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Stochastic averaging of quasi-integrable Hamiltonian systems with delayed feedback control

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Abstract

A stochastic averaging method for quasi-integrable Hamiltonian systems with time-delayed feedback control is proposed. First, a quasi-integrable Hamiltonian system with delayed feedback control subjected to Gaussian white noise excitations is formulated and then transformed into Itô stochastic differential equations without time delay. Then, the averaged Itô stochastic differential equations for the system are derived and the stationary solution of the averaged Fokker–Planck–Kolmogorov (FPK) equation associated with the averaged Itô equations is obtained for both non-resonant and resonant cases. Finally, three examples are worked out in detail to illustrate the application and effectiveness of the proposed method and the effect of time delayed feedback control on the response of the systems.

1. Introduction

In the implementation of feedback control of a dynamical system, time delay is usually unavoidable due to the time spent in measuring and estimating the system state, calculating and executing the control forces, etc. This time delay causes unsynchronized application of the control forces and this unsynchronization can not only deteriorate the control performance but also cause instability of the system. Thus, the time delay problem has drawn much attention of the control community.

Systems with time delay under deterministic excitation have been studied extensively [1–6]. The time-delayed systems under stochastic excitation have attracted many researches recently. The multiscale analysis has been adopted to study the effect of noise near critical delay in stochastic delay differential equations by Klosek and Kuske [7]. The center manifold reduction of delay differential equations was used by Fofana [8] to deal with machine-tool chatter problem. The linearly controlled system with deterministic and random time delays excited by Gaussian white noise has been treated by Grigoriu [9] and the stability of such a system has been investigated by means of Lyapunov exponent. The effects of time delay on the controlled linear systems under Gaussian random excitation has been studied by Di Paola and Pirrotta [10] using Taylor expansion of the

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control force. The effect of time delay on nonlinear systems under Gaussian white noise also has been studied by Bilello et al. [11] using the Taylor expansion.

In the present paper, a stochastic averaging method for quasi-integrable Hamiltonian systems with time-delayed feedback control under Gaussian white noise excitation is proposed. The delayed feedback control forces are approximated by control forces without time delay and the system is transformed into Itô stochastic differential equations without time delay, from which the averaged Itô equations are derived. The stationary solution of the averaged FPK equation associated with averaged Itô equations is obtained by using the technique proposed by the second present author and his co-worker [12]. Three examples are worked out in detail to illustrate the application and effectiveness of the proposed procedure and the effect of delayed feedback control on the response of the systems.

2. Quasi-integrable Hamiltonian systems with delayed feedback control

Consider an n-degree-of-freedom ($n$dof) quasi-Hamiltonian system with delayed feedback control forces governed by the following equations:

$$\dot{Q}_i = \frac{\partial H'}{\partial P_i},$$
$$\dot{P}_i = -\frac{\partial H'}{\partial Q_i} - \varepsilon c_{ij}\frac{\partial H'}{\partial P_j} - \varepsilon F_i(\mathbf{Q}_\tau, \mathbf{P}_\tau) + \varepsilon^{1/2} f_{ik} W_k(t), \quad i,j = 1,2,\ldots,n; \quad k = 1,2,\ldots,m, \tag{1}$$

where Q_i and P_i are generalized displacements and momenta, respectively; $H' = H'(\mathbf{Q},\mathbf{P})$ is twice differentiable Hamiltonian; ε is a small positive parameter; $\varepsilon c_{ij} = \varepsilon c_{ij}(\mathbf{Q},\mathbf{P})$ represent the coefficients of quasi-linear dampings; $\varepsilon^{1/2} f_{ik} = \varepsilon^{1/2} f_{ik}(\mathbf{Q},\mathbf{P})$ represent the amplitudes of stochastic excitations; $\varepsilon F_i(\mathbf{Q}_\tau, \mathbf{P}_\tau)$ with $\mathbf{Q}_\tau = \mathbf{Q}(t-\tau)$ and $\mathbf{P}_\tau = \mathbf{P}(t-\tau)$ denote delayed feedback control forces, τ is the time delay; $W_k(t)$ are Gaussian white noises in the sense of Stratonovich with correlation functions

$$E[W_k(t)W_l(t+T)] = 2D_{kl}\delta(T), \quad k,l = 1,2,\ldots,m. \tag{2}$$

When $\varepsilon = 0$, system (1) is reduced to $n$dof Hamiltonian system. It is called integrable or completely integrable if there exist n independent integrals of motion, $H_1 = H, H_2, \ldots, H_n$, which are in involution. The term "in involution" means that all H_i are commute with each other, i.e.,

$$[H_i, H_j] = 0, \quad i,j = 1,2,\ldots,n, \tag{3}$$

where

$$[H_i, H_j] = \frac{\partial H_i}{\partial p_k}\frac{\partial H_j}{\partial q_k} - \frac{\partial H_i}{\partial q_k}\frac{\partial H_j}{\partial p_k}, \quad k = 1,2,\ldots,n \tag{4}$$

is the Poisson bracket of H_i and H_j.

In principle, a canonical transformation

$$I_i = I_i(\mathbf{q},\mathbf{p}), \theta_i = \theta_i(\mathbf{q},\mathbf{p}), \quad i = 1,2,\ldots,n \tag{5}$$

can be introduced so that the Hamiltonian equations of an integrable Hamiltonian system are of the form

$$\dot{I}_i = -\frac{\partial}{\partial \theta_i} H(\mathbf{I}) = 0,$$
$$\dot{\theta}_i = \frac{\partial}{\partial I_i} H(\mathbf{I}) = \omega_i(\mathbf{I}), \tag{6}$$

where I_i and θ_i are action-angle variables and $\omega_i(\mathbf{I})$ are the frequencies of the system. Eq. (6) can be easily solved to yield

$$I_i = \text{const.},$$
$$\theta_i = \omega_i(\mathbf{I})t + \delta_i, \quad i = 1,2,\ldots,n, \tag{7}$$

where δ_i are constants. It is seen from Eq. (7) that the motion of an integrable Hamiltonian system is almost periodic or periodic depending upon the number of the strong resonant relations of the form

$$k_i^u \omega_i = 0, \quad u = 1, 2, \ldots, \alpha; \quad i = 1, 2, \ldots, n \tag{8}$$

among $\omega_i(\mathbf{I})$, where $k_i{}^u$ are integers and α is the number of resonant relationships. If there is no resonant relation, then the Hamiltonian system is called non-resonant. The motion of non-resonant integrable Hamiltonian system is almost periodic and a single orbit covers n-dimensional tore uniformly. If there are $n-1$ resonant relations, then the system is called completely resonant and the motion of the system is periodic. If the number of resonant relations is between 1 and $n-1$, then the system is called partially resonant and the motion of the system is almost periodic.

It is noted that n action variable I_i can be regarded as n independent integrals of motion in involution, satisfying Eq. (3), and the frequencies of truly nonlinear Hamiltonian systems are functions of integrals of motion or action variables.

If the Hamiltonian system associated with Eq. (1) is integrable, then system (1) is called quasi-integrable Hamiltonian system. This system can be modeled as Stratonovich stochastic differential equations and then converted into Itô stochastic differential equation by adding Wong–Zakai correction terms [13], i.e.,

$$\begin{aligned} dQ_i &= \frac{\partial H'}{\partial P_i}, \\ dP_i &= -\left[\frac{\partial H'}{\partial Q_i} + \varepsilon c_{ij}\frac{\partial H'}{\partial P_i} + \varepsilon F_i(\mathbf{Q}_\tau, \mathbf{P}_\tau) - \varepsilon D_{kl} f_{jl}\frac{\partial f_{ik}}{\partial P_j}\right] dt + \varepsilon^{1/2}\sigma_{ik}\, dB_k(t), \quad i,j = 1,2,\ldots,n; \quad k = 1,2,\ldots,m, \end{aligned} \tag{9}$$

where $B_i(t)$ are standard Wiener processes and $\boldsymbol{\sigma\sigma}^T = 2\mathbf{fDf}^T$. The double summation terms $\varepsilon D_{kl} f_{jl} \partial f_{ik}/\partial P_j$ in Eq. (9) are the Wong–Zakai correction terms.

Assume that the Hamiltonian H' associated with system (1) is of the form

$$H' = \sum_{i=1}^{n} H_i'(q_i, p_i), \quad H_i' = \frac{1}{2} p_i^2 + G(q_i), \tag{10}$$

where $G(q_i) \geqslant 0$ is symmetric with respect to the $q_i = 0$, and with minimum at $q_i = 0$. Then the associated Hamiltonian system has a family of periodic solutions around the origin and the solution to Eq. (9) is of the form [14,15]

$$Q_i(t) = A_i \cos \Phi_i(t), \quad P_i(t) = -A_i \frac{d\Theta_i}{dt} \sin \Phi_i(t), \quad \Phi_i(t) = \Theta_i(t) + \Gamma_i(t), \tag{11}$$

where $\cos \Phi(t)$ and $\sin \Phi(t)$ are called generalized harmonic functions. For quasi-integrable Hamiltonian systems, $A_i(t)$ and $\Gamma_i(t)$ are slowly varying processes and the average value of the instantaneous frequency $d\Theta_i/dt$ is equal to $\omega_i(A_i)$ [14,15]. If $A_i(t-\tau)$ and $\Gamma_i(t-\tau)$ are approximated by $A_i(t)$ and $\Gamma_i(t)$, respectively, and $\Theta_i(t-\tau)$ is approximated by $\Theta_i(t) - \omega_i\tau$, then we have the following approximate expressions:

$$\begin{aligned} Q_i(t-\tau) &= A_i(t-\tau)\cos \Phi_i(t-\tau) \doteq A_i(t) \cos[\omega_i(t-\tau) + \Gamma_i(t)] \\ &= A_i(t)\{\cos[\omega_i t + \Gamma_i(t)]\cos \omega_i\tau + \sin[\omega_i t + \Gamma_i(t)]\sin \omega_i\tau\} = Q_i(t)\cos \omega_i\tau - \frac{P_i}{\omega_i}\sin \omega_i\tau, \end{aligned}$$

$$\begin{aligned} P_i(t-\tau) &= -A_i(t-\tau)\frac{d\Theta_i(t-\tau)}{dt}\sin \Phi_i(t-\tau) \doteq -A_i(t)\omega_i \sin[\omega_i(t-\tau) + \Gamma_i(t)] \\ &= -A_i(t)\omega_i\{\sin[\omega_i t + \Gamma_i(t)]\cos \omega_i\tau + \cos[\omega_i t + \Gamma_i(t)]\sin \omega_i\tau\} = P_i \cos \omega_i\tau + Q_i(t)\omega_i \sin \omega_i\tau. \end{aligned} \tag{12}$$

The numerical results for three example described in Section 4 will show that Eq. (12) is acceptable even for some large time delay τ:

$\varepsilon F(\mathbf{Q}_\tau, \mathbf{P}_\tau) - \varepsilon D_{kl} f_{jl} \partial f_{ik}/\partial P_j$ in Eq. (9) can be split into two parts: one has the effect of modifying the conservative forces and the other modifying the damping forces. The first part can be combined with $-\partial H/\partial Q_i$ to form an overall effective conservative forces $-\partial H/\partial Q_i$ with a new Hamiltonian $H = H(\mathbf{Q}, \mathbf{P}; \tau)$ and with $\partial H/\partial P_i = \partial H/\partial P_i$. The second part may be combined with $-\varepsilon c_{ij}\,\partial H'/\partial P_j$ to constitute an effective damping forces $-\varepsilon m_{ij}\,\partial H/\partial P_i$ with $m_{ij} = m_{ij}(\mathbf{Q}, \mathbf{P}; \tau)$. With these accomplished, Eq. (9) can be rewritten as

$$\mathrm{d}Q_i = \frac{\partial H}{\partial P_i}\,\mathrm{d}t,$$
$$\mathrm{d}P_i = -\left(\frac{\partial H}{\partial Q_i} + \varepsilon m_{ij}\frac{\partial H}{\partial P_j}\right)\mathrm{d}t + \varepsilon^{1/2} f_{ik}\,\mathrm{d}B_k(t), \quad i,j = 1,2,\ldots,n; \quad k = 1,2,\ldots,m, \tag{13}$$

which is the Itô equations for regular quasi-Hamiltonian systems.

3. Averaged equations and stationary solutions

The stochastic averaging method for quasi-Hamiltonian systems has been well developed [12,16,17,19]. The dimension and form of the averaged Itô and FPK equations depend upon the integrability and resonance of the associated Hamiltonian system. Suppose that the Hamiltonian system with modified Hamiltonian H is still integrable. Then the stochastic averaging method for quasi-integrable Hamiltonian systems [12] can be applied to the system governed by Eq. (13). The dimension and form of the averaged Itô and FPK equations depend upon the resonance of the associated Hamiltonian system with modified Hamiltonian H.

3.1. Non-resonant case

In this case, the averaged Itô equations are of the form

$$\mathrm{d}I_r = \varepsilon U_r(\mathbf{I})\,\mathrm{d}t + \varepsilon^{1/2} V_{rk}(\mathbf{I})\,\mathrm{d}B_k(t), \quad r = 1,2,\ldots,n; \quad k = 1,2,\ldots,m \tag{14}$$

and the averaged FPK equation is of the form

$$\frac{\partial p}{\partial t} = \varepsilon\left\{-\frac{\partial}{\partial I_r}[a_r(I)p] + \frac{1}{2}\frac{\partial^2}{\partial I_r\,\partial I_s}[b_{rs}(I)p]\right\}. \tag{15}$$

In Eqs. (14) and (15),

$$a_r(\mathbf{I}) = U_r(\mathbf{I}) = \frac{1}{(2\pi)^n}\int_0^{2\pi}\left(-m_{ij}\frac{\partial H}{\partial p_j}\frac{\partial I_r}{\partial p_i} + D_{kl} f_{ik} f_{jl}\frac{\partial^2 I_r}{\partial p_i \partial p_j}\right)\mathrm{d}\boldsymbol{\theta},$$
$$b_{rs}(\mathbf{I}) = [\mathbf{V}\mathbf{V}^{\mathrm{T}}]_{rs} = \frac{1}{(2\pi)^n}\int_0^{2\pi}\left(2D_{kl} f_{ik} f_{jl}\frac{\partial I_r}{\partial p_i}\frac{\partial I_s}{\partial p_j}\right)\mathrm{d}\boldsymbol{\theta}, \quad r,s,i,j = 1,2,\ldots,n; \quad k,l = 1,2,\ldots,m \tag{16}$$

in which $\boldsymbol{\theta} = [\theta_1, \theta_2, ..., \theta_n]^{\mathrm{T}}$, $\mathbf{V} = [V_{rk}]$ and $\int_0^{2\pi}[\bullet]\mathrm{d}\boldsymbol{\theta}$ denotes an n-fold integral.

The exact stationary solution of FPK Eq. (15) with vanish probability potential flow at boundary is of the form

$$p(\mathbf{I}) = C\exp[-\lambda(\mathbf{I})], \tag{17}$$

where C is a normalization constant and $\lambda(\mathbf{I})$ is the so-called probability potential which is governed by equations

$$b_{rs}\frac{\partial\lambda}{\partial I_s} = \frac{\partial b_{rs}}{\partial I_s} - 2a_r, \quad r,s = 1,2,\ldots,n. \tag{18}$$

If diffusion matrix $\mathbf{B} = [b_{rs}]$ is not singular, i.e., its inverse matrix $\mathbf{B}^{-1} = \mathbf{G} = [g_{rs}]$ exists, then Eq. (18) can be converted into

$$\frac{\partial\lambda}{\partial I_i} = g_{ir}\left(\frac{\partial b_{rs}}{\partial I_s} - 2a_r\right). \tag{19}$$

Furthermore, if the following compatibility conditions

$$\frac{\partial}{\partial I_j} g_{ir}\left(\frac{\partial b_{rs}}{\partial I_s} - 2a_r\right) = \frac{\partial}{\partial I_i} g_{jr}\left(\frac{\partial b_{rs}}{\partial I_s} - 2a_r\right) \tag{20}$$

are satisfied, then the probability potential is

$$\lambda(\mathbf{I}) = \lambda_0 + \int_0^{\mathbf{I}} \frac{\partial \lambda}{\partial I_s} \, dI_s, \tag{21}$$

where $\lambda_0 = \lambda(0)$ and the second term is a summation of line integrals over $s = 1, 2, \ldots, n$. The exact stationary solution $p(\mathbf{I})$ of averaged FPK Eq. (15) is obtained by substituting Eq. (21) into Eq. (17). The approximate stationary probability density of system (1) is then

$$p(\mathbf{q}, \mathbf{p}) = p(\mathbf{I}, \boldsymbol{\theta}) \left| \frac{\partial(\mathbf{I}, \boldsymbol{\theta})}{\partial(\mathbf{q}, \mathbf{p})} \right| = p(\boldsymbol{\theta}|\mathbf{I}) p(\mathbf{I}) \left| \frac{\partial(\mathbf{I}, \boldsymbol{\theta})}{\partial(\mathbf{q}, \mathbf{p})} \right| = \frac{1}{(2\pi)^n} p(\mathbf{I}), \tag{22}$$

where $|\partial(\mathbf{I}, \boldsymbol{\theta})/\partial(\mathbf{q}, \mathbf{p})|$ is the absolute value of the Jacobian determinant of the canonical transformations from $\mathbf{q}$,$\mathbf{p}$ to $\mathbf{I}$,$\boldsymbol{\theta}$ which is always equal to unity.

If the action-angle variables $\mathbf{I}$,$\boldsymbol{\theta}$ for Hamiltonian system with Hamiltonian H can not be obtained, then the averaged Itô equations for independent integrals of motion may be derived. Suppose that Eq. (10) still holds for the modified Hamiltonian, i.e.,

$$H = \sum_{i=1}^{n} H_i(q_i, p_i), \quad H_i = p_i^2/2 + G_i(q_i) \tag{23}$$

then the averaged Itô equations are of the form

$$dH_r = \varepsilon \bar{m}_r(\mathbf{H}) dt + \varepsilon^{1/2} \bar{\sigma}_{rk}(\mathbf{H}) \, dB_k(t), \quad r = 1, 2, \ldots, n; \quad k = 1, 2, \ldots, m \tag{24}$$

and the averaged FPK equation is of the form

$$\frac{\partial p}{\partial t} = \varepsilon \left\{ -\frac{\partial}{\partial H_r} [a_r(\mathbf{H}) p] + \frac{1}{2} \frac{\partial^2}{\partial H_r \partial H_s} [b_{rs}(\mathbf{H}) p] \right\}, \tag{25}$$

where the averaged drift and diffusion coefficients are

$$a_r(\mathbf{H}) = \bar{m}_r(\mathbf{H}) = \frac{1}{T} \oint \left(-m_{ij} \frac{\partial H}{\partial p_j} \frac{\partial H_r}{\partial p_i} + \frac{1}{2} \sigma_{ik} \sigma_{jk} \frac{\partial^2 H_r}{\partial p_i \partial p_j} \right) \prod_{u=1}^{n} \left(1 \Big/ \frac{\partial H_u}{\partial p_u} \right) dq_u,$$

$$b_{rs}(\mathbf{H}) = \bar{\sigma}_{rk} \bar{\sigma}_{sk}(\mathbf{H}) = \frac{1}{T} \oint \prod_{u=1}^{n} \left(\sigma_{ik} \sigma_{jk} \frac{\partial H_r}{\partial p_i} \frac{\partial H_r}{\partial p_j} \Big/ \frac{\partial H_u}{\partial p_u} \right) dq_u \tag{26}$$

in which

$$T = T(\mathbf{H}) = \prod_{u=1}^{n} T_u(H_u) = \prod_{u=1}^{n} \oint \left(1 \Big/ \frac{\partial H_u}{\partial p_u} \right) dq_u. \tag{27}$$

The exact stationary solution $p(\mathbf{H})$ to averaged FPK Eq. (25) can be obtained similarly and the approximate stationary probability density of system (1) is then

$$p(\mathbf{q}, \mathbf{p}) = p(\mathbf{H})/T(\mathbf{H}). \tag{28}$$

3.2. Resonant case

Suppose that the modified Hamiltonian system with Hamiltonian H is integrable and resonant with α weak resonant relations of the form

$$k_r^u \omega_r = 0(\varepsilon), \quad u = 1, 2, \ldots, \alpha; \quad r = 1, 2, \ldots, n. \tag{29}$$

Then the averaged Itô equations are of the form

$$\begin{aligned} dI_r &= \varepsilon \bar{m}_r(\mathbf{I}, \boldsymbol{\Psi}) \, dt + \varepsilon^{1/2} \bar{\sigma}_{rk}(\mathbf{I}, \boldsymbol{\Psi}) \, dB_k(t), \\ d\Psi_u &= \varepsilon \bar{m}_{n+u}(\mathbf{I}, \boldsymbol{\Psi}) \, dt + \varepsilon^{1/2} \bar{\sigma}_{n+u,k}(\mathbf{I}, \boldsymbol{\Psi}) \, dB_k(t), \quad r = 1, 2, \ldots, n; \quad u = 1, 2, \ldots, \alpha; \quad k = 1, 2, \ldots, m \end{aligned} \tag{30}$$

and the averaged FPK equation is of the form

$$\frac{\partial p}{\partial t} = \varepsilon\left[-\frac{\partial}{\partial I_r}(a_r p) - \frac{\partial}{\partial \psi_u}(a_{n+u}p) + \frac{1}{2}\frac{\partial^2}{\partial I_r \partial I_s}(b_{rs}p) + \frac{\partial^2}{\partial I_r \partial \psi_u}(b_{r,n+u}p) + \frac{1}{2}\frac{\partial^2}{\partial \psi_u \partial \psi_v}(b_{n+u,n+v}p)\right], \tag{31}$$

where $\mathbf{\Psi} = [\Psi_1, \Psi_2, \ldots, \Psi_\alpha]^T$, $\psi_u = k_r^u \Theta_r$, $k = 1, 2, \ldots, \alpha$ and

$$\begin{aligned}
&\bar{m}_r(\mathbf{I}, \mathbf{\Psi}) = \frac{1}{(2\pi)^{n-\alpha}} \int_0^{2\pi} \left(-m_{ij}\frac{\partial H}{\partial p_j}\frac{\partial I_r}{\partial p_i} + \frac{1}{2}\sigma_{ik}\sigma_{jk}\frac{\partial^2 I_r}{\partial p_i \partial p_j}\right) \mathrm{d}\boldsymbol{\theta}_1, \\
&\bar{m}_{n+u}(\mathbf{I}, \mathbf{\Psi}) = \frac{1}{(2\pi)^{n-\alpha}} \int_0^{2\pi} \left(O_u(\varepsilon) - m_{ij}\frac{\partial H}{\partial p_j}\frac{\partial \Psi_r}{\partial p_i} + \frac{1}{2}\sigma_{ik}\sigma_{jk}\frac{\partial^2 \Psi_r}{\partial p_i \partial p_j}\right) \mathrm{d}\boldsymbol{\theta}_1, \\
&\bar{\sigma}_{ik}\bar{\sigma}_{sk}(\mathbf{I}, \mathbf{\Psi}) = \frac{1}{(2\pi)^{n-\alpha}} \int_0^{2\pi} \sigma_{ik}\sigma_{jk}\frac{\partial I_r}{\partial p_i}\frac{\partial I_s}{\partial p_j}\,\mathrm{d}\boldsymbol{\theta}_1, \\
&\bar{\sigma}_{rk}\bar{\sigma}_{n+u,k}(\mathbf{I}, \mathbf{\Psi}) = \frac{1}{(2\pi)^{n-\alpha}} \int_0^{2\pi} \sigma_{ik}\sigma_{jk}\frac{\partial I_r}{\partial p_i}\frac{\partial \Psi_u}{\partial p_j}\,\mathrm{d}\boldsymbol{\theta}_1, \\
&\bar{\sigma}_{n+u,k}\bar{\sigma}_{n+v,k}(\mathbf{I}, \mathbf{\Psi}) = \frac{1}{(2\pi)^{n-\alpha}} \int_0^{2\pi} \sigma_{ik}\sigma_{jk}\frac{\partial \Psi_u}{\partial p_i}\frac{\partial \Psi_v}{\partial p_j}\,\mathrm{d}\boldsymbol{\theta}_1, \\
&a_r = a_r(\mathbf{I}, \boldsymbol{\psi}) = \bar{m}_r(\mathbf{I}, \mathbf{\Psi}), \\
&a_{n+u} = a_{n+u}(\mathbf{I}, \boldsymbol{\psi}) = \bar{m}_{n+u}(\mathbf{I}, \mathbf{\Psi}), \\
&b_{rs} = b_{rs}(\mathbf{I}, \boldsymbol{\psi}) = \bar{\sigma}_{rk}\bar{\sigma}_{sk}(\mathbf{I}, \mathbf{\Psi}), \\
&b_{r,n+u} = b_{r,n+u}(\mathbf{I}, \boldsymbol{\psi}) = \bar{\sigma}_{rk}\bar{\sigma}_{n+u,k}(\mathbf{I}, \mathbf{\Psi}), \\
&b_{n+u,n+v} = b_{n+u,n+v}(\mathbf{I}, \boldsymbol{\psi}) = \bar{\sigma}_{n+u,k}\bar{\sigma}_{n+v,k}(\mathbf{I}, \mathbf{\Psi}), \quad r, s, i, j = 1, 2, \ldots, n; \quad u, v = 1, 2, \ldots, \alpha; \quad k, l = 1, 2, \ldots, m
\end{aligned} \tag{32}$$

in which $\boldsymbol{\theta}_1 = [\theta_1, \theta_2, \ldots, \theta_{n-r}]^T$. The exact stationary solution to averaged FPK Eq. (31) is of the form

$$p(\mathbf{I}, \boldsymbol{\psi}) = C \exp[-\lambda(\mathbf{I}, \boldsymbol{\psi})], \tag{33}$$

where $\lambda(\mathbf{I}, \boldsymbol{\psi})$ can be obtained by expanding $\lambda(\mathbf{I}, \boldsymbol{\psi})$, the averaged drift and diffusion coefficients into α-fold Fourier expansions of $\boldsymbol{\psi}$, substituting them into averaged FPK Eq. (31) with $\partial p/\partial t = 0$ and obtaining the Fourier coefficients of $\lambda(\mathbf{I},\boldsymbol{\psi})$. The approximate stationary solution of system (1) is then

$$p(\mathbf{q}, \mathbf{p}) = p(\mathbf{I}, \boldsymbol{\psi}, \boldsymbol{\theta}_1)\left|\frac{\partial(\mathbf{I}, \boldsymbol{\psi}, \boldsymbol{\theta}_1)}{\partial(\mathbf{q}, \mathbf{p})}\right| = p(\boldsymbol{\theta}_1|\mathbf{I}, \boldsymbol{\psi})p(\mathbf{I}, \boldsymbol{\psi})\left|\frac{\partial(\mathbf{I}, \boldsymbol{\psi}, \boldsymbol{\theta}_1)}{\partial(\mathbf{q}, \mathbf{p})}\right| = \frac{1}{(2\pi)^{(n-\alpha)}}p(\mathbf{I}, \boldsymbol{\psi})\left|\frac{\partial(\mathbf{I}, \boldsymbol{\psi}, \boldsymbol{\theta}_1)}{\partial(\mathbf{q}, \mathbf{p})}\right|, \tag{34}$$

where $|\partial(\mathbf{I}, \boldsymbol{\psi}, \boldsymbol{\theta}_1)/\partial(\mathbf{q}, \mathbf{p})|$ is the absolute value of the Jacobian determinant for the transformation from $\mathbf{q}$, $\mathbf{p}$ to $\mathbf{I}$, $\boldsymbol{\psi}$, $\boldsymbol{\theta}_1$, which is an integer.

4. Examples

Three examples are given to illustrate the application and effectiveness of the proposed method.

4.1. Example 1

Consider a van der Pol oscillator with time-delayed linear feedback control subject to Gaussian white noise excitation. The equation of motion is

$$\ddot{X} + \omega'^2 X = \varepsilon(1 - X^2)\dot{X} - \varepsilon(a_1 X_\tau + a_2 \dot{X}_\tau) + \varepsilon^{1/2} W(t), \tag{35}$$

where ε is a small positive parameter; $X_\tau = X(t - \tau)$ and $\dot{X}_\tau = \dot{X}(t - \tau)$ are delayed system state; a_1 and a_2 are feedback control gains; $W(t)$ is a Gaussian white noise with intensity $2D$. System (35) without stochastic excitation has been studied by Atay [18] using averaging method.

Note that there is no Wong–Zakai correction term for this example. Let $X = Q$, $\dot{X} = P$. The Hamiltonian associated with the system (35) is

$$H' = I'\omega' = (\omega'^2 q^2 + p^2)/2, \tag{36}$$

The time-delayed feedback control forces in system (35) can be approximately converted into one without time delay, i.e.,

$$\varepsilon(a_1 Q_\tau + a_2 P_\tau) \doteq \varepsilon\left[(a_1 \cos \omega'\tau + a_2\omega' \sin \omega'\tau)Q + \left(a_2 \cos \omega'\tau - \frac{a_1}{\omega'} \sin \omega'\tau\right)P\right]. \tag{37}$$

Note that on the right-hand side of Eq. (37), the term proportional to Q represents conservative control force while that proportional to P represents dissipative control force. The first term should be combined with the restoring force $\omega'^2 Q$ into a modified restoring force $\omega'^2 Q + \varepsilon(a_1 \cos \omega'\tau + a_2\omega' \sin \omega'\tau)Q$. Thus, the modified Hamiltonian is

$$H = I\omega = [p^2 + (\omega'^2 + \varepsilon a_1 \cos \omega'\tau + \varepsilon a_2\omega' \sin \omega'\tau)q^2]/2 \tag{38}$$

and the modified frequency is

$$\omega = (\omega'^2 + \varepsilon a_1 \cos \omega'\tau + \varepsilon a_2\omega' \sin \omega'\tau)^{1/2}. \tag{39}$$

The action variable is $I = H/\omega$.

Applying the stochastic averaging method to the modified system leads to the averaged Itô equation

$$\mathrm{d}I = \varepsilon U(I)\,\mathrm{d}t + \varepsilon^{1/2} V(I)\,\mathrm{d}B(t) \tag{40}$$

and averaged FPK equation

$$\frac{\partial p}{\partial t} = \varepsilon\left\{-\frac{\partial}{\partial I}[a(I)p] + \frac{1}{2}\frac{\partial^2}{\partial I\,\partial I}[b(I)p]\right\}, \tag{41}$$

where

$$a(I) = U(I) = \left(1 + \frac{a_1}{\omega}\sin \omega\tau - a_2 \cos \omega\tau\right)I - \frac{I^2}{2\omega} + \frac{D}{\omega},$$
$$b(I) = V^2(I) = \frac{2DI}{\omega}. \tag{42}$$

The exact stationary solutions to FPK Eq. (41) is

$$p(I) = C \exp[-\lambda(I)], \tag{43}$$

where

$$\lambda(I) = \frac{I^2}{4D} - \frac{\omega I}{D}\left(1 + \frac{a_1}{\omega}\sin \omega\tau - a_2 \cos \omega\tau\right). \tag{44}$$

The approximate stationary probability density of original system (35) is then

$$p(x, \dot{x}) = \frac{1}{2\pi} p(I)|_{I=(\dot{x}^2+\omega^2x^2)/2\omega}. \tag{45}$$

Some numerical results for stationary marginal probability density $p(x)$ of system (35) with displacement, velocity and both displacement and velocity feedback controls obtained by using the proposed stochastic averaging method and from digital simulation are shown in Figs. 1–3, respectively. It is seen that in all three cases the results obtained by using the proposed method agree well with those from digital simulation even for long delay time. From Eq. (37) it is seen that the time-delayed feedback control forces change both the nature frequency and damping coefficient of the oscillator in a manner of harmonic function with periodic $T = 2\pi/\omega'$. So, both the stability and response may be affected by the time-delayed feedback control. For example, in the case of displacement feedback control without time delay, the response of the system is a diffused limit cycle (see Fig. 1(a)) while it is random vibration around the origin when time delay $\tau = 5.0$ (see Fig. 1(f)). This implies that time-delayed feedback control may cause phenomenological bifurcation.

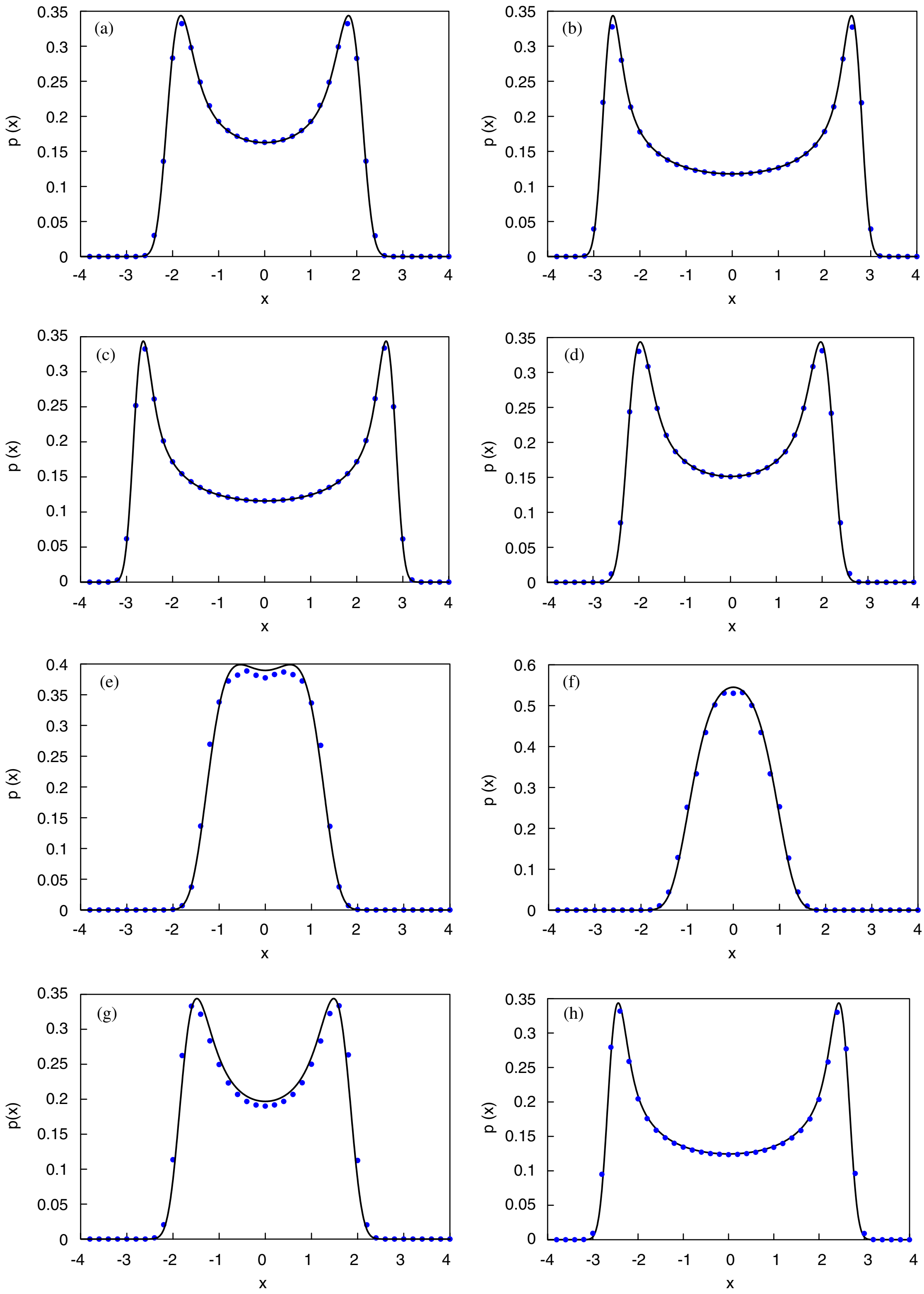

Fig. 1. Stationary marginal probability density $p(x)$ of system (35) with displacement feedback. The parameters are: $\omega' = 1$, $a_1 = 1$, $a_2 = 0$, $\varepsilon = 0.01$, $2D = 0.2$, (a) $\tau = 0$, (b) $\tau = 1$, (c) $\tau = 2$, (d) $\tau = 3$, (e) $\tau = 4$, (f) $\tau = 5$, (g) $\tau = 6$, (h) $\tau = 7$. — By using the proposed stochastic averaging method; ● from digital simulation.

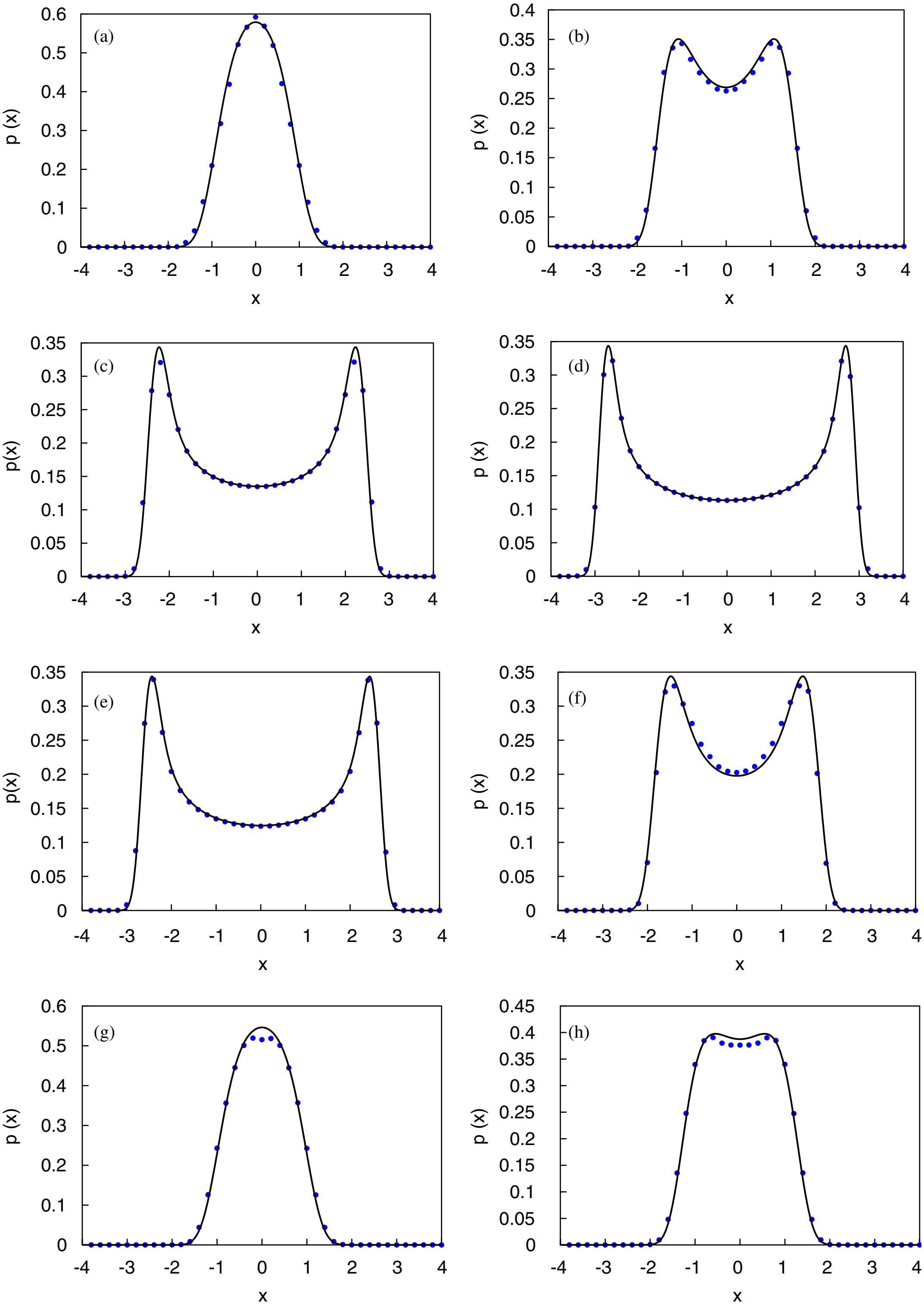

Fig. 2. Stationary marginal probability density $p(x)$ of system (35) with velocity feedback. The parameters are: $\omega' = 1$, $a_1 = 0$, $a_2 = 1$, $\varepsilon = 0.01$, $2D = 0.2$, (a) $\tau = 0$, (b) $\tau = 1$, (c) $\tau = 2$, (d) $\tau = 3$, (e) $\tau = 4$, (f) $\tau = 5$, (g) $\tau = 6$, (h) $\tau = 7$. — By using the proposed stochastic averaging method; ● from digital simulation.

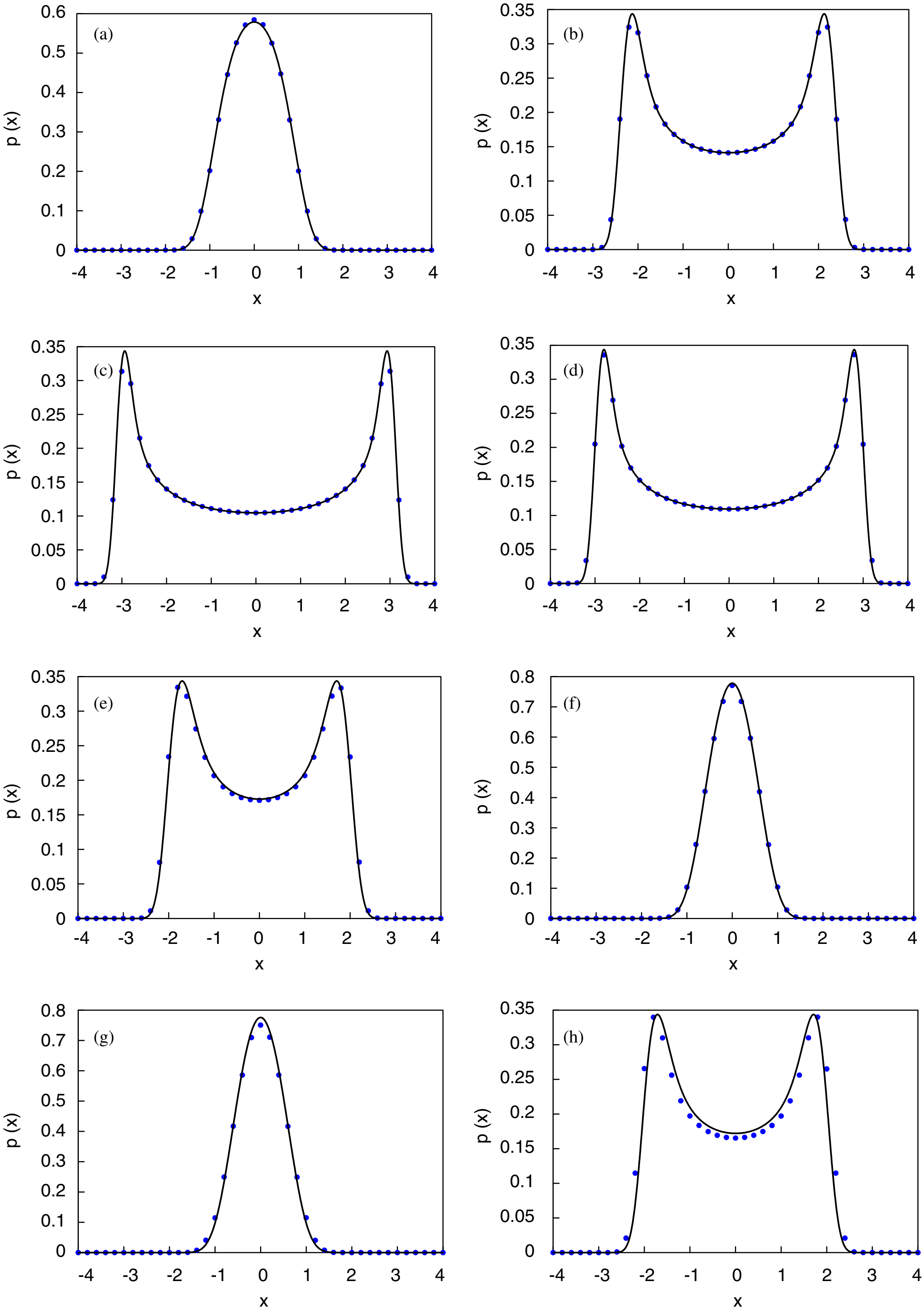

Fig. 3. Stationary marginal probability density $p(x)$ of system (35) with both displacement and velocity feedback. The parameters are: $\omega' = 1$, $a_1 = 1$, $a_2 = 1$, $\varepsilon = 0.01$, $2D = 0.2$, (a) $\tau = 0$, (b) $\tau = 1$, (c) $\tau = 2$, (d) $\tau = 3$, (e) $\tau = 4$, (f) $\tau = 5$, (g) $\tau = 6$, (h) $\tau = 7$. — By using the proposed stochastic averaging method; ● from digital simulation.

It happens also in the case of velocity feedback control (see Fig. 2) and in the case of both displacement and velocity feedback controls (see Fig. 3).

4.2. Example 2

Consider two linear oscillators coupled by linear and polynomial type nonlinear dampings subject to external excitations of two uncorrelated Gaussian white noises and delayed velocity feedback control. The equations of motion of the system are of the form

$$\begin{aligned}&\ddot{X}_1+\varepsilon[\alpha'_{11}\dot{X}_1+\alpha_{12}\dot{X}_2+\beta_1\dot{X}_1(X_1^2+X_2^2)]+\omega'^2_1X_1=-\varepsilon\eta_1\dot{X}_{1\tau}+\varepsilon^{1/2}W_1(t),\\&\ddot{X}_2+\varepsilon[\alpha_{21}\dot{X}_1+\alpha'_{22}\dot{X}_2+\beta_2\dot{X}_2(X_1^2+X_2^2)]+\omega'^2_2X_2=-\varepsilon\eta_2\dot{X}_{2\tau}+\varepsilon^{1/2}W_2(t),\end{aligned}\tag{46}$$

where $\alpha'_{ii},\alpha_{ij},\beta_i,\eta_i,\omega'_i(i,j=1,2)$ are constants; $\varepsilon\eta_i(i=1,2)$ are the feedback gain and $\varepsilon\eta_i\dot{X}_{i\tau}=\varepsilon\eta_i\dot{X}_i(t-\tau)$ are time-delayed feedback control forces; $W_i(t)(i=1,2)$ are uncorrelated Gaussian white noises with intensities $2D_{ii}$.

The Hamiltonian system associated with Eq. (46) is linear and the Hamiltonian can be expressed in terms of action variables as

$$H'=\sum_{i=1}^{2}\omega'_iI'_i,\quad I'_i=\frac{1}{2\omega'_i}(\dot{X}_i^2+\omega'^2_iX_i^2),\quad \theta'_i=-\tan^{-1}\left(\frac{\dot{X}_i}{\omega'_iX_i}\right).\tag{47}$$

The time-delayed system state in system (46) can be approximately converted into that without time delay as

$$\dot{X}_{i\tau}=\dot{X}_i\cos\omega'_i\tau+X_i\omega'_i\sin\omega'_i\tau.\tag{48}$$

The modified Hamiltonian is of the form

$$H=\sum_{i=1}^{2}\omega_iI_i,\tag{49}$$

where $\omega_i^2=\omega'^2_i+\varepsilon\eta_i\omega'_i\sin\omega'\tau$. I_i and θ_i are of the same form as I'_i and θ'_i in Eq. (47) with ω'_i replaced by ω_i. Also, the damping coefficients α'_{ii} become $\alpha_{ii}=\alpha'_{ii}+\varepsilon\eta_i\omega_i\sin\omega_i\tau$. Eq. (46) can be rewritten as the following Itô stochastic differential equations:

$$\begin{aligned}\mathrm{d}I_i&=\varepsilon\left\{-[(\alpha_{i1}\dot{X}_1+\alpha_{i2}\dot{X}_2+\beta_i\dot{X}_i(X_1^2+X_2^2)]\frac{\dot{X}_i}{\omega_i}+\frac{D_{ii}}{\omega_i}\right\}\mathrm{d}t+\varepsilon^{1/2}\frac{\dot{X}_i}{\omega_i}\mathrm{d}B_i(t),\\\mathrm{d}\theta_i&=\omega_i+\varepsilon\left\{[(\alpha_{i1}\dot{X}_1+\alpha_{i2}\dot{X}_2+\beta_i\dot{X}_i(X_1^2+X_2^2)]\frac{\omega_iX_i}{\omega_i^2X_i^2+\dot{X}_i^2}+D_{ii}\frac{2\omega_iX_i\dot{X}_i}{(\omega_i^2X_i^2+\dot{X}_i^2)^2}\right\}\mathrm{d}t\\&\quad-\varepsilon^{1/2}\frac{\omega_iX_i}{\omega_i^2X_i^2+\dot{X}_i^2}\mathrm{d}B_i(t).\end{aligned}\tag{50}$$

Note that the repeated subscripts in Eq. (50) do not imply a summation. Two cases are considered in the following.

Nonresonant case: $r\omega_1+s\omega_2\neq0,r,s$ are integers. In this case, the averaged FPK equation is of the form of Eq. (15) with the following drift and diffusion coefficients

$$\begin{aligned}&a_1=-\alpha_{11}I_1-\frac{\beta_1}{2\omega_1}I_1^2-\frac{\beta_1}{\omega_2}I_1I_2+\frac{D_{11}}{\omega_1},\\&a_2=-\alpha_{22}I_2-\frac{\beta_2}{2\omega_2}I_2^2-\frac{\beta_2}{\omega_1}I_1I_2+\frac{D_{22}}{\omega_2},\\&b_{11}=\frac{2}{\omega_1}D_{11}I_1,\quad b_{22}=\frac{2}{\omega_2}D_{22}I_2,\quad b_{12}=b_{21}=0.\end{aligned}\tag{51}$$

The stationary solution of the averaged FPK equation is of the form of Eq. (17), where $\partial\lambda/\partial I_s$ satisfy the following equations:

$$\frac{2D_{11}I_1}{\omega_1}\frac{\partial\lambda}{\partial I_1}=\frac{2D_{11}}{\omega_1}-2\left(-\alpha_{11}I_1-\frac{\beta_1}{2\omega_1}I_1^2-\frac{\beta_1}{\omega_2}I_1I_2+\frac{D_{11}}{\omega_1}\right),$$
$$\frac{2D_{22}I_2}{\omega_2}\frac{\partial\lambda}{\partial I_2}=\frac{2D_{22}}{\omega_2}-2\left(-\alpha_{22}I_2-\frac{\beta_2}{2\omega_2}I_2^2-\frac{\beta_2}{\omega_1}I_1I_2+\frac{D_{22}}{\omega_2}\right). \tag{52}$$

If $(\beta_1/D_{11})(\omega_1/\omega_2)=(\beta_2/D_{22})(\omega_2/\omega_1)=\gamma$, the averaged FPK equation has an exact stationary solution

$$p(I_1,I_2)=C\,\exp[-\lambda(I_1,I_2)], \tag{53}$$

where

$$\lambda(I_1,I_2)=\frac{1}{D_{11}}\left(\alpha_{11}\omega_1I_1+\frac{\beta_1}{4}I_1^2\right)+\frac{1}{D_{22}}\left(\alpha_{22}\omega_2I_2+\frac{\beta_2}{4}I_2^2\right)+\gamma I_1I_2. \tag{54}$$

The approximate stationary probability density of the displacements and velocities of original system (46) is then

$$p(x_1,x_2,\dot{x}_1,\dot{x}_2)=\frac{1}{4\pi^2}\,p(I_1,I_2)|_{I_i=(\dot{x}_i^2+\omega_i^2x_i^2)/2\omega_i}. \tag{55}$$

Primary resonant case: $\omega_1=\omega_2=\omega$. Let $\theta_1-\theta_2=\psi$. The averaged FPK equation in this case is of the form of Eq. (31) with the following drift and diffusion coefficients:

$$a_1=-\alpha_{11}I_1-\alpha_{12}\sqrt{I_1I_2}\cos\psi-\frac{\beta_1}{2\omega}I_1^2-\frac{\beta_1}{\omega}I_1I_2(1-\frac{1}{2}\cos 2\psi)+\frac{D_{11}}{\omega},$$
$$a_2=-\alpha_{22}I_2-\alpha_{21}\sqrt{I_1I_2}\cos\psi-\frac{\beta_2}{2\omega}I_2^2-\frac{\beta_2}{\omega}I_1I_2(1-\frac{1}{2}\cos 2\psi)+\frac{D_{22}}{\omega},$$
$$a_3=\frac{1}{2}\left(\alpha_{12}\sqrt{\frac{I_2}{I_1}}+\alpha_{21}\sqrt{\frac{I_1}{I_2}}\right)\sin\psi-\frac{1}{4\omega}(\beta_1I_2+\beta_2I_1)\sin 2\psi,$$
$$b_{11}=\frac{2}{\omega}D_{11}I_1,\quad b_{22}=\frac{2}{\omega}D_{22}I_2,\quad b_{33}=\frac{1}{2\omega}\left(\frac{D_{11}}{I_1}+\frac{D_{22}}{I_2}\right),$$
$$b_{12}=b_{21}=b_{13}=b_{31}=b_{23}=b_{32}=0. \tag{56}$$

The stationary solution of the averaged FPK equation is of the form

$$p(I_1,I_2,\psi)=C\,\exp[-\lambda(I_1,I_2,\psi)], \tag{57}$$

where $\lambda(I_1,I_2,\psi)$ satisfies the following partial differential equations:

$$\frac{2D_{11}I_1}{\omega}\frac{\partial\lambda}{\partial I_1}=\frac{2D_{11}}{\omega}-2[-\alpha_{11}I_1-\alpha_{12}\sqrt{I_1I_2}\cos\psi-\frac{\beta_1}{2\omega}I_1^2-\frac{\beta_1}{\omega}I_1I_2\left(1-\frac{1}{2}\cos 2\psi\right)+\frac{D_{11}}{\omega}],$$

$$\frac{2D_{22}I_2}{\omega}\frac{\partial\lambda}{\partial I_2}=\frac{2D_{22}}{\omega}-2[-\alpha_{22}I_2-\alpha_{21}\sqrt{I_1I_2}\cos\psi-\frac{\beta_2}{2\omega}I_2^2-\frac{\beta_2}{\omega}I_1I_2(1-\frac{1}{2}\cos 2\psi)+\frac{D_{22}}{\omega}],$$

$$\left(\frac{D_{11}}{2\omega I_1}+\frac{D_{22}}{2\omega I_2}\right)\frac{\partial\lambda}{\partial\psi}=-\left(\alpha_{12}\sqrt{\frac{I_2}{I_1}}+\alpha_{21}\sqrt{\frac{I_1}{I_2}}\right)\sin\psi+\frac{1}{2\omega}(\beta_1I_2+\beta_2I_1)\sin\psi. \tag{58}$$

Let

$$\lambda(I_1,I_2,\psi)=\lambda_0(I_1,I_2)+\lambda_1(I_1,I_2)\cos\psi+\lambda_2(I_1,I_2)\cos 2\psi. \tag{59}$$

Substituting Eq. (59) into Eq. (58), we obtain three sets of partial differential equations for λ_0, λ_1 and λ_2. In the case that $\beta_1/D_{11}=\beta_2/D_{22}=\gamma_1$, $\alpha_{12}/D_{11}=\alpha_{21}/D_{22}=\gamma_2$, we obtain the exact stationary solution (57) with

$$\lambda(I_1,I_2,\psi)=\frac{\alpha_{11}\omega}{D_{11}}I_1+\frac{\alpha_{22}\omega}{D_{22}}I_2+\frac{\beta_1}{4D_{11}}I_1^2+\frac{\beta_2}{4D_{22}}I_2^2+\gamma_1I_1I_2-\frac{\gamma_1}{2}I_1I_2\cos 2\psi+2\gamma_2\omega\sqrt{I_1I_2}\cos\psi. \tag{60}$$

The approximate stationary probability density of the displacements and velocities of original system (46) is then

$$p(x_1, \dot{x}_1, x_2, \dot{x}_2) = \frac{1}{2\pi} p(I_1, I_2, \psi), \tag{61}$$

where $\psi = \theta_1 - \theta_2$; I_i and θ_i are functions of x_i and $\dot{x}_i$.

Let q_1, p_1 represent the displacement and velocity of the first oscillator, respectively. Some numerical results for stationary probability density $p(q_1, p_1)$ obtained by using the proposed stochastic averaging method and from digital simulation are shown in Fig. 4 for non-resonant case and in Fig. 5 for primary resonant case. It is seen that the proposed method yields very good prediction even the time delay approaches to one period. For both non-resonant and resonant cases, the results for several τ values are given to illustrate the effect of time delay in control forces on the response of the system. It is seen that the time delay in control forces may affect the response of the system greatly, and may even cause phenomenological bifurcation.

4.3. Example 3

As an example of strongly nonlinear stochastic system, consider a Duffing–van der Pol oscillator with delayed linear feedback control subject to Gaussian white noise excitation. The equation of motion is

$$\ddot{X} + \omega_0'^2 X + \alpha X^3 = \varepsilon(b - X^2)\dot{X} - \varepsilon(a\dot{X}_\tau) + \varepsilon^{1/2} W(t), \tag{62}$$

where ε is a small positive parameter; $\dot{X}_\tau = \dot{X}(t-\tau)$ is delayed system velocity; εa is feedback control gain; $W(t)$ is a Gaussian white noise with intensity $2D$.

Note that there is no Wong–Zakai correction term for this example. Let $X = Q$, $\dot{X} = P$. The Hamiltonian associated with the system (62) is

$$H' = \frac{1}{2} p^2 + \frac{1}{2} \omega_0'^2 q^2 + \frac{1}{4} \alpha q^4. \tag{63}$$

The time-delayed feedback control force in system (62) can be approximately converted into a control force without time delay, i.e.,

$$\varepsilon a P_\tau \doteq \varepsilon a(P \cos \omega'\tau + Q\omega' \sin \omega'\tau), \tag{64}$$

where the average frequency ω' is

$$\omega'(H') = \frac{\pi\sqrt{\alpha}}{2\sqrt{2}} \frac{\sqrt{A^2 + B^2}}{K(r)}, \tag{65}$$

where $K(r)$ is complete elliptic integral of the first kind; $r = A/\sqrt{A^2 + B^2}$, $A^2 = \omega_0'^2/\alpha\left(\sqrt{1 + 4\alpha H'/\omega_0'^4} - 1\right)$, $B^2 = \omega_0'^2/\alpha\left(\sqrt{1 + 4\alpha H'/\omega_0'^4} + 1\right)$.

After the term proportional to Q in Eq. (64) is combined with the restoring force to form a modified restoring force, the new Hamiltonian is

$$H = \frac{1}{2} p^2 + \frac{1}{2} \omega_0^2 q^2 + \frac{1}{4} \alpha q^4, \tag{66}$$

where $\omega_0^2 = \omega_0'^2 + \varepsilon\omega' \sin \omega'\tau$. Also, we can obtain modified frequency $\omega(H)$ of the nonlinear oscillator, which is of the same form as $\omega'(H')$ in Eq. (65) but with ω_0' is replaced by ω_0.

The Itô equation for H can be obtained from Eq. (62) by using Itô differential rule as follows:

$$\mathrm{d}H = \varepsilon\left\{[(b - q^2 - a \cos \omega\tau]p^2 + D\right\} \mathrm{d}t + \varepsilon^{1/2} p \, \mathrm{d}B(t). \tag{67}$$

Applying the stochastic averaging method to Eq. (67) leads to the following averaged FPK equation

$$\frac{\partial p}{\partial t} = \varepsilon\left\{-\frac{\partial}{\partial H}[a(H)p] + \frac{1}{2} \frac{\partial^2}{\partial H \, \partial H}[b(H)p]\right\}, \tag{68}$$

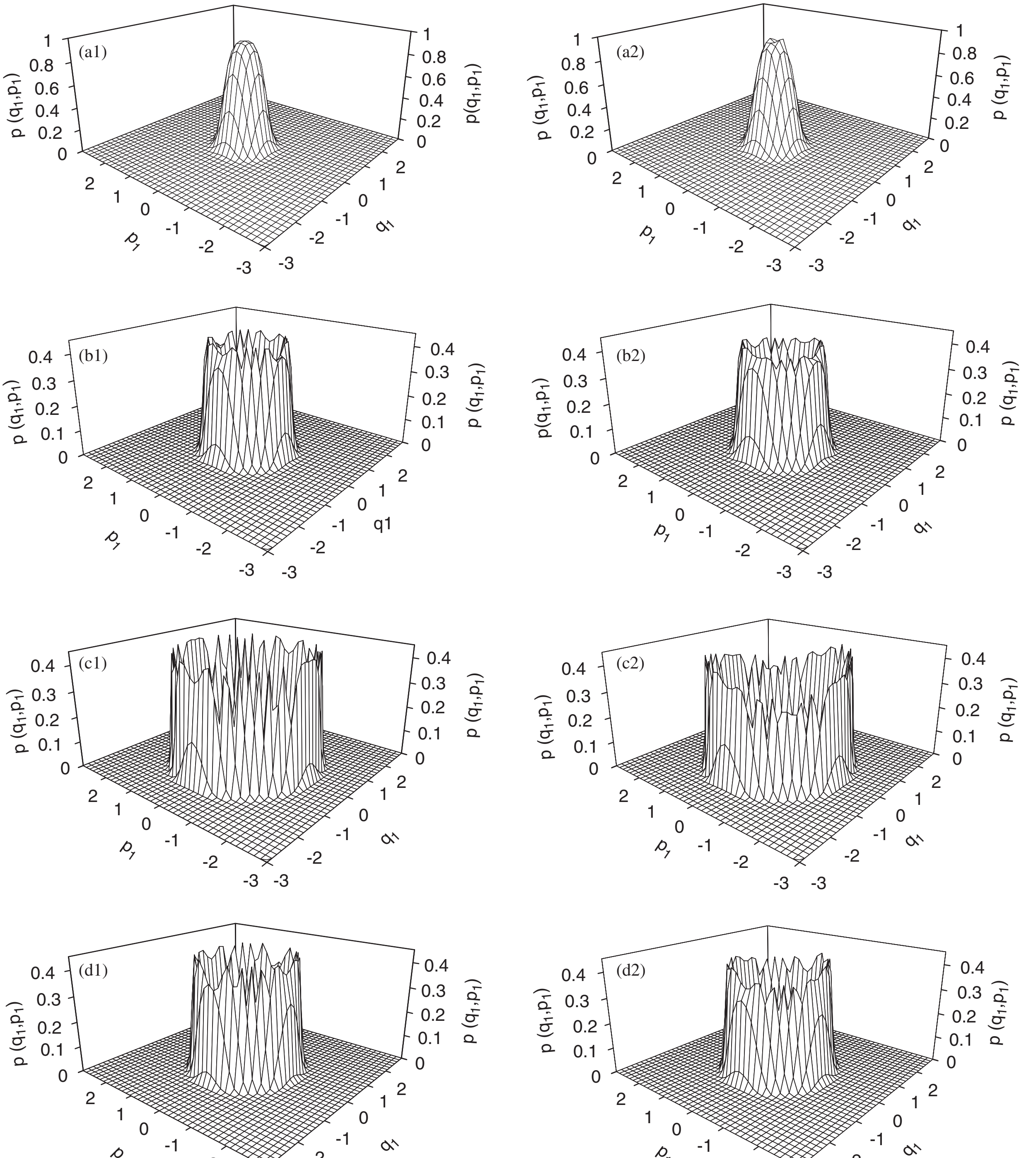

Fig. 4. Stationary probability density $p(q_1, p_1)$ of system (46) with velocity feedback in non-resonant case. The parameters are: (a1) $\tau = 0$, $\beta_2 = 5.0$, result obtained by using proposed stochastic averaging method; (a2) $\tau = 0$, $\beta_2 = 5.0$, result obtained by using digital simulation; (b1) $\tau = 1$, $\beta_2 = 5.04$, result obtained by using proposed stochastic averaging method; (b2) $\tau = 1$, $\beta_2 = 5.04$, result obtained by using digital simulation; (c1) $\tau = 4$, $\beta_2 = 4.91$, result obtained by using proposed stochastic averaging method; (c2) $\tau = 4$, $\beta_2 = 4.91$, result obtained by using digital simulation; (d1) $\tau = 5$, $\beta_2 = 4.65$, result obtained by using proposed stochastic averaging method; (d2) $\tau = 5$, $\beta_2 = 4.65$, result obtained by using digital simulation; The other parameters are: $\varepsilon = 0.01, \omega'_1 = 1.0, 2D_{11} = 0.2, \alpha'_{11} = -5.0, \alpha_{12} = 5.0, \beta_1 = 10.0, \eta_1 = 5.0$, $\omega'_2 = 1.414$, $2D_{22} = 0.2, \alpha_{21} = 5.0, \alpha'_{22} = 5.0, \eta_1 = 5.0$.

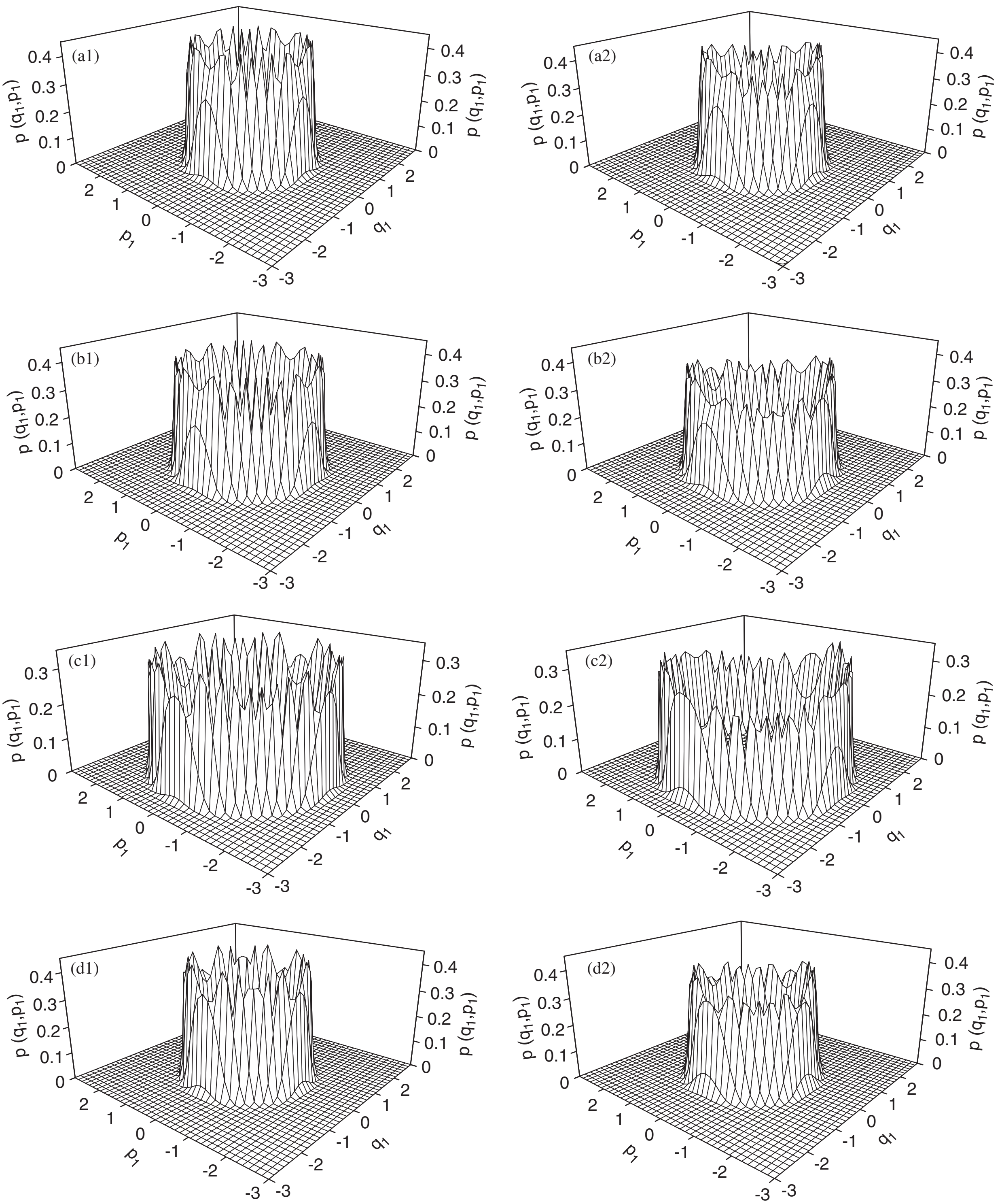

Fig. 5. Stationary probability density $p(q_1, p_1)$ of system (46) with velocity feedback in resonant case. The parameters are: (a1) $\tau = 0$, result obtained by using proposed stochastic averaging method; (a2) $\tau = 0$, result obtained by using digital simulation; (b1) $\tau = 1$, result obtained by using proposed stochastic averaging method; (b2) $\tau = 1$, result obtained by using digital simulation; (c1) $\tau = 2$, result obtained by using proposed stochastic averaging method; (c2) $\tau = 2$, result obtained by using digital simulation; (d1) $\tau = 6$, result obtained by using proposed stochastic averaging method; (d2) $\tau = 6$, result obtained by using digital simulation; The other parameters are: $\varepsilon = 0.01, \omega_1' = 1.0, 2D_{11} = 0.2, \alpha_{11}' = -5.0, \alpha_{12} = 5.0, \beta_1 = 5.0, \eta_1 = 5.0$, $\omega_2' = 1.0$, $2D_{22} = 0.2, \alpha_{21} = 5.0, \alpha_{22}' = -5.0, \beta_2 = 5.0, \eta_2 = 5.0$.

where the drift and diffusion coefficients are

$$a(H) = \frac{1}{T(H)} \oint_{\Omega} \left[[(b - q^2 - a \cos \omega\tau] p^2 + D \right] / p \mathrm{d}q,$$

$$b(H) = \frac{1}{T(H)} \oint_{\Omega} \left[[2Dp^2] / p \, \mathrm{d}q, \right.$$

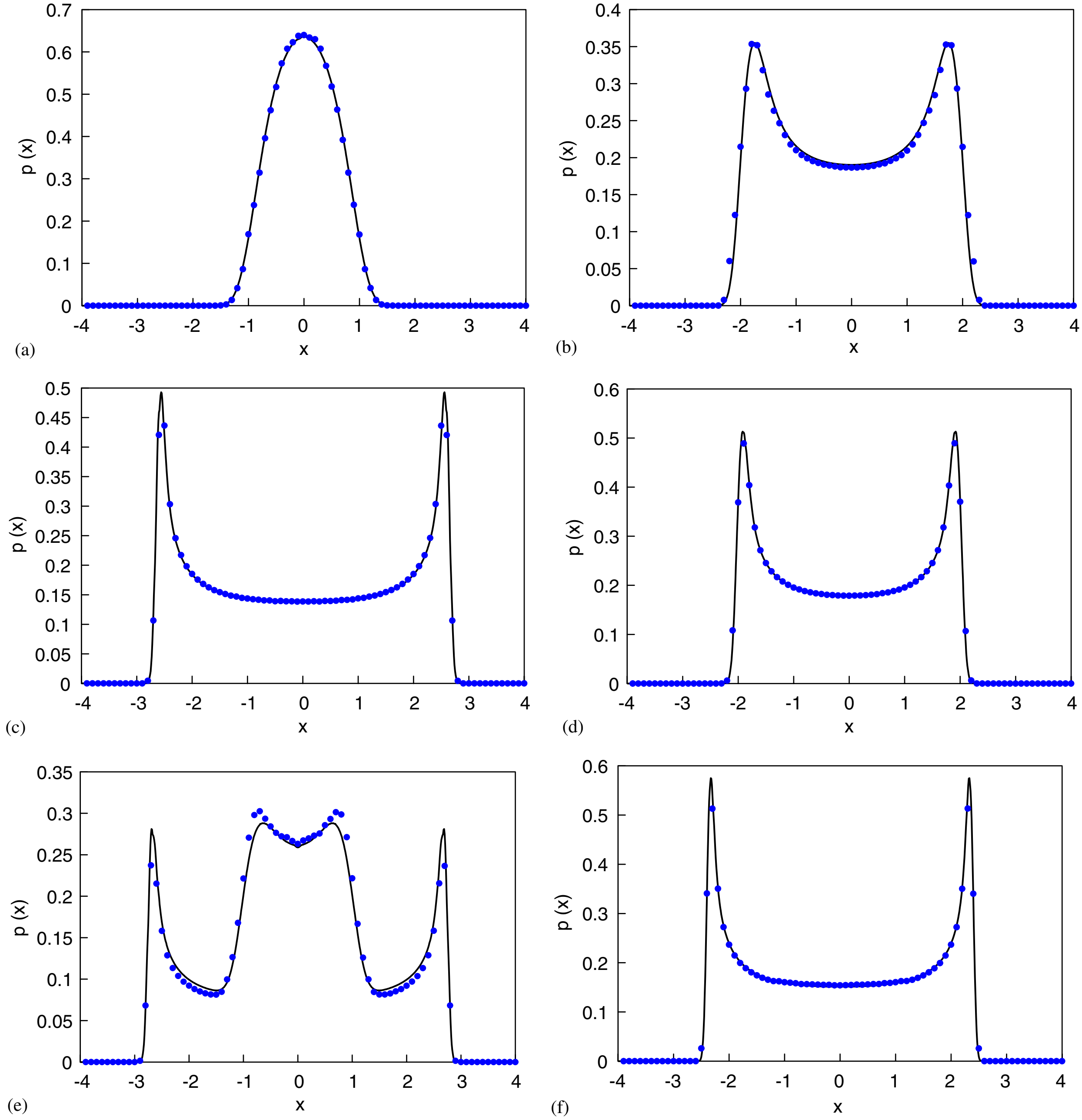

Fig. 6. Stationary marginal probability density $p(x)$ of system (62) with velocity feedback. The parameters are: $a = 1.0$, $b = 1.0$, $\omega_0' = 1.0$, $\alpha = 0.5$, $\varepsilon = 0.01$, $2D = 0.2$, (a) $\tau = 0$, (b) $\tau = 1$, (c) $\tau = 2$, (d) $\tau = 3$, (e) $\tau = 5$, (f) $\tau = 6$. — By using the proposed stochastic averaging method; ● from digital simulation.

$$T(H) = \oint_{\Omega} \left(1 \Big/ p\right) \mathrm{d}q,$$
$$\Omega = \{q | H = \omega_0^2 q^2/2 + \alpha q^4/4\}. \tag{69}$$

The stationary solutions to averaged FPK Eq. (68) is

$$p(H) = C \exp[-\lambda(H)], \tag{70}$$

where

$$\lambda(H) = \lambda_0 + \int_0^H \frac{1}{b(H)} \left(\frac{\mathrm{d}b(H)}{\mathrm{d}H} - 2a(H)\right) \mathrm{d}H. \tag{71}$$

The approximate stationary probability density of original system (62) is then

$$p(q,p) = \frac{1}{2\pi} p(H)|_{H=\left(\frac{1}{2}p^2+\frac{1}{2}\omega_0^2 q^2+\frac{1}{4}\alpha q^4\right)}. \tag{72}$$

Some numerical results for stationary marginal probability density $p(x)$ obtained by using the proposed stochastic averaging method and from digital simulation are shown in Fig. 6. From the figures, it is seen that the analytical results obtained by using the proposed method agree well with those from digital simulation even for long delay time. From the figures, we can also see the phenomenological bifurcation in the response of the system caused by the delayed feedback control.

5. Conclusion

In the present paper, a stochastic averaging method for quasi-integrable Hamiltonian systems with time-delayed feedback control has been proposed. After the time-delayed feedback control forces are approximated by control forces without time delay, the original stochastic averaging method for quasi-integrable Hamiltonian systems proposed by the present second author and his co-workers can be directly applied to the systems with time-delayed feedback control. The analytical results obtained for three examples agree well with those from digital simulation even for large time delay. The numerical results show that the delayed feedback control may affect the response of a system greatly and even may cause phenomenological bifurcation.

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