar or a vector potential

$$\mathbf{u}_l = \mathrm{grad}\,\phi, \quad \mathbf{f}_l = \mathrm{grad}\,\gamma_l, \tag{10a}$$

$$\mathbf{u}_t = \mathrm{curl}\,\boldsymbol{\Psi}, \quad \mathbf{f}_t = \mathrm{curl}\,\boldsymbol{\gamma}_t, \tag{10b}$$

where $\mathrm{div}\,\boldsymbol{\Psi} = 0$ and $\mathrm{div}\,\boldsymbol{\gamma}_t = 0$, such that the displacement formulation Eq. (6) can be split into two uncoupled equations [12,18], expressed for their respective displacement- or potential-components

$$\rho\omega^2\mathbf{u}_l + (\lambda+2\mu)\mathrm{grad}\,\mathrm{div}\,\mathbf{u}_l + f_l = \mathbf{0}, \quad \rho\omega^2\phi + (\lambda+2\mu)\nabla^2\phi + \gamma_l = 0, \tag{11a}$$

$$\rho\omega^2\mathbf{u}_t - \mu\,\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}_t + \mathbf{f}_t = 0, \quad \rho\omega^2\boldsymbol{\Psi} + \mu\nabla^2\boldsymbol{\Psi} + \boldsymbol{\gamma}_t = \mathbf{0}. \tag{11b}$$

Terms with l subscript represent the *longitudinal* waves governed by Eqs. (11a), and terms with t subscript represent the shear (*transverse*) waves governed by Eqs. (11b). Each of the harmonic linear variables of Eqs. (1)–(2) is then the summation of a longitudinal (subscript l) and a shear (subscript t) term:

- the strain tensor $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_l + \boldsymbol{\varepsilon}_t$ where $\boldsymbol{\varepsilon}_\mathbf{l}$ and $\boldsymbol{\varepsilon}_\mathbf{t}$ are derived (1), respectively, from $\mathbf{u}_l$ et $\mathbf{u}_t$,
- the stress tensor $\boldsymbol{\sigma} = \boldsymbol{\sigma}_l + \boldsymbol{\sigma}_t$, such that $\operatorname{div}\boldsymbol{\sigma}_l = (\lambda + 2\mu)\operatorname{grad}\operatorname{div}\mathbf{u}_l$ and $\operatorname{div}\boldsymbol{\sigma}_t = -\mu\operatorname{curl}\operatorname{curl}\mathbf{u}_t$.

From these linear longitudinal- and shear-components, the energy variables can then be written

$$\mathbf{I} = \mathbf{I}_l + \mathbf{I}_t + \mathbf{I}_{lt} + \mathbf{I}_{tl}, \tag{12a}$$

$$T = T_l + T_t + T_{lt} + T_{tl}, \tag{12b}$$

$$U = U_l + U_t + U_{lt} + U_{tl}, \tag{12c}$$

where each component is defined as

$$\mathbf{I}_l = \mathrm{j}\omega\boldsymbol{\sigma}_l\mathbf{u}_l^*/2, \quad \mathbf{I}_t = \mathrm{j}\omega\boldsymbol{\sigma}_t\mathbf{u}_t^*/2, \quad \mathbf{I}_{lt} = \mathrm{j}\omega\boldsymbol{\sigma}_l\mathbf{u}_t^*/2, \quad \mathbf{I}_{tl} = \mathrm{j}\omega\boldsymbol{\sigma}_t\mathbf{u}_l^*/2, \tag{13a}$$

$$T_l = \rho\omega^2\mathbf{u}_l\cdot\mathbf{u}_l^*/4, \quad T_t = \rho\omega^2\mathbf{u}_t\cdot\mathbf{u}_t^*/4, \quad T_{lt} = \rho\omega^2\mathbf{u}_l\cdot\mathbf{u}_t^*/4, \quad T_{tl} = \rho\omega^2\mathbf{u}_t\cdot\mathbf{u}_l^*/4, \tag{13b}$$

$$U_l = \boldsymbol{\sigma}_l : \boldsymbol{\varepsilon}_l^*/4, \quad U_t = \boldsymbol{\sigma}_t : \boldsymbol{\varepsilon}_t^*/4, \quad U_{lt} = \boldsymbol{\sigma}_l : \boldsymbol{\varepsilon}_t^*/4, \quad U_{tl} = \boldsymbol{\sigma}_t : \boldsymbol{\varepsilon}_l^*/4. \tag{13c}$$

Longitudinal and shear decomposition then gives rise to four terms for energy- or quadratic-variables:

- a pure longitudinal component (subscript l),
- a pure shear component (subscript t),
- two mixed components (subscripts lt and tl), accounting for longitudinal and shear interactions.

Because of the uniqueness of the longitudinal and shear decomposition for the displacement vector, the separation according to (12)–(13) is also unique.

The physical meaning of this decomposition for quadratic variables is clear for pure components: l terms account for the longitudinal wave contribution, and t terms for the longitudinal wave contribution. The physical sense of mixed components is, however, not obvious: the total kinetic energy density T (12b) is real and positive, both T_l and T_t are real and positive, but T_{tl} and T_{lt} are complex conjugates such that $T_{tl} + T_{tl}$ is real and either positive or negative. Similarly, in non-dissipative materials, the strain energy density U (12c), U_l and U_t are real and positive, but U_{tl} and U_{lt} are complex conjugates and $U_{tl} + U_{tl}$ is real and positive or negative. Negative energy densities associated with mixed components can be explained as follows for the kinetic energy density: the latter is proportional to the squared modulus $\mathbf{u}\cdot\mathbf{u}^*$ of the displacement vector; for local displacement vectors $\mathbf{u}_l$ and $\mathbf{u}_t$ of same direction but opposite orientations, the squared modulus of the summation $\mathbf{u} = \mathbf{u}_l + \mathbf{u}_t$ is smaller than the addition $\mathbf{u}_l\cdot\mathbf{u}_l^* + \mathbf{u}_t\cdot\mathbf{u}_t^*$ of the squared modulus of $\mathbf{u}_l$ and $\mathbf{u}_t$, so that the negative mixed kinetic energy accounts for the difference, $T_{lt} + T_{tl} = T - T_l - T_t$. Working with the mixed components of decomposition (13) is not brought by power-transfer considerations, but by the mathematical curl-free and divergence-free decomposition of the displacement vector. However, each of the four components, including the mixed ones, contribute to energy transfers. In particular the power balance Eq. (4) comes in the form

$$\operatorname{div}\mathbf{I}_l = -2\mathrm{j}\omega(T_l - U_l) - \mathrm{j}\omega\mathbf{f}_l\cdot\mathbf{u}_l^*/2, \tag{14a}$$

$$\operatorname{div}\mathbf{I}_t = -2\mathrm{j}\omega(T_t - U_t) - \mathrm{j}\omega\mathbf{f}_t\cdot\mathbf{u}_t^*/2, \tag{14b}$$

$$\operatorname{div}\mathbf{I}_{lt} = -2\mathrm{j}\omega(T_{lt} - U_{lt}) - \mathrm{j}\omega\mathbf{f}_l\cdot\mathbf{u}_t^*/2, \tag{14c}$$

$$\operatorname{div}\mathbf{I}_{tl} = -2\mathrm{j}\omega(T_{tl} - U_{tl}) - \mathrm{j}\omega\mathbf{f}_t\cdot\mathbf{u}_l^*/2, \tag{14d}$$

where the summation of Eqs. (14) gives (4).

The relevance of decomposition (12) comes from the fact that the different components (13) have simpler expressions than the general energy variables (8)–(9), as a result of the different mathematical properties satisfied by each separate component.

3.2. Pure longitudinal component and discussion

The following analysis considers, in the general case in an isotropic medium the pure longitudinal component (subscript l) of energy variables. This component represents the total field when only pure longitudinal waves are present (i.e. when $\mathbf{u} = \mathbf{u}_l$, $\mathbf{u}_t = 0$, curl $\mathbf{u} = 0$, curl $\mathbf{f} = 0$). The following equations then apply to linear acoustics in fluids, with $\lambda = \rho c_0^2$ and $\mu = 0$, where c_0 is the (possibly complex) adiabatic speed of sound. Expressions for acoustic fields are very simplified: letting $\mu = 0$ in the expression (15a), acoustic intensity $\mathbf{I}_l = p\mathbf{v}^*/2$ reduces to the first term, where $p = -\lambda \operatorname{div} \mathbf{u}_l$ is the acoustic pressure and $\mathbf{v}^* = -\mathrm{j}\omega\mathbf{u}^*$ is the acoustic velocity conjugate.

Pure longitudinal energy variables take the form

$$\mathbf{I}_l = \frac{\mathrm{j}\omega}{2}((\lambda + \mu)\operatorname{div} \mathbf{u}_l \mathbf{u}_l^* - \mu \operatorname{div} \mathbf{u}_l^* \mathbf{u}_l + \mu \operatorname{grad}(\mathbf{u}_l \cdot \mathbf{u}_l^*) + \mu \operatorname{curl}(\mathbf{u}_l \times \mathbf{u}_l^*)), \tag{15a}$$

$$T_l = \rho\omega^2 \mathbf{u}_l \cdot \mathbf{u}_l^*/4, \tag{15b}$$

$$U_l = \frac{1}{4}(\lambda \operatorname{div} \mathbf{u}_l \operatorname{div} \mathbf{u}_l^* + \mu(\nabla^2(\mathbf{u}_l \cdot \mathbf{u}_l^*) - \mathbf{u}_l \cdot \nabla^2 \mathbf{u}_l^* - \nabla^2 \mathbf{u}_l \cdot \mathbf{u}_l^*)). \tag{15c}$$

From Eq. (15a), the property $\operatorname{div}(\mathbf{u}_l \times \mathbf{u}_l^*) = 0$ and relations (11a), the divergence and the curl of $\mathbf{I}_l$ can be written

$$\operatorname{div} \mathbf{I}_l = \frac{\mathrm{j}\omega}{2}\left(\begin{array}{c} \lambda \operatorname{div} \mathbf{u}_l \operatorname{div} \mathbf{u}_l^* + \mu\nabla^2(\mathbf{u}_l \cdot \mathbf{u}_l^*) + \rho\omega^2\left(\frac{\mu}{\lambda^*+2\mu^*} - \frac{\lambda+\mu}{\lambda+2\mu}\right)\mathbf{u}_l \cdot \mathbf{u}_l^* \\ -\frac{\lambda+\mu}{\lambda+2\mu}\,\mathbf{f}_l \cdot \mathbf{u}_l^* + \frac{\mu}{\lambda^*+2\mu^*}\,\mathbf{f}_l^* \cdot \mathbf{u}_l \end{array}\right) \tag{15d}$$

$$\operatorname{curl} \mathbf{I}_l = \frac{\mathrm{j}\omega}{2}\left(\begin{array}{c} -\mu\nabla^2(\mathbf{u}_l \times \mathbf{u}_l^*) - \rho\omega^2\left(\frac{\mu}{\lambda^*+2\mu^*} + \frac{\lambda+\mu}{\lambda+2\mu}\right)(\mathbf{u}_l \times \mathbf{u}_l^*) \\ -\frac{\lambda+\mu}{\lambda+2\mu}\,\mathbf{f}_l \times \mathbf{u}_l^* + \frac{\mu}{\lambda^*+2\mu^*}\,\mathbf{f}_l^* \times \mathbf{u}_l \end{array}\right). \tag{15e}$$

Substituting for Eqs. (11a) into Eq. (15c), the strain energy density may be expressed as

$$U_l = \frac{1}{4}\left(\begin{array}{c} \lambda \operatorname{div} \mathbf{u}_l \operatorname{div} \mathbf{u}_l^* + \mu\nabla^2(\mathbf{u}_l \cdot \mathbf{u}_l^*) + \mu\rho\omega^2\left(\frac{1}{\lambda+2\mu} + \frac{1}{\lambda^*+2\mu^*}\right)\mathbf{u}_l \cdot \mathbf{u}_l^* \\ +\frac{\mu}{\lambda+2\mu}\,\mathbf{f}_l \cdot \mathbf{u}_l^* + \frac{\mu^*}{\lambda^*+2\mu^*}\,\mathbf{f}_l^* \cdot \mathbf{u}_l \end{array}\right). \tag{15f}$$

Working with one of the quadratic components reduces the number of quadratic variables to be used: the quadratic variables of Eqs. (15) are the real positive scalars $\mathbf{u}_l \cdot \mathbf{u}_l^*$ and $\operatorname{div} \mathbf{u}_l \operatorname{div} \mathbf{u}_l^*$, the imaginary vector $\mathbf{u}_l \times \mathbf{u}_l^*$, the complex vector $\operatorname{div} \mathbf{u}_l \mathbf{u}_l^*$ and its conjugate. Only the simplest ones (the first three) are used for the derivatives of structural intensity (15d)–(15e) and energy densities (15b) and (15f). In these expressions, the contribution of external loads is clearly written as scalar- or cross-products with displacement: $\mathbf{f}_l \cdot \mathbf{u}_l^*$ in expressions of $\operatorname{div} \mathbf{I}_l$ and U_l, and $\mathbf{f}_l \times \mathbf{u}_l^*$ in expressions of curl $\mathbf{I}_l$. Moreover, other relationships may be obtained, linking quadratic variables. For example, setting $p = \phi$ in Eq. (32c) and substituting $\nabla^2\phi$ by its expression obtained in Eq. (11a) gives

$$\nabla^2(\phi\phi^*) + \rho\omega^2\left(\frac{1}{\lambda + 2\mu} + \frac{1}{\lambda^* + 2\mu^*}\right)\phi\phi^* = 2\mathbf{u}_l \cdot \mathbf{u}_l^* - \left(\frac{1}{\lambda^* + 2\mu^*}\,\phi\gamma_l^* + \frac{1}{\lambda + 2\mu}\,\phi^*\gamma_l\right). \tag{15g}$$

This relation will be useful for solving 1D fields (Sections 4 and 5).

Similar relationships, but involving more quadratic variables, can be written for the pure-shear or mixed components (Appendix B, Eqs. (33)–(35)). Since a closed equation set cannot be derived from relationships

(15), a general quadratic formulation is not obtained for pure longitudinal components, nor for other shear- or mixed-ones. But the decomposition for quadratic variables is well suited to obtain an exact formulation in the special case of wave energy transfers in 1D systems.

4. Exact 1D quadratic formulation

4.1. Energy transfers in 1D fields

The above relations are considerably simplified in the case of displacement and quadratic variables which depend on only one direction of the 3D space. The material being isotropic (a4), we assume a dependence along the x direction. The potentials of displacement and force are then in the form $\phi = \phi(x)$, $\gamma_l = \gamma_l(x)$, $\mathbf{\Psi} = \Psi_y(x)\mathbf{e}_y + \Psi_z(x)\mathbf{e}_z$, $\boldsymbol{\gamma}_t = \gamma_{ty}(x)\mathbf{e}_y + \gamma_{tz}(x)\mathbf{e}_z$. Under these particular conditions, the direction of propagation being fixed and known, longitudinal and shear components of displacement are orthogonal, and expressions for the different components of energy quantities reduce to

$$
\begin{array}{lll}
\text{longitudinal} & \text{shear} & \text{mixed} \\
\mathbf{I}_l = \frac{j\omega}{2}(\lambda+2\mu)\operatorname{div}\mathbf{u}_l\mathbf{u}_l^* & I_t = -\frac{j\omega}{2}\,\mu\operatorname{curl}\mathbf{u}_t \times \mathbf{u}_t^* & \mathbf{I}_{lt} = \frac{j\omega}{2}\,\lambda\operatorname{div}\mathbf{u}_l\mathbf{u}_t^* \\
 & & \mathbf{I}_{tl} = \frac{j\omega}{2}\,\mu\operatorname{curl}\mathbf{u}_t \times \mathbf{u}_l^*
\end{array}
\tag{16a}
$$

$$
T_l = \rho\omega^2\mathbf{u}_l\cdot\mathbf{u}_l^*/4 \qquad T_t = \rho\omega^2\mathbf{u}_t\cdot\mathbf{u}_t^*/4 \qquad T_{lt} = T_{tl} = 0 \tag{16b}
$$

$$
U_l = \frac{1}{4}(\lambda+2\mu)\operatorname{div}\mathbf{u}_l\operatorname{div}\mathbf{u}_l^* \quad U_t = \frac{1}{4}\,\mu\operatorname{curl}\mathbf{u}_t\cdot\operatorname{curl}\mathbf{u}_t^* \quad U_{lt} = U_{tl} = 0. \tag{16c}
$$

The pure longitudinal and pure shear components for structural intensity are then x-oriented, curl-free, and their divergence expressed as

$$
\operatorname{curl}\mathbf{I}_l = \mathbf{0}, \quad \operatorname{curl}\mathbf{I}_t = \mathbf{0}, \tag{17a}
$$

$$
\operatorname{div}\mathbf{I}_l = \frac{j\omega}{4}(\lambda+2\mu)\left(\nabla^2(\mathbf{u}_l\cdot\mathbf{u}_l^*) + \rho\omega^2\left(\frac{1}{\lambda^*+2\mu^*} - \frac{1}{\lambda+2\mu}\right)\mathbf{u}_l\cdot\mathbf{u}_l^* - \frac{1}{\lambda+2\mu}\,\mathbf{f}_l\cdot\mathbf{u}_l^* + \frac{1}{\lambda^*+2\mu^*}\,\mathbf{f}_l^*\cdot\mathbf{u}_l\right), \tag{17b}
$$

$$
\operatorname{div}\mathbf{I}_t = \frac{j\omega}{4}\,\mu\left(\nabla^2(\mathbf{u}_t\cdot\mathbf{u}_t^*) + \rho\omega^2\left(\frac{1}{\mu^*} - \frac{1}{\mu}\right)\mathbf{u}_t\cdot\mathbf{u}_t^* - \frac{1}{\mu}\,\mathbf{f}_t\cdot\mathbf{u}_t^* + \frac{1}{\mu^*}\,\mathbf{f}_t^*\cdot\mathbf{u}_t\right). \tag{17c}
$$

In contrast, mixed components of structural intensity are perpendicular to x, divergence-free, and characterized by their curl

$$
\operatorname{div}\mathbf{I}_{lt} = 0, \quad \operatorname{div}\mathbf{I}_{tl} = 0, \tag{18a}
$$

$$
\operatorname{curl}\mathbf{I}_{lt} = \frac{j\omega}{4}\,\lambda\left(\nabla^2(\mathbf{u}_l\times\mathbf{u}_t^*) + \rho\omega^2\left(\frac{1}{\mu^*} - \frac{1}{\lambda+2\mu}\right)\mathbf{u}_l\times\mathbf{u}_t^* - \frac{1}{\lambda+2\mu}\,\mathbf{f}_l\times\mathbf{u}_t^* - \frac{1}{\mu^*}\,\mathbf{f}_t^*\times\mathbf{u}_l\right), \tag{18b}
$$

$$
\operatorname{curl}\mathbf{I}_{tl} = -\frac{j\omega}{4}\,\mu\left(\nabla^2(\mathbf{u}_t\times\mathbf{u}_l^*) + \rho\omega^2\left(\frac{1}{\lambda^*+2\mu^*} - \frac{1}{\mu}\right)\mathbf{u}_t\times\mathbf{u}_l^* + \frac{1}{\mu}\,\mathbf{f}_t\times\mathbf{u}_l^* - \frac{1}{\lambda^*+2\mu^*}\,\mathbf{f}_l^*\times\mathbf{u}_t\right). \tag{18c}
$$

Energy transfer in 1D fields can be summarized as follows:

- energy densities and the x-component of structural intensity are simply the addition of the pure longitudinal and of the pure shear components $T = T_l + T_t$, $U = U_l + U_t$. From (17a), these components of structural intensities derive from a scalar potential,
- due to the mixed components, longitudinal and shear waves along x contribute to energy transfers in the directions (y,z) perpendicular to x; the corresponding divergence-free (18a) intensities $\mathbf{I}_{lt}$ and $\mathbf{I}_{tl}$ derive from a vector potential, and no energy density is associated with these mixed components (16b)–(16c).

In 1D systems, energy transfers are considerably simplified, and the decomposition (12) focuses on only a few quadratic variables: the relations (17b)–(17c) involve the dot-products $\mathbf{u}_l \cdot \mathbf{u}_l^*$ and $\mathbf{u}_t \cdot \mathbf{u}_t^*$, proportional to kinetic energy densities (16b), and terms proportional to the power developed by external loads. Similarly, expressions (18b)–(18c) involve the cross-product $\mathbf{u}_l \times \mathbf{u}_t^*$ or its opposite conjugate, and terms related to external loads. The energy density and structural intensity divergence or curl of each component is then characterized by only one quadratic variable and external loads.

4.2. Quadratic formulation for 1D fields

The 1D quadratic formulation starts with the key observation that, comparing expressions (17b)–(17c) and (14a)–(14b), the *strain* energy densities U_l and U_t may be expressed from *kinetic* energy densities and external loads. And, reciprocally, the equation (15g) gives, for longitudinal components, a linear expression of kinetic energy density T_l as a function of strain energy density U_l and terms proportional to the power of external loads. These different relations linking kinetic- and strain-energy densities provide a closed set of relationships, giving the opportunity of an exact quadratic formulation for 1D fields.

4.2.1. Exact formulation

Expressing $\operatorname{div}\mathbf{u}_l = \nabla^2\phi$, $\operatorname{curl}\mathbf{u}_t = \nabla^2\mathbf{\Psi}$ and their conjugates from Eqs. (11) for potentials ϕ and Ψ, the strain energy densities (16c) in 1D systems take the form

$$U_l = \frac{\lambda + 2\mu}{4}\operatorname{div}\mathbf{u}_l \operatorname{div}\mathbf{u}_l^* = \frac{k_l^{*2}}{4}\left(\rho\omega^2\phi\phi^* + \phi\gamma_l^* + \gamma_l\phi^* + \frac{\gamma_l\gamma_l^*}{\rho\omega^2}\right), \tag{19a}$$

$$U_t = \frac{\mu}{4}\operatorname{curl}\mathbf{u}_t \cdot \operatorname{curl}\mathbf{u}_t^* = \frac{k_t^{*2}}{4}\left(\rho\omega^2\mathbf{\Psi}\cdot\mathbf{\Psi}^* + \mathbf{\Psi}\cdot\boldsymbol{\gamma}_t^* + \boldsymbol{\gamma}_t\cdot\mathbf{\Psi}_t^* + \frac{\boldsymbol{\gamma}_t\cdot\boldsymbol{\gamma}_t^*}{\rho\omega^2}\right), \tag{19b}$$

where $k_l = \sqrt{\rho\omega^2/\lambda + 2\mu}$ and $k_t = \sqrt{\rho\omega^2/\mu}$ are, respectively, the longitudinal and shear wavenumbers. As a consequence, the kinetic- and strain-energy densities of longitudinal components are thus linked by the following equation set:

$$\begin{cases} \nabla^2 T_l + \left(k_l^2 + k_l^{*2}\right)T_l = 2k_l^2 U_l - \frac{k_l^2}{4}\mathbf{f}_l\mathbf{u}_l^* - \frac{k_l^{*2}}{4}\mathbf{f}_l^*\mathbf{u}_l, \\ \nabla^2 U_l + \left(k_l^2 + k_l^{*2}\right)U_l = k_l^{*2}\begin{pmatrix} 2T_l + \frac{1}{2}\mathbf{f}_l\cdot\mathbf{u}_l^* + \frac{1}{2}\mathbf{f}_l^*\cdot\mathbf{u}_l \\ -\frac{1}{4k_l^2}\operatorname{div}\mathbf{u}_l\operatorname{div}\mathbf{f}_l^* - \frac{1}{4k_l^{*2}}\operatorname{div}\mathbf{u}_l^*\operatorname{div}\mathbf{f}_l + \frac{1}{2\rho\omega^2}\mathbf{f}_l\cdot\mathbf{f}_l^* \end{pmatrix}. \end{cases} \tag{20a}$$

The first equation is obtained in identifying Eqs. (14a) and (17b) according to the definition (16b) of T_l; the expression of U_l from variable $\phi\phi^*$ (right-hand side of Eq. (19a)), when combined with Eq. (15g) gives the second equation of the set (20a). Due to the similar simple expressions (16), (19b) for shear energy variables in 1D systems, a similar equation set is obtained for shear energy densities:

$$\begin{cases} \nabla^2 T_t + \left(k_t^2 + k_t^{*2}\right)T_t = 2k_t^2 U_t - \frac{k_t^2}{4}\mathbf{f}_t\cdot\mathbf{u}_t^* - \frac{k_t^{*2}}{4}\mathbf{f}_t^*\cdot\mathbf{u}_t \\ \nabla^2 U_t + \left(k_t^2 + k_t^{*2}\right)U_t = k_t^{*2}\begin{pmatrix} 2T_t + \frac{1}{2}\mathbf{f}_t\cdot\mathbf{u}_t^* + \frac{1}{2}\mathbf{f}_t^*\cdot\mathbf{u}_t - \frac{1}{4k_t^2}\operatorname{curl}\mathbf{f}_t^*\cdot\operatorname{curl}\mathbf{u}_t - \frac{1}{4k_t^{*2}}\operatorname{curl}\mathbf{f}_t\cdot\operatorname{curl}\mathbf{u}_t^* \\ +\frac{1}{2\rho\omega^2}\mathbf{f}_t\cdot\mathbf{f}_t^* \end{pmatrix} \end{cases} \tag{20b}$$

For the mixed components, a very similar set is obtained for the quadratic variables $\operatorname{div}\mathbf{u}_l^*\operatorname{curl}\mathbf{u}_t$ and $\mathbf{u}_t \times \mathbf{u}_l^*$ of Eqs. (18b)–(18c)

$$\begin{cases} \nabla^2(\mathbf{u}_t\times\mathbf{u}_l^*) + \left(k_t^2 + k_l^{*2}\right)\mathbf{u}_t\times\mathbf{u}_l^* = -2\operatorname{div}\mathbf{u}_l^*\operatorname{curl}\mathbf{u}_t - \frac{k_t^2}{\rho\omega^2}\mathbf{f}_t\times\mathbf{u}_l^* + \frac{k_l^{*2}}{\rho\omega^2}\mathbf{f}_l^*\times\mathbf{u}_t \\ \nabla^2(\operatorname{div}\mathbf{u}_l^*\operatorname{curl}\mathbf{u}_t) + \left(k_t^2 + k_l^{*2}\right)\operatorname{div}\mathbf{u}_l^*\operatorname{curl}\mathbf{u}_t \\ = k_t^2 k_l^{*2}\begin{pmatrix} -2\mathbf{u}_t\times\mathbf{u}_l^* - \frac{2}{\rho\omega^2}\mathbf{f}_t\times\mathbf{u}_l^* + \frac{2}{\rho\omega^2}\mathbf{f}_l^*\times\mathbf{u}_t \\ -\frac{1}{k_t^2\rho\omega^2}\operatorname{div}\mathbf{f}_l^*\operatorname{curl}\mathbf{u}_t - \frac{1}{k_l^{*2}\rho\omega^2}\operatorname{div}\mathbf{u}_l^*\operatorname{curl}\mathbf{f}_t - \frac{2}{\rho^2\omega^4}\mathbf{f}_t\times\mathbf{f}_l^* \end{pmatrix}. \end{cases} \tag{20c}$$

An equivalent expression for this system can be obtained by replacing the variable $\operatorname{div}\mathbf{u}_l^* \operatorname{curl}\mathbf{u}_t = \nabla^2\phi^*\nabla^2\mathbf{\Psi}$ by $k_t^2 k_l^{*2}\phi\mathbf{\Psi}$ and modifying the contribution of external loads according to Eq. (11).

4.2.2. Exact homogeneous solutions

It is interesting to consider the homogeneous solutions for the sets (20), satisfied by quadratic variables in a portion of medium free of external load. Removing external loads, the sets (20) reduce, respectively, to

$$\begin{cases} \nabla^2 T_l + \left(k_l^2 + k_l^{*2}\right) T_l = 2k_l^2 U_l, \\ \nabla^2 U_l + \left(k_l^2 + k_l^{*2}\right) U_l = 2k_l^{*2} T_l, \end{cases} \tag{21a}$$

$$\begin{cases} \nabla^2 T_t + \left(k_t^2 + k_t^{*2}\right) T_t = 2k_t^2 U_t, \\ \nabla^2 U_t + \left(k_t^2 + k_t^{*2}\right) U_t = 2k_t^{*2} T_t, \end{cases} \tag{21b}$$

$$\begin{aligned} &\begin{cases} \nabla^2(\mathbf{u}_l \times \mathbf{u}_t^*) + \left(k_t^2 + k_l^{*2}\right)\mathbf{u}_l \times \mathbf{u}_t^* = 2k_t^2 k_l^{*2}\phi^*\mathbf{\Psi}, \\ \nabla^2(\phi^*\mathbf{\Psi}) + \left(k_t^2 + k_l^{*2}\right)\phi^*\mathbf{\Psi} = 2\mathbf{u}_t \times \mathbf{u}_l^*, \end{cases} \\ &\Leftrightarrow \begin{cases} \nabla^2(\mathbf{u}_l \times \mathbf{u}_t^*) + \left(k_t^2 + k_l^{*2}\right)\mathbf{u}_l \times \mathbf{u}_t^* = -2\operatorname{div}\mathbf{u}_l^* \operatorname{curl}\mathbf{u}_t, \\ \nabla^2(\operatorname{div}\mathbf{u}_l^* \operatorname{curl}\mathbf{u}_t) + \left(k_t^2 + k_l^{*2}\right)\operatorname{div}\mathbf{u}_l^* \operatorname{curl}\mathbf{u}_t = -2k_t^2 k_l^{*2}\mathbf{u}_t \times \mathbf{u}_l^* \end{cases}. \end{aligned} \tag{21c}$$

When written for energy densities, the first equation of the sets (21a)–(21b) was obtained for real wavenumbers in the absence of damping (Eqs. (10) in Ref. [19]) for membranes. The summation, or the difference betwen the two equations constituting the sets (21a)–(21b) give the following relations, linking the total energy density $W = T + U$ and the Lagrangian density $L = T - U$

$$\begin{cases} \nabla^2 W_l = -\left(k_l^2 - k_l^{*2}\right) L_l, \\ \nabla^2 L_l + 2\left(k_l^2 + k_l^{*2}\right) L_l = \left(k_l^2 - k_l^{*2}\right) W_l, \end{cases} \quad \begin{cases} \nabla^2 W_t = -\left(k_t^2 - k_t^{*2}\right) L_t, \\ \nabla^2 L_t + 2\left(k_t^2 + k_t^{*2}\right) L_t = \left(k_t^2 - k_t^{*2}\right) W_t. \end{cases}$$

Substituting one equation into the other for each of the above sets give the following fourth-order equations, satisfied by any variable (kinetic-, strain- or total-energy densities and Lagrangian density) for pure longitudinal or shear components,

$$\nabla^4\xi_l + 2\left(k_l^2 + k_l^{*2}\right)\nabla^2\xi_l + \left(k_l^2 - k_l^{*2}\right)^2\xi_l = 0 \quad \text{for} \quad \xi_l = T_l, U_l, W_l \text{ or } L_l, \tag{22a}$$

$$\nabla^4\xi_t + 2\left(k_t^2 + k_t^{*2}\right)\nabla^2\xi_t + \left(k_t^2 - k_t^{*2}\right)^2\xi_t = 0 \quad \text{for} \quad \xi_t = T_t, U_t, W_t \text{ or } L_t \tag{22b}$$

and by any quadratic variable $\xi_{lt} = \mathbf{u}_t \wedge \mathbf{u}_l^*$, $\phi^*\mathbf{\Psi}$ or $\operatorname{div}\mathbf{u}_l^* \operatorname{curl}\mathbf{u}_t$ for mixed components

$$\nabla^4\xi_{lt} + 2\left(k_t^2 + k_l^{*2}\right)\nabla^2\xi_{lt} + \left(k_t^2 - k_l^{*2}\right)^2\xi_{lt} = 0. \tag{22c}$$

In the case of light damping, Mencik [20] obtained (i) a coupled equation set for energy densities T and U corresponding to Eq. (21a) for quasi-longitudinal waves in rods, and (ii) the equations of order four (22a) satisfied by T and U.

Each of the fourth-order Eqs. (22) is in the form

$$\left(\nabla^4 + 2\left(k_A^2 + k_B^{*2}\right)\nabla^2 + \left(k_A^2 - k_B^{*2}\right)^2\right)\xi = 0, \tag{23}$$

where

(i) $A = B = l$ or $A = B = t$, for the energy- and Lagrangian-density variables of (22a)–(22b) concerning, respectively, pure longitudinal or pure shear components; Eq. (23) is then satisfied for any quadratic variable (kinetic-T_A, strain-$\mathrm{Re}(U_A)$ or total-$\mathrm{Re}(W_A)$ energy densities, Lagrangian density $\mathrm{Re}(L_A)$, hysteretic dissipated power proportional to $\mathrm{Im}(U_A) = \mathrm{Im}(W_A) = -\mathrm{Im}(L_A)$, ($\phi\phi^*$, $\mathbf{u}_l \cdot \mathbf{u}_l^*$, $\operatorname{div}\mathbf{u}_l \operatorname{div}\mathbf{u}_l^*$) or ($\mathbf{\Psi} \cdot \mathbf{\Psi}^*$, $\mathbf{u}_t \cdot \mathbf{u}_t^*$, $\operatorname{curl}\mathbf{u}_t \cdot \operatorname{curl}\mathbf{u}_t^*$),

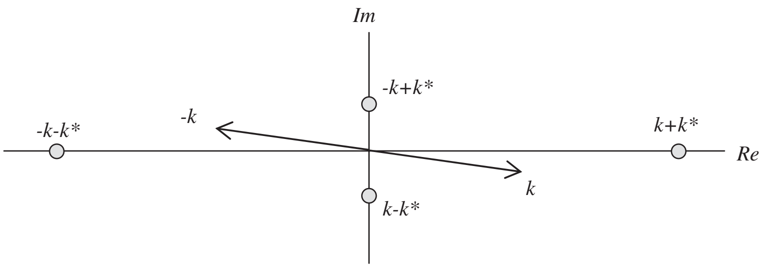

Fig. 1. Representation of the wavenumbers of pure longitudinal or pure shear components, in the complex plane, by arrows $\pm k$ for displacement forward and backward-propagative waves, by dots $\pm(k \pm k^*)$ for the corresponding quadratic variables.

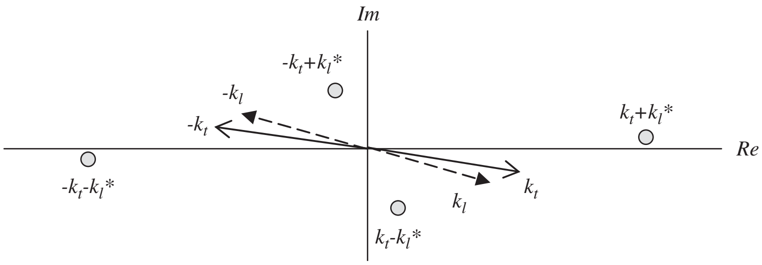

Fig. 2. Representation of the wavenumbers of the mixed component of type tl (for example $\mathbf{u}_t \times \mathbf{u}_l^*$), in the complex plane, by arrows $\pm k_l$ and $\pm k_t$ for displacement forward and backward-propagative waves, by dots $\pm(k_t \pm k_l^*)$ for the corresponding tl quadratic variables. The wavenumbers for quadratic components of type lt (like $\mathbf{u}_l \times \mathbf{u}_t^*$) are the complex conjugate $\pm(k_t^* \pm k_l)$. Wavenumbers k_l and k_t are here represented for Lamé coefficients presenting different complex arguments $\arg(\lambda) \neq \arg(\mu)$.

(ii) $A = t$ and $B = l$ for the last Eqs. (22c) concerning mixed components.

Eq. (23) can be factorized as

$$(\nabla^2 - r_1)(\nabla^2 - r_2)\xi = 0 \quad \text{where} \quad r_1 = \left(\pm \mathrm{j}(k_A - k_B^*)\right)^2 \quad \text{and} \quad r_2 = \left(\pm \mathrm{j}(k_A + k_B^*)\right)^2 \tag{24}$$

such that the solution for the homogeneous systems highlights the properties of quadratic variables for 1D fields: the displacement- or potential-fields in a medium free of external loads, and depending on the only x-coordinate may be described as the superposition of $(+x)$ forward- and $(-x)$ backward-propagating waves. Let us consider waves A and B, presenting, respectively, the wavenumbers k_A and k_B. The time averaged quadratic variable, being the product of a displacement A (or its derivative of any order) and the conjugate of another B, consists in a term of wavenumber $k_A + k_B^*$. Combining any propagative or counter-propagative waves $\pm k_A$ and $\pm k_B$ gives the four terms of wavenumbers $\pm(k_A \pm k_B^*)$ that are enhanced by the factorization (24). Then,

(i) when the quadratic variable is such that $k_A = \pm k_B$, as encountered for a pure-longitudinal- or shear-component, the four resulting quadratic components present two opposite real and two conjugate imaginary wavenumbers (Fig. 1), whereas
(ii) for mixed component ones, k_A differs from k_B, such that the resulting quadratic component are complex, pair opposed. The resulting wavenumbers are presented in Fig. 2 for quadratic variables of type ξ_{tl}; the conjugate quadratic wavenumbers are obtained for quadratic variables ξ_{lt}, because mixed components with subscripts tl and lt imply conjugate quadratic variables (Appendix B2).

4.2.3. Approximate solutions

The additional approximations usually introduced in energy flow methods, like light damping, are unnecessary to model energy transfers in 1D systems. Some of these approximations result of the removal of some of the components of quadratic variables: for example, ignoring interference among different waves removes the mixed quadratic variables and the associated energy transfer.

Another example is the following, in the case where only one type of wave is considered, either pure longitudinal or pure shear, with displacement wavenumber $\pm k$ such that $k^2 = k_R^2/(1+j\eta)$. Any quadratic variable involves the four components with wavenumbers $\pm(k \pm k^*)$ (Fig. 1), and terms r_1 and r_2 of the factorized Eq. (24) take the form of the Taylor series expansion $r_1 = k_R^2(\eta^2 - 5\eta^4/4 + O(\eta^6))$, $r_2 = -k_R^2(4 - 3\eta^2 + 11\eta^4/4 + O(\eta^6))$, giving, for small hysteretic damping $\eta \ll 1$, the approximate factorized Eq. (24)

$$\left(\nabla^2 - k_R^2\eta^2\right)\left(\nabla^2 + 4k_R^2\right)\xi = 0. \tag{25}$$

This equation is satisfied by any quadratic variable ξ, among others W and L. The General Energy Method [21] focuses only on the first factor of the factorized Eq. (25) when considering energy density W, and only the second factor when considering Lagrangian density L (Eq. (20) in Ref. [21]). This approach leads to the consideration of only imaginary quadratic wavenumbers $\pm(k - k^*)$ for W, and of only real ones $\pm(k + k^*)$ for L (Eq. (21) in Ref. [21]).

The present work confirms that the energy variables are then significantly simplified in 1D systems. These simplifications enable the development of efficient energy models [6,7,21,22] in such configurations, even if unnecessarily restrictive conditions are assumed. For example, the thermal analogy assumes that the structural intensity derives from a scalar potential, this being demonstrated in the case of a simple propagating plane wave [6]. If the mixed components which combine longitudinal and shear displacement component are ignored, structural intensity actually derives from a scalar potential for 1D fields, even if wave interference between forward and backward propagative waves is considered. However, this scalar potential cannot be written as a combination of the total-, the kinetic- and the strain-energy densities.

5. Agreement of the displacement- and the 1D quadratic-formulations

The following application is presented to illustrate the ability of the quadratic formulation (20) to obtain exact 1D solutions. No approximation is necessary, the model is suited to 1D problems, for any frequency range; the assumptions are those (a1–a4) presented in Section 1, external loads are applied and the different types of waves are coupled due to the boundary conditions involved.

We consider the semi-infinite medium $-L_1 \leqslant x \leqslant L_2$, $-\infty < y < +\infty$, $-\infty < z < +\infty$, where forward and backward-propagating waves in only the $\pm x$ direction are present, and the interference between these waves is considered. Hysteretic damping and the use of an isotropic material are considered; these properties are expressed in terms of complex Lamé coefficients (λ, μ), having the same complex argument $\eta_\lambda = \eta_\mu$ (5). Each Cartesian component of the displacement then corresponds to a particular type of wave: the x-displacement component represents longitudinal waves $\mathrm{u}_l = u_{lx}\mathbf{e}_x$, and the two remaining y- and z-components correspond, respectively, to y- and z-oriented shear waves $\mathbf{u}_t = u_{ty}\mathbf{e}_y + u_{tz}\mathbf{e}_z$. External loads are applied in the plane $x = 0$ by forces, in the x and y directions; these loads are expressed by the local surface forces $\mathbf{f}_\mathbf{l} = f_x\delta(x)\mathbf{e}_x$ and $\mathbf{f}_t = f_y\delta(x)\mathbf{e}_y$ where $\delta(x)$ is the Dirac function at $x = 0$, or, equivalently, by their potentials

$$\gamma_l = f_x H(x), \quad \boldsymbol{\gamma}_t = -f_y H(x)\mathbf{e}_y, \tag{26}$$

where $H(x)$ is the Heaviside function. The different components of external loads excite different types of waves: x-directed $\mathbf{f}_l = f_x\mathbf{e}_x$ and y-directed $\mathbf{f}_t = f_y\mathbf{e}_y$ loads correspond, respectively, to longitudinal and y-shear waves propagating in the $\pm x$ direction. The boundary conditions are:

- on the left end, at $x = -L_1$: the x-component for displacement vanishes, y and z displacement are free and the corresponding shear strains vanish,
- the right end ($x = L_2$) consists in a vanishing y-component for displacement and a frictionless slide surface condition of angle θ: the x- and z-displacement components are linked by the tangent of the angle $u_{tz}(L) = \mathrm{tg}\theta u_{lx}(L)$ and the shear stress related to this θ-inclined surface vanishes.

Due to this last boundary condition, a coupling between x-longitudinal and z-shear components is present at $x = L_2$, such that, when combined with the external loads considered, longitunal, y- and z-shear waves are excited in both directions $\pm x$.

To compute energy densities and energy transfers in this system, the two following methods are available: one using the displacement formulation, and the other using the quadratic formulation. They are compared below.

5.1. Displacement formulation

The usual solution, based on the displacement formulation (11), consists in determining the displacement field, and successively express strain, stress, and finally energy densities and structural intensity. We present here the results obtained from the displacement and load potentials. Considering excitations (26), solutions for the potential Eqs. (11) are in the form:

$$\phi = C_1 \cos(k_l x) + C_2 \sin(k_l x) + \frac{f_x H_0}{\rho\omega^2}(\cos(k_l x) - 1), \tag{27a}$$

$$\mathbf{\Psi} = (C_3 \cos(k_t x) + C_4 \sin(k_t x))\mathbf{e}_y + \left(C_5 \cos(k_t x) + C_6 \sin(k_t x) + \frac{f_y H_0}{\rho\omega^2}(\cos(k_t x) - 1)\right)\mathbf{e}_z, \tag{27b}$$

where k_l and k_t are, respectively, the wavenumbers for longitudinal and shear waves. The six constants (C_1, C_2, C_3, C_4, C_5, C_6) are determined to satisfy the boundary conditions (Table 2): $u_{lx}|_{-L_1} = 0$, $\mu u_{ty,x}|_{-L_1} = 0$, $\mu u_{tz,x}|_{-L_1} = 0$ (vanishing x-displacement and shear stress at $x = -L_1$) $u_{ty}|_{L_2} = 0$, $(u_{tz} - \operatorname{tg}\theta u_{lx})\big|_{L_2} = 0$, $((\lambda + 2\mu)\cos\theta u_{lx,x} + \mu \sin\theta u_{tz,x})|_{L_2} = 0$ (vanishing y-displacement and slide condition of angle θ at $x = L_2$).

Deriving this solution for displacement potentials gives the displacement vectors (10), strain and stress tensors; energy quantities (12)–(13) can be obtained by products of these linear variables. This example shows that, for 1D waves, the displacement formulation:

(i) presents three components of displacement (u_{lx}, u_{ty} and u_{tz}, or alternatively the longitudinal scalar potential ϕ and the two-components vector potential $\Psi_y \Psi_z$),
(ii) requires six boundary conditions to determine the displacement wave field,
(iii) characterizes the excitation of each of the wave component by a local force (f_x and f_y in this case), i.e. one discontinuity for the potential (27).

5.2. Quadratic formulation for 1D problems

Another way to determine energy variables is to directly compute quadratic variables, using relations (20): energy densities are first obtained, and then the known right-hand side term of Eqs. (17)–(18) is integrated to obtain structural intensities.

Except at the boundaries, longitudinal-, y- and z-directed shear waves are uncoupled in the bulk of the domain. When expressed in terms of quadratic variables, they give the following systems of equations for quadratic variables ${}^nQ_T$ and ${}^nQ_U$, $n = 1-5$ (summarized in Table 1):

- pure longitudinal components ($n = 1$ in Table 1), denoted by subscript l in Section 3.1, are energy densities T_l and U_l, satisfying Eqs. (20a),
- pure shear components ($n = 2, 3$ in Table 1, subscript t) are related by the energy densities T_t and U_t, i.e. the scalar products $\operatorname{curl}\mathbf{u}_t \cdot \operatorname{curl}\mathbf{u}_t^*$ and $\mathbf{u}_t \cdot \mathbf{u}_t^*$, satisfying Eqs. (20b). Such scalar products for vectors are the summation of two products, each containing a y- or z-displacement component, these differently oriented shear waves being uncoupled. Energy densities can then be presented as the summation of two terms $T_t = T_{ty} + T_{tz}$, $U_t = U_{ty} + U_{tz}$, where

$$T_{ty} = \rho\omega^2 u_{ty} \cdot u_{ty}^*/4, \quad U_{ty} = \frac{\mu}{4} u_{ty,x} u_{ty,x}^*, \tag{28a}$$

Table 1
Quadratic variables of Eqs. (30), which are the different components of the variables of equation sets (20)

Component	n	^nQ_U	^nQ_T	k_1	k_2
Longitudinal	1	U_l	T_l	k_l	k_l
Shear y	2	U_{ty}	T_{ty}	k_t	k_t
Shear z	3	U_{tz}	T_{tz}	k_t	k_t
Mixed longitudinal by shear z	4	$Q_{Uxy} = -u_{tz,x}u^*_{lx,x}$	$Q_{Txy} = u_{tz}u^*_{lx}$	k_l	k_t
Mixed longitudinal by shear y	5	$Q_{Uxz} = u_{ty,x}u^*_{lx,x}$	$Q_{Txz} = -u_{ty}u^*_{lx}$	k_l	k_t

$T_t = T_{ty} + T_{tz}$, $\mathbf{u}_t \times \mathbf{u}_l^* = Q_{Txy}\mathbf{e}_y + Q_{Txz}\mathbf{e}_z$

$U_t = U_{ty} + U_{tz}$, $\operatorname{div} \mathbf{u}_l^* \operatorname{curl} \mathbf{u}_t = Q_{Uxy}\mathbf{e}_y + Q_{Uxz}\mathbf{e}_z$

$$T_{tz} = \rho\omega^2 u_{tz} \cdot u^*_{tz}/4, \quad U_{tz} = \frac{\mu}{4} u_{tz,x}u^*_{tz,x}. \tag{28b}$$

The system (20b) can then be split into two similar systems concerning, respectively, the y- and z-oriented shear waves; the former is written

$$\begin{cases} \nabla^2 T_{ty} + \left(k_t^2 + k_t^{*2}\right) T_{ty} = 2k_t^2 U_{ty} - \frac{k_t^2}{4} f_{ty}u^*_{ty} - \frac{k_t^{*2}}{4} f^*_{ty}u_{ty} \\ \nabla^2 U_{ty} + \left(k_t^2 + k_t^{*2}\right) U_{ty} = k_t^{*2} \begin{pmatrix} 2T_{ty} + \frac{1}{2} f_{ty}u^*_{ty} + \frac{1}{2} f^*_{ty}u_{ty} - \frac{1}{4k_t^2} f^*_{ty,x}u_{ty,x} \\ -\frac{1}{4k_t^{*2}} f_{ty,x}u^*_{ty,x} + \frac{1}{2\rho\omega^2} f_{ty} f^*_{ty} \end{pmatrix} \end{cases} \tag{29}$$

and the latter is obtained in substituting variables of y-components by these of z-components,

- mixed quadratic components ($n = 4, 5$ in Table 1, subscripts lt and tl) satisfy the same vectorial system (20c). The quadratic variables involved present a vanishing x-component, such that the y- and z-components of this vectorial system can also be split into two scalar systems, expressing, respectively, interactions between longitudinal and y-shear or longitudinal and z-shear waves.

Working with quadratic variables leads to solve five scalar systems, because the interaction between y- and z-shear waves does not contribute to energy densities nor to power flow. The first three ((20a), (29) and the corresponding system for tz variables) involving pure longitudinal or pure transverse components, are in the form

$$\begin{cases} \nabla^2\, {}^nQ_T + \left(k_2^2 + k_1^{*2}\right){}^nQ_T - 2k_2^2\, {}^nQ_U = s_0\delta_{(0)}, \\ \nabla^2\, {}^nQ_U + \left(k_2^2 + k_1^{*2}\right){}^nQ_U - 2k_1^{*2}\, {}^nQ_T = s_1\delta_{(0)} + s_2\delta_{(0),x}, \end{cases} \quad n = 1-3, \tag{30a}$$

where the quadratic variables ^nQ_T and ^nQ_U are the energy density components and $k = k_1 = k_2$ are the corresponding wavenumbers, presented in Table 1 ($n = 1-3$). The right-hand sides represent excitation of the system by external loads at $x = 0$:

s_0 is the discontinuity of $T_{,x}$ $\quad s_0 = -(k^2 f|_0 u^*|_0 + k^{*2} f^*|_0\, u|_0)/4,$
s_1 the discontinuity of $U_{,x}$ $\quad s_1 = k^{*2}(f|_0 u|_0 + f^*|_0\, u^*|_0)/4,$
and s_2 the discontinuity of U $\quad s_2 = \frac{k^{*2}}{4}\left(\frac{f|_0 f^*|_0}{\rho\omega^2} - \frac{f^*|_0 u_{,x}|_{0-}}{k^2} - \frac{f|_0 u^*_{,x}|_{0-}}{k^{*2}}\right),$

where $\xi|_0$ (respectively $\xi|_{0-}$, $\xi|_{0+}$) is the notation for the value of ξ at $x = 0$ (respectively left, right limit).

The two last scalar systems, concerning mixed components, are in the form

$$\begin{cases} \nabla^2\, {}^nQ_T + \left(k_2^2 + k_1^{*2}\right){}^nQ_T + 2\, {}^nQ_U = s_0\delta_{(0)}, \\ \nabla^2\, {}^nQ_U + \left(k_2^2 + k_1^{*2}\right){}^nQ_U + 2k_2^2 k_1^{*2}\, {}^nQ_T = s_1\delta_{(0)} + s_2\delta_{(0),x} \end{cases} \quad n = 4, 5, \tag{30b}$$

where the quadratic variables ${}^nQ_T$ and ${}^nQ_U$ and the wavenumbers k_1 and k_2 are presented in Table 1 ($n = 4, 5$). The right-hand side of Eq. (30b) represents excitation of the system by external loads: s_0, s_1 and s_2 are, respectively, the discontinuity of ${}^nQ_{T,x}$, ${}^nQ_{U,x}$ and ${}^nQ_U$ at $x = 0$.

For each of these five scalar sets ($n = 1-5$), the solution in the bulk of the domain is expressed as the summation of a general solution, in the form

$$\begin{cases} {}^nQ_{Um} = c_{1m}\cos((k_1 - k_2^*)x) + c_{2m}\sin((k_1 - k_2^*)x) + c_{3m}\cos((k_1 + k_2^*)x) + c_{4m}\sin((k_1 + k_2^*)x), \\ {}^nQ_{Tm} = k_1 k_2^*\big(c_{1m}\cos((k_1 - k_2^*)x) + c_{2m}\sin((k_1 - k_2^*)x) - c_{3m}\cos((k_1 + k_2^*)x) - c_{4m}\sin((k_1 + k_2^*)x)\big) \end{cases}$$

and a particular solution of Eq. (30a) or Eq. (30b) which accounts for the external loads.

Solution of the quadratic formulation can then be obtained in determining the 20 constants c_{im} ($i = 1-4$, $m = 1-5$) in order to satisfy the 20 relations for boundary conditions, when expressed using quadratic variables, as presented in Table 2. The boundary conditions for quadratic variables are obtained from the displacement boundary conditions, according to the definition of the quadratic variables (Table 1). Energy densities and other quadratic variables for the different type of waves (pure longitudinal, y- and z-shear, mixed components) are then obtained directly.

The use of the 1D quadratic formulation requires an important computational effort:

(i) five equation sets of two variables are considered,
(ii) one end is characterized by two conditions for each of these systems, i.e. a total amount of 20 boundary conditions for the 1D system,
(iii) a local excitation is characterized, for each of these five components, by the three discontinuities of ${}^nQ_{T,x}$, ${}^nQ_{U,x}$ and ${}^nQ_U$; since the displacement components are continuous, the continuity of the variables ${}^nQ_T$ (proportional to the squared modulus of the displacement) are ensured at the point of concentrated excitation.

The different structural intensity components can be next computed as follows: the total structural intensity $\mathbf{I}$ is the summation (12a) of pure longitudinal $\mathbf{I}_l$ and shear $\mathbf{I}_t$ components, and mixed components $\mathbf{I}_{lt}$ and $\mathbf{I}_{tl}$. As discussed above for energy densities, the pure shear component $\mathbf{I}_t = (-\mathrm{j}\omega/2)\,\mu\,\mathrm{curl}\,\mathbf{u}_t \times \mathbf{u}_t^*$ may be presented as the summation of two contributions, $\mathbf{I}_t = \mathbf{I}_{ty} + \mathbf{I}_{tz}$, each of them pointing in the $\pm x$ direction, but

Table 2
Boundary conditions for the displacement (left) and the quadratic (right) formulations (notation: $,x = \partial/\partial x$)

Displacement	Quadratic variables			
At $x = -L_1$:				
$u_{lx} = 0$	$T_l = 0$	$U_{l,x} = 0$	$U_{ty} = 0$	$U_{tz} = 0$
$\mu(\partial u_{ty}/\partial x) = 0$	$T_{ty,x} = 0$	$Q_{Uy} = 0$	$Q_{Uxz} = 0$	
$\mu(\partial u_{tz}/\partial x) = 0$	$T_{tz,x} = 0$	$Q_{Uz} = 0$	$Q_{Uxy} = 0$	
At $x = L_2$:				
$u_{ty}(L) = 0$	$T_{ty} = 0$	$U_{ty,x} = 0$	$Q_{Txz} = 0$	
$\begin{cases} u_{tz} = \tan\theta u_{lx} \\ u_{lx,x} + \frac{\mu}{\lambda+2\mu}\tan\theta u_{tz,x} = 0 \end{cases}$	$\left\{\begin{array}{c} T_{tz} = \tan^2\theta T_l \\ \rho\omega^2 Q_{Txy}/4 = \tan\theta T_l \\ T_{tz,x} + \frac{\lambda+2\mu}{\mu} T_{l,x} = 0 \\ U_l = \tan^2\theta \frac{\mu^*}{\lambda^*+2\mu^*} U_{tz} \\ U_{l,x} + \frac{\mu^*}{\lambda^*+2\mu^*} U_{tz,x} = 0 \\ T_{l,x} + \frac{\mu k_l^*}{4(k_l^{*2}-k_t^2)}\left(\left(\tan\theta + \frac{\lambda+2\mu}{\mu\tan\theta}\right)Q_{Uxy,x} - \left(\frac{(\lambda+2\mu)k_l^{*2}}{\mu\tan\theta} + k_t^2\tan\theta\right)Q_{Txy,x}\right) = 0 \\ \frac{\rho\omega^2}{\lambda^*+2\mu^*} Q_{Txz,x} - Q_{Uxz,x} = 0 \end{array}\right.$			

corresponding, respectively, to the y- or z-displacement component of the differently polarized shear waves. These two components of pure shear structural intensity are linked to the corresponding energy density components (T_{ty}, T_{tz}, U_{ty}, U_{tz}), such that the scalar Eq. (17c) may also be split into the two scalar equations

$$\begin{aligned}\operatorname{div}\mathbf{I}_{ty} &= \frac{\mathrm{j}\omega}{4}\,\mu\left(\frac{4}{\rho\omega^2}\left(\Delta T_{ty} + (k_t^{*2} - k_t^2)T_{ty}\right) - \frac{1}{\mu} f_{ty}u_{ty}^* + \frac{1}{\mu^*} f_{ty}^* u_{ty}\right),\\ \operatorname{div}\mathbf{I}_{tz} &= \frac{\mathrm{j}\omega}{4}\,\mu\left(\frac{4}{\rho\omega^2}\left(\Delta T_{tz} + (k_t^{*2} - k_t^2)T_{tz}\right) - \frac{1}{\mu} f_{tz}u_{tz}^* + \frac{1}{\mu^*} f_{tz}^* u_{tz}\right).\end{aligned} \tag{31}$$

The vectorial Eqs. (18) present the only two y- and z-non vanishing components; each of them can be treated separately, giving rise to four scalar equations, applied, respectively, to the y- and z-components of $\mathbf{I}_{lt}$ and $\mathbf{I}_{tl}$.

From the quadratic variables (energy densities), determined above, and external loads terms, the right-hand side of Eqs. (17b) and (31) and the y- and z-components of Eqs. (18b)–(18c) are known; the structural intensity components corresponding to each wave interaction (pure longitudinal, y- or z-shear, mixed (longitudinal by y-shear) or (longitudinal by z-shear) of components $\mathbf{I}_{lt}$ and $\mathbf{I}_{tl}$) is computed by integrating these equations, with adequate boundary conditions:

- pure longitudinal, pure y-shear and pure z-shear waves contribute to energy flow in the x direction. Due to the boundary conditions, each of these components of structural intensity vanishes at $x = -L_1$ ($\mathbf{I}_l|_{-L_1} = \mathbf{0}$, $\mathbf{I}_{ty}|_{-L_1} = \mathbf{0}$, $\mathbf{I}_{tz}|_{-L_1} = \mathbf{0}$). At $x = L_2$, $\mathbf{I}_t|_{L_2} = \mathbf{0}$ and the power flow of longitudinal and z-shear waves is reflected and converted in the opposite wave type, $(\mathbf{I}_l + \mathbf{I}_{tz})|_{L_2} = \mathbf{0}$. Only the left ($x = -L_1$) or the right ($x = L_2$) boundary condition is necessary to integrate structural intensity components, and the boundary conditions at the opposite end are naturally satisfied by the result of this computation,
- mixed components contribute to energy transfers in the y- and z-directions; although waves are propagating in the $\pm x$ direction only, structural intensity presents non-zero y- and z-components, locally, and when averaged over the thickness of the domain, $\int_{-L_1}^{-L_2} \mathbf{I}_{lt} + \mathbf{I}_{tl}\,\mathrm{d}x \neq \mathbf{0}$. The y- and z-components of Eqs. (18b)–(18c) are integrated, with the boundary conditions

$$(\mathbf{I}_{tl}\cdot\mathbf{e}_y)|_{-L_1} = \mathbf{0},\quad (\mathbf{I}_{tl}\cdot\mathbf{e}_z)|_{-L_1} = \mathbf{0},\quad (\mathbf{I}_{lt}\cdot\mathbf{e}_y)|_{L_2} = \mathbf{0},\quad \left.\left(\mathbf{I}_{lt}\cdot\mathbf{e}_z - \frac{\lambda\tan\theta}{\lambda+2\mu}\,\mathbf{I}_l\cdot\mathbf{e}_x\right)\right|_{L_2} = \mathbf{0}.$$

5.3. Numerical results for the displacement- and the quadratic-formulations

The displacement and energy field at a frequency of 10 kHz are presented for longitudinal component (Fig. 3), y- and z-shear component (Figs. 4 and 5) and mixed components (Figs. 6 and 7), in a medium of density $\rho = 7800\,\mathrm{kg\,m^{-3}}$, Young modulus $E = 210(1 + 0.01j)$ GPa and Poisson ratio $v = 0.3$, $L_1 = L_2 = 10$ m, $f_x = f_y = 1\,\mathrm{N\,m^{-2}}$. The different components of displacement, obtained using Eqs. (27), are presented in Figs. 3a, 4a and 5a. As presented above, the different components for kinetic- and strain-energy densities may be obtained from the displacement formulation (27) and definitions (13) and (28) (Section 5.1), or using the quadratic formulation in directly solving the solution sets (30) for energy densities, from which structural intensity may be integrated (Section 5.2). The graphs of energy variables obtained form both the displacement and the quadratic formulations are superimposed in Figs. 3–5 (b–f) and in Figs. 6 and 7. Since no approximation has been introduced in the quadratic formulation, these two different approaches are strictly equivalent: the corresponding graphs are undistinguishable, even when detailed views are observed in Figs. 3–5(e–f) and in Fig. 7.

In the system studied, the power of excitation is provided by the loading forces at $x = 0$ for longitudinal (Fig. 3) and y-shear (Fig. 4) waves, highlighted by the discontinuity of the real part of structural intensity (d, e). This power is dissipated in the structure, giving a vanishing structural intensity on both ends $x = -L_1$ and $x = L_2$, except, due to the slide condition at $x = L_2$ where the longitudinal intensity is converted into z-shear form (Fig. 5e), and dissipated in this z-shear form (Fig. 5d).

In a section of unloaded structure, any energy quantity, as a quadratic variable, is the combination of four components with different wave-numbers (Figs. 1 and 2). In the case of pure longitudinal or pure shear

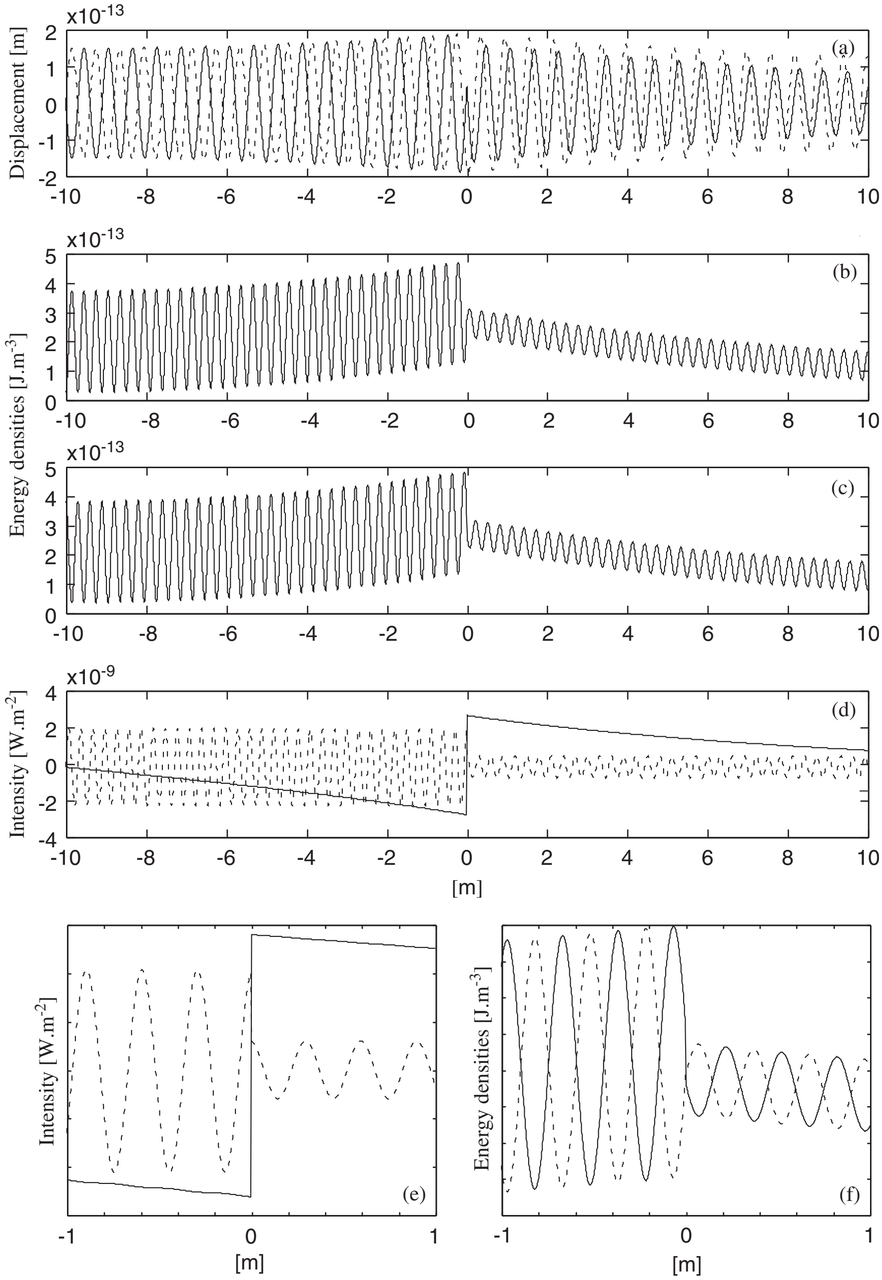

Fig. 3. Agreement of the displacement- and the quadratic-formulations: longitudinal wave fields in a 1D system (Section 5, $L_1 = -10$ m, $L_2 = 10$ m, frequency 8 kHz), (a) x-component of displacement (real part — , imaginary part - - -) obtained from displacement formulation (27a); energy variables obtained from Eqs. (13) in using the displacement field (a) and from the quadratic formulation (30a) are superimposed : (b) kinetic energy density T_l, (c) strain energy density $\mathrm{Re}(U_l)$, (d) structural intensity $\mathbf{I}_l$ (x-oriented, real part — , imaginary part - - -). Details for $-1 \leqslant x \leqslant 1$: (e) structural intensity, (f) energy densities (T_l - - - , $\mathrm{Re}(U_l)$ —).

components, terms with wavenumbers in the form $\pm(k+k^*)$ show spatial variations close to the half displacement wave length π/k, while the others, of wavenumbers $\pm(k-k^*)$, highlight variations at larger space scale, determined by the hysteretic damping properties of the material. These two very different scale length for the energy variables make the local representation of energy transfers difficult. However, all these four components are necessary to accurately represent energy densities of pure longitudinal (Fig. 3b and c), pure

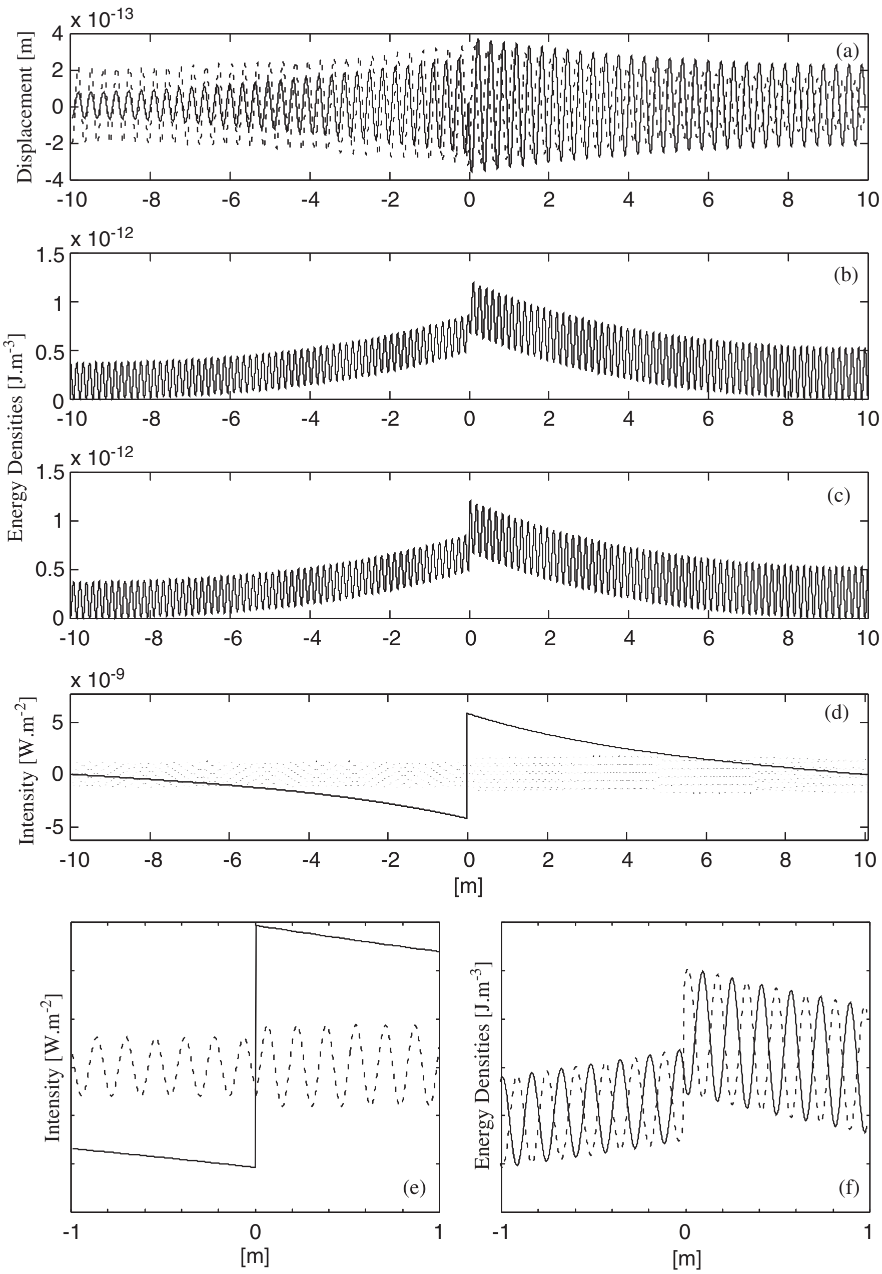

Fig. 4. Agreement of the displacement- and the quadratic-formulations: y-shear wave fields in a 1D system (Section 5, $L_1 = -10\,\text{m}$, $L_2 = 10\,\text{m}$, frequency 8 kHz): (a) y-component of displacement (real part — , imaginary part - - -) obtained from displacement formulation (27b); energy variables obtained from Eqs. (13) in using the displacement field (a) and from the quadratic formulation (30a) are superimposed: (b) kinetic energy density T_{ty}, (c) strain energy density $\mathrm{Re}(U_{ty})$, (d) structural intensity $\mathbf{I}_{ty}$ (x-oriented, real part — , imaginary part - - -). Details for $-1 \leqslant x \leqslant 1$: (e) structural intensity, (f) energy densities (T_{ty} — , $\mathrm{Re}(U_{ty})$ - - -).

shear (Figs. 4 and 5(b,c)) components and mixed quadratic variables (Figs. 6 and 7). The structural intensity involves these components separately: the active power flow is mainly described by components of wavenumbers $\pm(k-k^*)$, and the reactive one by components of wavenumbers $\pm(k+k^*)$ (Figs. 3–5(d,e)). As discussed in Section 4.2, approximate energy models use to focus on large scale energy transfers, considering only quadratic components showing smooth spatial variations, i.e. wavenumbers $\pm(k-k^*)$.

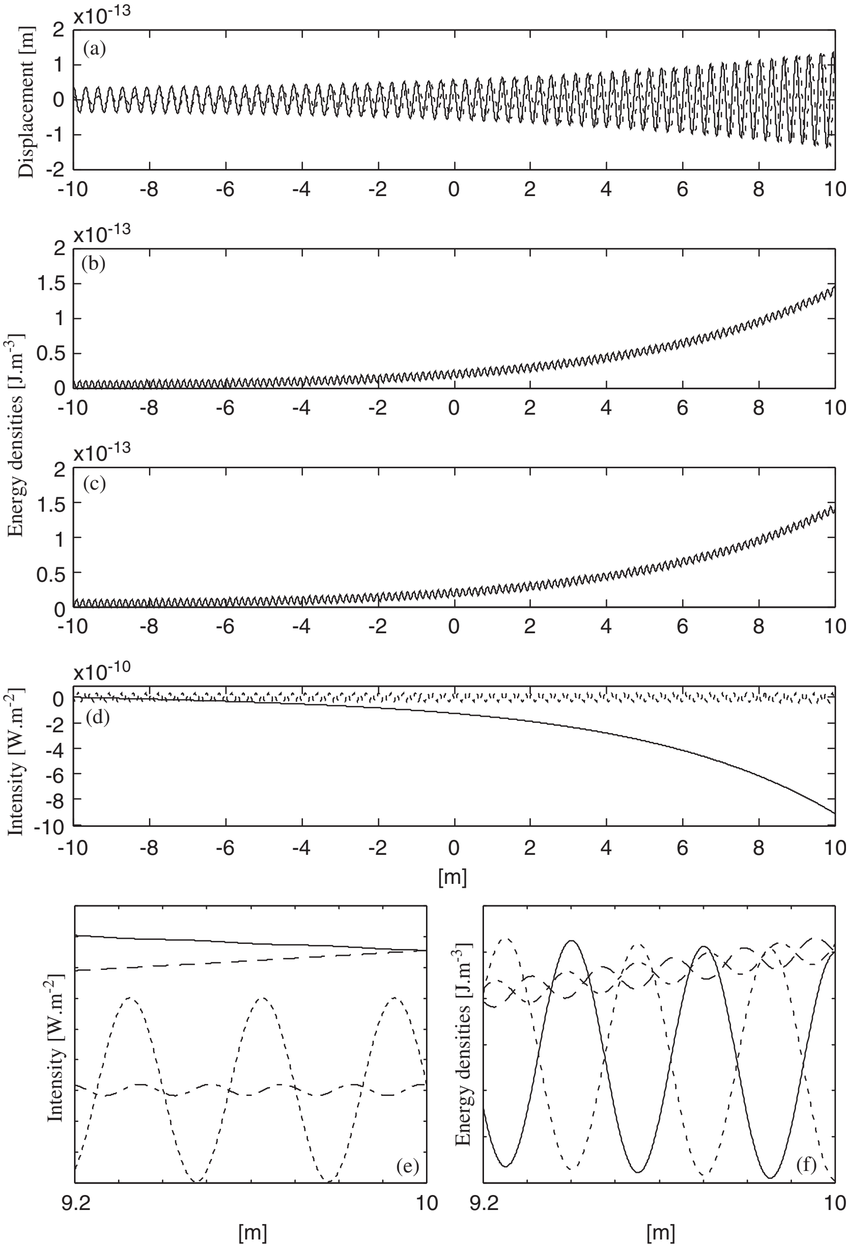

Fig. 5. Agreement of the displacement- and the quadratic-formulations: z-shear wave fields in a 1D system (Section 5, $L_1 = -10$ m, $L_2 = 10$ m, frequency 8 kHz), (a) z-component of displacement (real part —, imaginary part - - -) obtained from displacement formulation (27b); energy variables obtained from Eqs. (13) in using the displacement field (a) and from the quadratic formulation (30a) are superimposed: (b) kinetic energy density T_{tz}, (c) strain energy density $\mathrm{Re}(U_{tz})$, (d) structural intensity $\mathbf{I}_{tz}$ (x-oriented, real part—, imaginary part - -). Detailed views for the wave conversion at the right end $9.2 \leqslant x \leqslant 10$: (e) structural intensity ($\mathrm{Re}(\mathbf{I}_l)$ —, $\mathrm{Im}(\mathbf{I}_l)$ ……, $\mathrm{Re}(-\mathbf{I}_{tz})$ - - -, $\mathrm{Im}(-\mathbf{I}_{tz})$ ·—·—·), (f) energy densities (T_l —, $\mathrm{Re}(U_l)$ ……, T_{tz} - - -, $\mathrm{Re}(U_{tz})$ ·—·—).

6. Conclusion

For harmonic wave fields in homogeneous, isotropic and hysteretic damped materials, the time-averaged energy densities and the structural intensity are not linked by a closed set of equations. Energy variables can be written as linear expressions, provided other quadratic variables are used. From the longitudinal and shear components for displacement, these variables can be presented as the summation of four quadratic

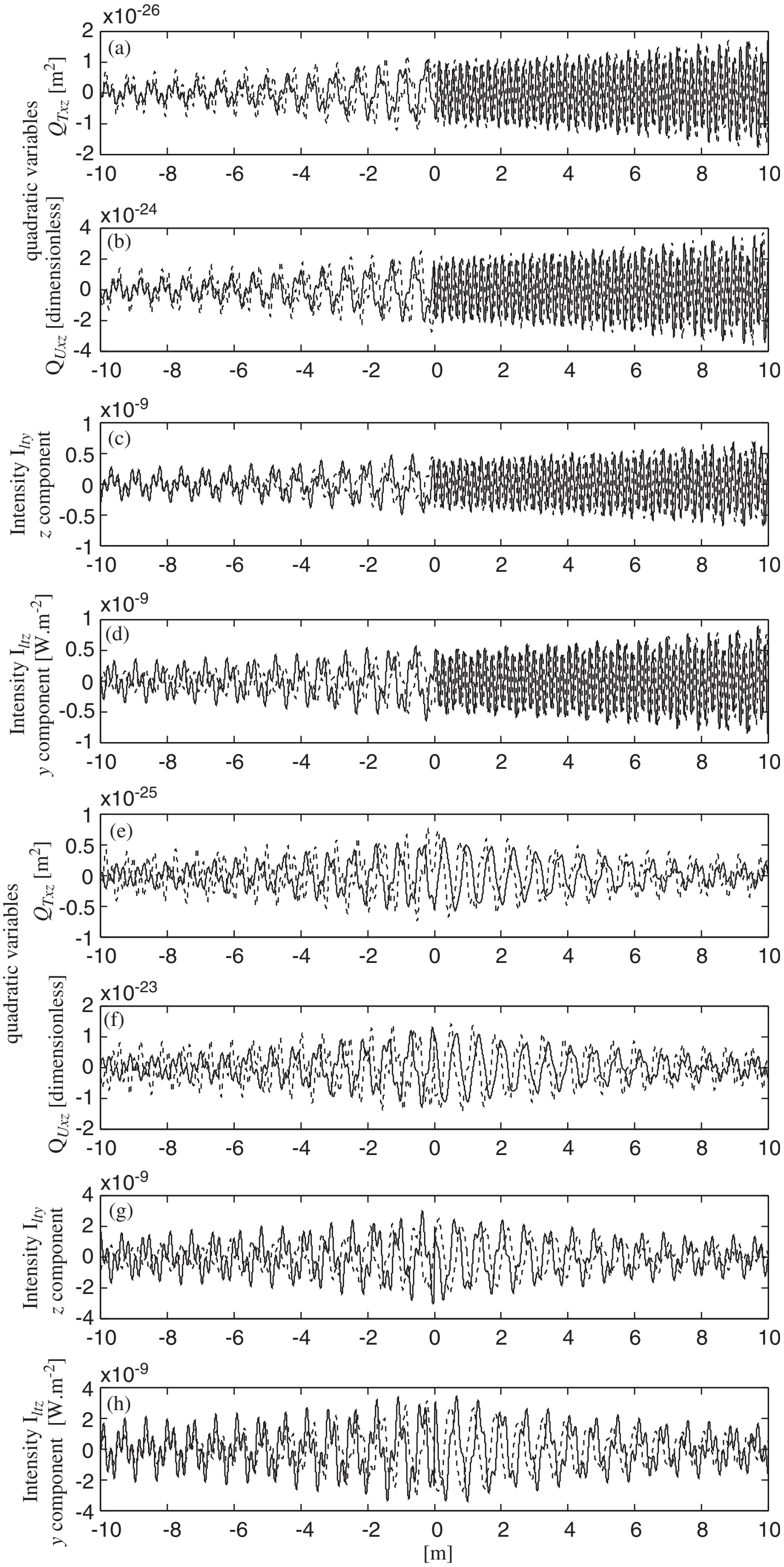

Fig. 6. Agreement of the displacement- and the quadratic-formulations: mixed quadratic components in a 1D system (Section 5, $L_1 = -10\,\text{m}$, $L_2 = 10\,\text{m}$, frequency 8 kHz). The results obtained from Eqs. (13) in using the displacement field and from the quadratic formulation (30b) are superimposed (real part —, imaginary part - - -): (a) Q_{Txy}, (b) Q_{Uxy}, (c) $\mathbf{I}_{ltz}$ (z-oriented), (d) $\mathbf{I}_{tlz}$ (z-oriented), (e) Q_{Txz}, (f) Q_{Uxz}, (g) $\mathbf{I}_{lty}$ (y-oriented), (h) $\mathbf{I}_{tly}$ (y-oriented).

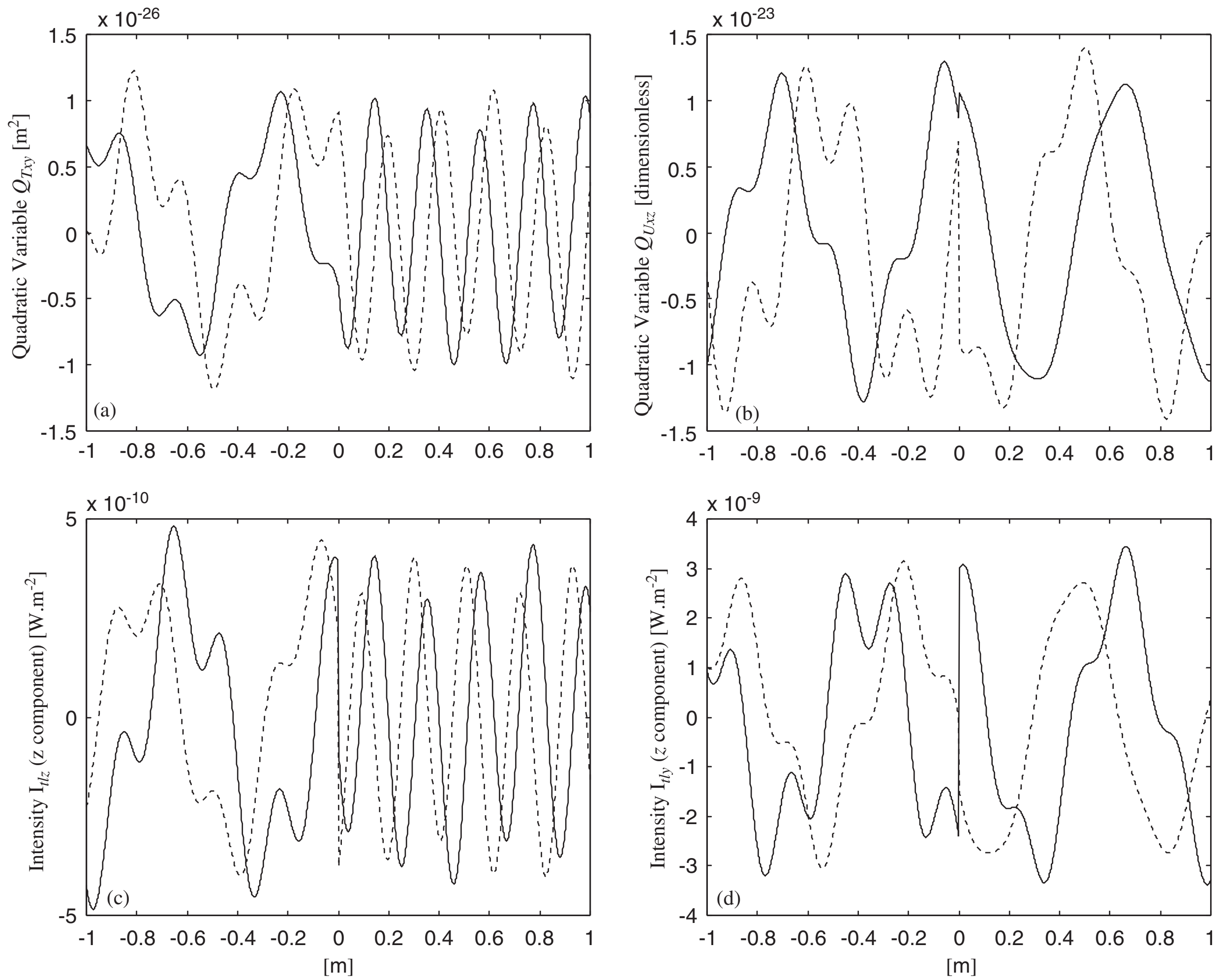

Fig. 7. Detailed views of Fig. 6 for mixed quadratic components for $-1 \leqslant x \leqslant 1$ (real part—, imaginary part - - -): (a) Q_{Txy}, (b) Q_{Uxz}, (c) $\mathbf{I}_{tlz}$ (z-oriented), (d) $\mathbf{I}_{tly}$ (y-oriented).

components, pure longitudinal, pure shear and two mixed ones, each satisfying specific relationships. However, these additional equations are not sufficient to obtain a quadratic formulation, except in the case of 1D systems, where particular simplifications are satisfied by the energy field.

This exact formulation for 1D systems excited by external sources is suited to a wide frequency range, without any other simplifying assumptions such as light damping being necessary. The equivalence of the displacement and the quadratic formulations is illustrated in a system with wave conversion at one end, and excited by concentrated loads. The solution for this quadratic formulation is analyzed in terms of wavenumbers for free waves. However when compared to the displacement formulation, the quadratic formulation for 1D energy fields involves differential equations of higher order, more variables and more complicated expressions for boundary conditions. Another difference is the introduction of the external loads: excitations are characterized by a concentrated force in the displacement formulation, but by many different terms in the quadratic formulation. For example, a concentrated load is characterized by discontinuities of the strain energy density and of the first spatial derivative of kinetic- and strain-energy densities. Moreover, the solutions for the quadratic formulation evidence variations at a spatial scale that is half of the displacement wavelength. As a result of this additional complexity, this complete 1D quadratic formulation is not practically suited to compute the energy field. Nevertheless the present work confirms that 1D systems are the most favorable configuration to develop energy models, and proves that very general assumptions are sufficient to obtain an exact quadratic formulation in such systems.

Acknowledgements

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Appendix A

Formulas

For any vector $\mathbf{q}$, with $\boldsymbol{\varepsilon}$ denoting the symmetric part of the $\mathbf{u}$ vector gradient (Eq. (1)),

$$2\boldsymbol{\varepsilon}\mathbf{q} = \mathrm{grad}(\mathbf{u}\cdot\mathbf{q}) + \mathrm{curl}(\mathbf{u}\times\mathbf{q}) + (\mathrm{curl}\,\mathbf{q})\times\mathbf{u} - \mathbf{u}\,\mathrm{div}\,\mathbf{q} + \mathbf{q}\,\mathrm{div}\,\mathbf{u}, \tag{32a}$$

and

$$2\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon}^* = \nabla^2(\mathbf{u}\cdot\mathbf{u}^*) - \mathbf{u}\cdot\nabla^2\mathbf{u}^* - \nabla^2\mathbf{u}\cdot\mathbf{u}^* - \mathrm{curl}\,\mathbf{u}\cdot\mathrm{curl}\,\mathbf{u}^*. \tag{32b}$$

for any scalar p or vector $\mathbf{q}$

$$\nabla^2(pp^*) = p\nabla^2 p^* + p^*\nabla^2 p + 2\,\mathrm{grad}\,p\cdot\mathrm{grad}\,p^*. \tag{32c}$$

$$\nabla^2(\mathbf{q}\cdot\mathbf{q}^*) = \mathbf{q}\cdot\nabla^2\mathbf{q}^* + \mathbf{q}^*\cdot\nabla^2\mathbf{q} + 2\mathrm{grad}\,\mathbf{q} : \mathrm{grad}^{\mathrm{T}}\,\mathbf{q}^*. \tag{32d}$$

Appendix B

B1. Pure shear component

Expressions for structural intensity and energy densities for pure shear displacement are

$$\mathbf{I}_t = \mathrm{j}\omega\mu\big(\mathrm{curl}\,\mathbf{u}_t^*\times\mathbf{u}_t + \mathrm{grad}(\mathbf{u}_t\cdot\mathbf{u}_t^*) + \mathrm{curl}(\mathbf{u}_t\times\mathbf{u}_t^*)\big)/2, \tag{33a}$$

$$T_t = \rho\omega^2\mathbf{u}_t\cdot\mathbf{u}_t^*/4. \tag{33b}$$

$$U_t = \mu\big(\nabla^2(\mathbf{u}_t\cdot\mathbf{u}_t^*) - \mathbf{u}_t\cdot\nabla^2\mathbf{u}_t^* - \nabla^2\mathbf{u}_t\cdot\mathbf{u}_t^* - \mathrm{curl}\,\mathbf{u}_t\cdot\mathrm{curl}\,\mathbf{u}_t^*\big)/4. \tag{33c}$$

Substituting for shear displacement Eq. (11b) in the divergence and the curl of Eq. (33a) gives

$$\mathrm{div}\,\mathbf{I}_t = \frac{\mathrm{j}\omega}{2}\big(-\mu\,\mathrm{curl}\,\mathbf{u}_t\cdot\mathrm{curl}\,\mathbf{u}_t^* + \mu\nabla^2(\mathbf{u}_t\cdot\mathbf{u}_t^*) + (\mu/\mu^*)\rho\omega^2\mathbf{u}_t\cdot\mathbf{u}_t^* + (\mu/\mu^*)\mathbf{f}_t^*\cdot\mathbf{u}_t\big) \tag{33d}$$

$$\mathrm{curl}\,\mathbf{I}_t = \frac{\mathrm{j}\omega}{2}\begin{pmatrix}\mu\,\mathrm{curl}\,\mathrm{curl}(\mathbf{u}_t\times\mathbf{u}_t^*) - \rho\omega^2(\mu/\mu^*)\mathbf{u}_t\times\mathbf{u}_t^* + \mu\,\mathrm{grad}\big(\mathrm{curl}\,\mathbf{u}_t^*\cdot\mathbf{u}_t\big)\\ -2\mu\boldsymbol{\varepsilon}_t\,\mathrm{curl}\,\mathbf{u}_t^* + (\mu/\mu^*)\mathbf{f}_t^*\times\mathbf{u}_t\end{pmatrix}. \tag{33e}$$

The quadratic variables of Eqs. (33d)–(33e) are the real positive scalars $\mathbf{u}_t\cdot\mathbf{u}_t^*$ and $\mathrm{curl}\,\mathbf{u}_t\cdot\mathrm{curl}\,\mathbf{u}_t^*$, the vectors $\mathbf{u}_t\times\mathbf{u}_t^*$ (imaginary) and $\mathrm{curl}\,\mathbf{u}_t^*\times\mathbf{u}_t$ (complex), and more complicated variables such as $\mathrm{grad}\big(\mathrm{curl}\,\mathbf{u}_t^*\cdot\mathbf{u}_t\big)$ and $\boldsymbol{\varepsilon}_t\,\mathrm{curl}\,\mathbf{u}_t^*$ for $\mathrm{curl}\,\mathbf{I}_t$. These expressions governing shear components then imply more variables than relations (15) linking longitudinal variables. A general simple relationship like (15g) is not available for general shear fields, because the expression of the dot-product Laplacian (32d) for $\mathbf{q} = \boldsymbol{\Psi}$ reveals more complex than expression (32c) for the product of scalars.

B2. Mixed components

Mixed components present similar, but more complicated expressions

$$\mathbf{I}_{lt} = \frac{j\omega}{2}\left((\lambda+\mu)\operatorname{div}\mathbf{u}_l\mathbf{u}_t^* + \mu\left(\operatorname{curl}\mathbf{u}_t^* \times \mathbf{u}_l\right) + \mu\,\operatorname{grad}(\mathbf{u}_l\cdot\mathbf{u}_t^*) + \mu\,\operatorname{curl}(\mathbf{u}_l\times\mathbf{u}_t^*)\right), \tag{34a}$$

$$T_{lt} = \rho\omega^2\mathbf{u}_l\cdot\mathbf{u}_t^*/4, \tag{34b}$$

$$U_{lt} = \frac{\mu}{4}\left(\nabla^2(\mathbf{u}_l\cdot\mathbf{u}_t^*) - \mathbf{u}_l\cdot\nabla^2\mathbf{u}_t^* - \nabla^2\mathbf{u}_l\cdot\mathbf{u}_t^*\right), \tag{34c}$$

$$\operatorname{div}\mathbf{I}_{lt} = \frac{j\omega}{2}\left(\mu\nabla^2(\mathbf{u}_l\cdot\mathbf{u}_t^*) + \left(\frac{\mu}{\lambda+2\mu} - 1 + \frac{\mu}{\mu^*}\right)\rho\omega^2\mathbf{u}_l\cdot\mathbf{u}_t^* - \frac{\lambda+\mu}{\lambda+2\mu}\,\mathbf{f}_l\cdot\mathbf{u}_t^* + \frac{\mu}{\mu^*}\,\mathbf{f}_t^*\cdot\mathbf{u}_l\right), \tag{34d}$$

$$\operatorname{curl}\mathbf{I}_{lt} = \frac{j\omega}{2}\begin{pmatrix}(\lambda+2\mu)\operatorname{div}\mathbf{u}_l\operatorname{curl}\mathbf{u}_t^* - \left(\frac{2\lambda+3\mu}{\lambda+2\mu} + \frac{\mu-\mu^*}{\mu^*}\right)\rho\omega^2\mathbf{u}_l\times\mathbf{u}_t^* - \mu\nabla^2(\mathbf{u}_l\times\mathbf{u}_t^*) - 2\mu\boldsymbol{\varepsilon}_l\operatorname{curl}\mathbf{u}_t^* \\ -\frac{\lambda+\mu}{\lambda+2\mu}\,\mathbf{f}_l\times\mathbf{u}_t^* + \frac{\mu}{\mu^*}\,\mathbf{f}_t^*\times\mathbf{u}_l\end{pmatrix} \tag{34e}$$

and

$$\mathbf{I}_{tl} = \frac{j\omega}{2}\left(-\mu\operatorname{div}\mathbf{u}_l^*\mathbf{u}_t + \mu\,\operatorname{grad}(\mathbf{u}_t\cdot\mathbf{u}_l^*) + \mu\,\operatorname{curl}(\mathbf{u}_t\times\mathbf{u}_l^*)\right). \tag{35a}$$

$$T_{tl} = \rho\omega^2\mathbf{u}_t\cdot\mathbf{u}_l^*/4. \tag{35b}$$

$$U_{tl} = \frac{\mu}{4}\left(\nabla^2(\mathbf{u}_t\cdot\mathbf{u}_l^*) - \mathbf{u}_t\cdot\nabla^2\mathbf{u}_l^* - \nabla^2\mathbf{u}_t\cdot\mathbf{u}_l^*\right), \tag{35c}$$

$$\operatorname{div}\mathbf{I}_{tl} = \frac{j\omega}{2}\left(\mu\nabla^2(\mathbf{u}_t\cdot\mathbf{u}_l^*) + \frac{\mu}{\lambda^*+2\mu^*}\,\rho\omega^2\mathbf{u}_t\cdot\mathbf{u}_l^* + \frac{\mu}{\lambda^*+2\mu^*}\,\mathbf{f}_l^*\cdot\mathbf{u}_t\right), \tag{35d}$$

$$\operatorname{curl}\mathbf{I}_{tl} = \frac{j\omega}{2}\begin{pmatrix}-\mu\operatorname{div}\mathbf{u}_l^*\operatorname{curl}\mathbf{u}_t - \mu\nabla^2(\mathbf{u}_t\times\mathbf{u}_l^*) - \frac{\mu}{\lambda^*+2\mu^*}\,\rho\omega^2\mathbf{u}_t\times\mathbf{u}_l^* \\ +\mu\,\operatorname{grad}\left(\operatorname{curl}\mathbf{u}_t\cdot\mathbf{u}_l^*\right) + \frac{\mu}{\lambda^*+2\mu^*}\,\mathbf{f}_l^*\times\mathbf{u}_t\end{pmatrix}. \tag{35e}$$

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