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Variational correctness and Timoshenko beam finite element elastodynamics

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Abstract

The finite element discretisation of the two-noded Timoshenko beam element for elastodynamics offers very interesting insights into the error analysis aspects of the formulation. In this paper, the relatively different order of convergence of the two spectra of the Timoshenko beam theory, and the extra-variational aspect of the use of reduced integration to free the element of locking, are investigated. The correct variational basis for finite element analysis of elastodynamic problems is presumed to originate from the principle of virtual work, with a simultaneous consideration of errors in both displacement and strains. A variationally correct element would lock; to make an element free of locking, some degree of variational incorrectness must be brought in. The present paper also demonstrates that reduced integration violates the virtual work principle which in turn causes the loss of boundedness of the finite element eigenvalues with the exact solution.

1. Introduction

The Timoshenko beam theory has features that make it an interesting case study for the examination of errors that appear when a finite element discretisation of its elastodynamics is made [1]. Unlike the classical (or engineering) theory of beams, the Timoshenko theory incorporates shear flexibility and rotary inertia. As a consequence, it shows two distinct spectra, a basic flexural (or bending dominated) spectra, and a shear dominated spectra [2,3]. Also, in the case of low order formulations (e.g. a two noded element), there is the possibility of locking, which has to be relieved using an extra-variational step such as reduced integration [4–9]. In this paper, we shall investigate carefully, the error analysis implications for the two spectra, and that due to the extra-variational nature of reduced integration. Tong et al. [10] have studied the convergence rate of the eigenvalues for lumped and consistent mass finite element models. They have shown that for the lumped and consistent mass models the eigenvalues converge to the exact solution at the same rate. Cook et al.

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[6,11,12] have shown that lumped mass finite element model produces lower bound eigenvalues with respect to the corresponding exact solution. Throughout, in what follows, it is assumed that mass matrices are computed using a consistent mass approach, so that no confusion in this regard enters into the picture. The accuracy of the solution is then contingent only on the quality of the stiffness matrix.

Quite recently, attempts were made to derive error convergence rates and estimates for the finite element elastodynamics of one-dimensional (1D) elements like bar and Euler–Bernoulli beam [13] and Timoshenko beam [14,15]. Some recent works show how computations in finite element method for free-vibration problems can be interpreted using the Function Space Approach [16,17], where the use of the principle of virtual work, leads to a projection theorem for elastodynamics [17]. This allows one to see that for arbitrary meshing and for a given mode the approximate values for the natural frequencies computed through variationally correct formulations are always higher than the exact values, whenever consistent mass approaches are used [17,18]. This assurance of boundedness is lost in any extra-variational formulation, such as in the Timoshenko beam element based on reduced integration. Fuenmayor et al. [19] have presented an *h*-adaptive technique in finite element elastodynamics considering only the error in the modal displacement. Wiberg et al. [20] used the concept of super convergent patch recovery [5] with the strain energy as the error indicator for another version of *h*-adaptive meshing for free vibration problems.

The present paper shows that reduced integration is variationally incorrect as it violates the virtual work principle. The eigenvalues obtained using reduced integration does not offer the boundedness to the exact solution and hence the use of adaptive meshing may lead to wrong results with reduced integration.

2. A brief introduction to Timoshenko beam elastodynamics

The dynamic equilibrium of beams in flexure derived based on Timoshenko theory includes the separate effects of transverse shear deformation and rotary inertia. Basically, this theory is an extension of the classical or Euler–Bernoulli theory, which is based on the assumption that plane sections, initially normal to the neutral axis remain plane and normal to that axis even after deformation. Timoshenko theory includes the shear strain by relaxing the above assumption made in Euler–Bernoulli theory. The influence of these effects is significant for natural frequencies of higher modes and also for thick beams, where the length to depth ratio (L/d) of the beam is relatively smaller.

In case of Timoshenko theory, both bending and transverse shear energies are accounted for evaluating the total elastic strain energy; for a beam of length L it can be calculated as

$$U = \frac{1}{2}\int_0^L \left(EI\theta_x^2 + kGA(w_x - \theta)^2\right) \mathrm{d}x, \tag{1}$$

where E and G are Young's modulus and shear modulus of the material of the beam respectively, and I and A are the sectional moment of inertia and sectional area of the beam, respectively. Here θ is the slope of the deflection curve when the shear deformation is neglected and (w_x–θ) is the shear strain. The shear correction factor k depends upon the shape of the cross-section [21] and strictly upon the mode of vibration. For the present analysis the value of k is considered as a constant ($k = 5/6$, for rectangular cross section) for all modes of vibration.

The total kinetic energy T is due to lateral vibration and rotary inertia and can be written as

$$T = \frac{1}{2}\int_0^L \left(\rho I\,\dot{\theta}^2 + \rho A\dot{w}^2\right) \mathrm{d}x, \tag{2}$$

where ρ is mass density of the material of the beam. Here, a dot over the symbol indicates the first derivative with respect to time.

The governing differential equation for the free vibration of beams can be obtained by applying Hamilton's principle. It can be seen that the governing equations thus obtained is a coupled differential equation involving bending and shear terms as given by

$$\begin{aligned} &\rho A\ddot{w} - kGA(w_{xx} - \theta_x) = 0, \\ &\rho I\,\ddot{\theta} - EI\theta_{xx} - kGA(w_x - \theta) = 0, \end{aligned} \tag{3}$$

where $w_x - \theta = 0$ or w and $\theta_x = 0$ or θ are to be specified as the boundary conditions. One can easily derive the relation between the displacement $w(x)$ and the rotation of the section $\theta(x)$ from Eq. (3).

A solution to the differential equation (3), shows two distinct spectra for the natural frequency of the structure [2,3] interpreted as a basic flexural (or bending dominated) spectra, and a shear dominated spectra. For a hinged–hinged boundary condition [3], the solution of Eq. (3) reduces to a quadratic polynomial equation in ω^2 as

$$\frac{\rho^2 L^4}{EkG}\omega_n^4 - \left\{\frac{\rho A L^4}{EI} + \frac{\rho L^2}{E}\left(1 + \frac{E}{kG}\right)(n\pi)^2\right\}\omega_n^2 + (n\pi)^4 = 0, \tag{4}$$

where n is the mode number and ω is the natural frequency of the structure.

3. Finite element formulations

A number of finite element beam models based on Timoshenko's theory have been proposed in the literature [9,22]. In general, these elements differ from one another in the choice of the nodal variables. The widely used two noded linear Timoshenko beam element with two degrees of freedom (transverse displacement w and rotation θ) at each node, is considered for the present study. Such an element of length l is shown in Fig. 1. The linear Lagrangian interpolation functions are used to represent the C^0 approximate displacement field $\lfloor w^h \ \theta^h \rfloor$, of the transverse displacement component w and the rotation component θ (of the plane normal to neutral axis). This is given by

$$\begin{Bmatrix} w^h \\ \theta^h \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 \\ 0 & N_1 & 0 & N_2 \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = [N]\{\delta^e\}, \tag{5}$$

where the linear Lagrangian shape functions are $N_1 = (1-\xi)/2$ and $N_2 = (1+\xi)/2$. The non-dimensional co-ordinate ξ varies between -1 and $+1$, and $\{\delta^e\}$ is the nodal displacement vector, given by $\{\delta^e\} = [w_1\theta_1 w_2\theta_2]^{\mathrm{T}}$.

3.1. Spurious stiffening due to locking

It is well known that the stiffness matrix formulated for this element will lead to shear locking [4–9], if the shear energy is computed exactly. This is due to the presence of spurious constraint [9] in the limiting case of a thin beam simulation, which requires that the shear energy must vanish. The spurious term thus caused effectively enhances the element bending stiffness to $EI^* = EI + kGAl^2/12$. In other words, the bending stiffness becomes infinitely large when the beam becomes very thin. Here EI and kGA are the bending and shear rigidities, respectively.

The element can be made lock-free by employing an extra-variational trick, such as reduced integration for the shear energy term. The stiffness matrix obtained after reduced integration gives very good convergence rate for the frequencies of the flexural mode [14,15]. Unfortunately, there is no work reported so far highlighting the implications of reduced integration on the shear spectra. The present paper addresses these issues. It will be shown in the later section that, the shear locking affects the results of the bending dominated

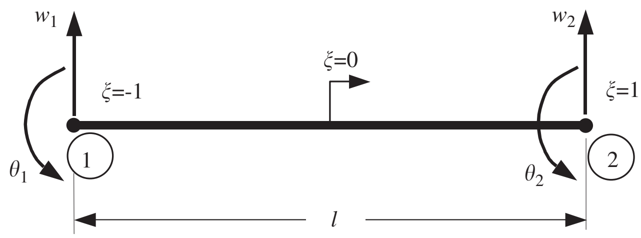

Fig. 1. The linear Timoshenko beam element.

spectra, but not for the shear spectra. In fact, a reduced integration scheme improves the convergence for the bending dominated spectra but invites additional errors in the shear energy term which in turn reduces the convergence rate of the frequencies of the shear dominated spectra. This paper also examines how the extra-variational nature of reduced integration will cause the violation of the virtual work principle. This will be shown in the later sections. It has been demonstrated with the help of a ''sweep test'', that the upper-bound nature of the approximate finite element natural frequency to the exact solution, especially for the shear dominated spectra, is lost if reduced integration is employed. This is attributed to the violation of the virtual work principle (or the extra-variational nature of reduced integration).

4. The projection theorem and the energy-error rules for finite element elastodynamics—a function space review [17]

Free vibration problems (or eigenvalue problems) in structural mechanics can be expressed as variational problems using Rayleigh quotient:

$$\bar{\omega}^2 = \frac{\|\bar{\varepsilon}\|^2}{|\bar{u}|^2}, \tag{6}$$

where $\{\bar{u}\}$ is the approximate modal displacement (satisfying the kinematic boundary conditions), $\{\bar{\varepsilon}\}$ is the resulting approximate modal strain vector and $\bar{\omega}^2$ is the approximate eigenvalue. The inner products used here to define the energy norms (both strain energy and kinetic energy) are defined in the Appendix. Replacing the approximate modal displacement (and resulting modal strain) vector in Eq. (6) by corresponding exact functions, we have

$$\omega^2 = \frac{\|\varepsilon\|^2}{|u|^2}. \tag{7}$$

Here $\{u\}$, $\{\varepsilon\}$ and ω^2 are the analytical modal displacement vector, modal strain vector and the corresponding (exact) eigenvalue, respectively. Also, through the virtual work principle [16] we can deduce an equation for the exact eigenvalue in terms of the exact and approximate solutions as

$$\omega^2 = \frac{\langle \bar{\varepsilon}, \varepsilon \rangle}{(\bar{u}, u)}. \tag{8}$$

Combining Eqs. (6)–(8), the projection theorem (consequence of virtual work principle) and the energy error rule for elastodynamics, which governs the error in eigenvalues due to discretisation, can be derived [17] in the form

$$\langle \bar{\varepsilon}, \varepsilon - \bar{\varepsilon} \rangle = (\bar{u}, \omega^2 u - \bar{\omega}^2 \bar{u}), \tag{9}$$

and

$$\|\varepsilon\|^2 - \|\bar{\varepsilon}\|^2 = \omega^2 |u|^2 - \bar{\omega}^2 |\bar{u}|^2. \tag{10}$$

The energy rules for elastodynamics presented above are valid only at the global level. This is expected because of the global character of natural frequency. Note that it has been shown earlier for the linear elastostatic case that these rules also govern the error due to discretisation at the element level [7]. Using the virtual work rule (Eq. (9)) and the energy-error rule (Eq. (10)) one can obtain a relation between strain error and frequency error:

$$\|\varepsilon - \bar{\varepsilon}\|^2 = \omega^2 |u - \bar{u}|^2 + [\bar{\omega}^2 - \omega^2] |\bar{u}|^2. \tag{11}$$

The above equation has been presented earlier using the weak form by Strang and Fix [16], but with normalising the approximate displacement norm $\left(|\bar{u}|^2 = 1\right)$.

The error in the eigenvalue is given by the expression derived from above as

$$\left[\frac{\bar{\omega}^2}{\omega^2} - 1\right] = \frac{\|\varepsilon - \bar{\varepsilon}\|^2 - \omega^2 |u - \bar{u}|^2}{\omega^2 |\bar{u}|^2}. \tag{12}$$

It is clear from the above relation that the error in the approximate finite element eigenvalue is affected by errors in displacements and strains simultaneously. From Eq. (11), one can derive the equation for frequency-error-hyperboloid [17] as

$$\frac{|u-\bar{u}|^2}{|\bar{u}|^2}+\frac{\bar{\omega}^2}{\omega^2}-\frac{\|\varepsilon-\bar{\varepsilon}\|^2}{\omega^2|\bar{u}|^2}=1. \tag{13}$$

Studies conducted recently [17,18] have shown that for a variationally correct formulation, *consistent* with the *weak form* of the differential equation, the computed approximate eigenvalue for a given mode is always greater than the corresponding exact eigenvalue. In the present paper we shall show that reduced integration technique used to avoid shear locking violates Eqs. (8) and (13), and hence is an extra-variational technique. Due to violation of the above rules, the reduced integration finite element model disturbs the upper-bound nature of the approximate eigenvalue which is expected in a variationally correct formulation. The loss of boundedness may cause wrong interpretation while using adaptive meshing for optimisation of the finite element mesh size.

5. The "sweep test" and the "egg cup" profile

The boundedness aspect of the variationally correct finite element solution can be very elegantly demonstrated using a two-element moving node sweep test for a 1D problem. Fig. 2(a) shows that the mid node can be located anywhere along the length. Each such configuration gives one possible global test function from the global function space. The global stiffness and mass matrices are now a function of the node location. Each case would then give a Rayleigh–Ritz solution to the eigenvalue problem.

The frequencies obtained as the node is moved are as typically shown in Fig. 2(b). If the formulation had been variationally correct, the frequencies would vary in the same convex "egg cup" profile described in an abstract sense in Strang and Fix [16] for the Rayleigh–Ritz problem, as the nodal position is moved from a highly asymmetric mesh to a perfectly symmetric mesh. For a uniform beam with symmetric boundary conditions the globally optimal solution would occur when equal length elements are used, i.e. the mid node is placed exactly at the centre of the beam. Interestingly this is also the case where the global errors are equi-partitioned between the two elements. Any other nodal position, as common sense indicates, will be sub-optimal and would produce frequencies with higher errors.

This benchmark will now serve the very useful purpose of examining how this boundedness aspect will suffer if some kind of extra-variational relaxation is introduced, either with the formulation of the stiffness matrix, or the mass matrix or both. In this investigation, we will confine attention to changes in the stiffness matrix due to the use of reduced integration. Therefore, a consistent mass matrix will be used in all the computations here.

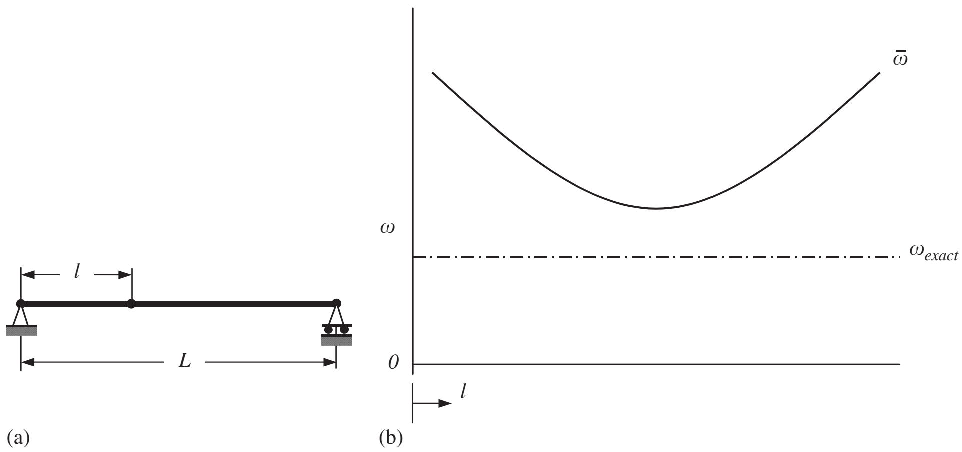

Fig. 2. (a) The two element moving node sweep test. (b) Eigenvalues $\bar{\omega}$ as node location in sweep test.

6. Accuracy and convergence tests—a brief note

In order to assess the quality of approximate finite element eigenvalue it is necessary to consider the errors in strains and displacements simultaneously, as we have seen in Eq. (12). It can be argued from the best-fit paradigm [7] that, for the linear elastostatics, a finite element computation produces approximate strains which are accurate to $O(h^n)$, consequently the strain energy will be accurate to the $O(h^{2n})$. This is also true for elastodynamics where the finite element strains are approximately the best-fit strains. Also, by using the idea of generalised mass in Eq. (10), the errors in the eigenvalues can be expressed as

$$-\left(\|\varepsilon\|^2 - \|\bar{\varepsilon}\|^2\right) = \left(\bar{\omega}^2 - \omega^2\right), \tag{14}$$

in which the modal displacement norms are normalised using

$$|u|^2 = |\bar{u}|^2 = 1. \tag{15}$$

Eq. (14) shows that the accuracy of finite element eigenvalue can still be assessed by the accuracy of the strain energy.

We now analyse the accuracy of the linear two noded Timoshenko beam element, which uses linear functions to interpolate displacement w and rotation θ. In this case, the bending strain θ_x, which is a constant for linear θ, is accurate to $O(h)$ and hence the bending energy will be accurate to $O(h^2)$. So, from Eq. (14), it is expected that the natural frequencies of the bending dominated spectra will have a rate of convergence of $O(h^2)$. This is true for both locked and lock-free models. However, the actual error in the locked solution is much greater due to the additional stiffening [7,9] by the ratio, $I^*/I = 1 + kGAl^2/12EI$. A reduced integration technique for the shear energy terms avoids spurious bending stiffening and makes the element lock-free. This will now bring down the error in the bending dominated spectra. It is interesting to see what will happen to the shear dominated spectra due to reduced integration.

The quality of the natural frequencies of shear dominated spectra is governed primarily by the shear energy. For the linear Timoshenko beam element the shear strain $\theta - w_x$ (a linear function) is accurate to the $O(h^2)$ and the shear energy is expected to converge at $O(h^4)$. The effect of bending is negligible in the shear dominated spectra. So shear locking will not affect the convergence of the eigenvalues of the shear dominated spectra. In other words, locking is expected only for flexural spectra, but not for shear spectra [9]. It would be interesting to examine the implications of reduced integration used to avoid shear locking (only in bending dominated spectra) on convergence of eigenvalues of the shear dominated spectra. Reduced integration effectively eliminates the linear term in the interpolation function leaving behind only a constant term. That is, the accuracy of shear strain is now only of $O(h)$ and hence the accuracy in estimating the shear energy is of $O(h^2)$. So, with reduced integration, the order of convergence of natural frequencies of shear dominated spectra is expected to be of $O(h^2)$. These predictions shall be demonstrated with numerical examples in the following section.

7. Numerical experiments, results and discussions

In the present paper, a simply supported beam with the following properties has been considered for numerical experiments:

Length = 1000, breadth = 10, $E = 2 \times 10^5$, $v = 0$, and $k = 5/6$, where v is the Poisson's ratio and k is the shear correction factor. To study the effect of thickness on convergence rates of the eigenvalues of both 1st (bending dominated) and 2nd (shear dominated) spectra, a thin beam, with length to depth ratio $L/d = 100$, and a thick beam with $L/d = 10$ have been considered.

7.1. Timoshenko beam analysis based on analytical solution, the virtual work principle and the finite element model

A simply supported beam is considered here for the comparative study of the locked and lock-free finite element Timoshenko beam elastodynamics.

The solution $W(x) = \sin(n\pi x/l)$ for the mode shape will exactly satisfy the differential equation for the Timoshenko beam model presented in Eq. (3) with a hinged-hinged end conditions. It should be noted that for the Timoshenko beam model the displacement field contains the transverse displacement $W(x)$ and the rotation of the section $\theta(x)$, i.e., $\{u\} = [W(x)\ \theta(x)]^T$, where $\{u\}$ is a multi-component displacement field.

The relation between $W(x)$ and $\theta(x)$ for this case can be easily derived from Eq. (3). Let us consider a system which is oscillating at a frequency ω with a coupled mode shape $W(x)$ and $\theta(x)$, i.e. a separation of variables can be assumed in the form $w(x,t) = W(x)\cos\omega t$ and $\theta(x,t) = \theta(x)\cos\omega t$ in Eq. (3). Then Eq. (3) readily gives

$$\theta(x) = \frac{EI}{(kGA - \omega^2 mr^2)}\left\{\frac{d^3W}{dx^3} + \left(\frac{m\omega^2}{kGA} + \frac{kGA}{EI}\right)\frac{dW}{dx}\right\}, \tag{16}$$

where $r^2 = I/A$ is the radius of gyration, and $m = \rho A$ is the mass per unit length.

7.1.1. Pure shear vibration of hinged–hinged beam

Hinged–hinged Timoshenko beam permits the pure shear vibration where the section rotates without any transverse displacement and the rotation of the section is constant throughout the length of the beam; i.e., $W(x) = 0$ and $\theta(x) = C$, a constant. Table 1 shows the results of the analysis for thin and thick beams using analytical solution, virtual work principle and the finite element solution with two-element discretisation. The finite element model predicts the pure shear frequency exactly in both the locked and lock-free solutions. In fact, the virtual work principle (Eq. (8)) is satisfied by both the exact and reduced integration finite element model. It is also seen that the pure shear mode satisfies the frequency-error-hyperboloid equation for both reduced and exact integration. This is a special case, because the finite element model predicts the constant shear strain behaviour of the hinged-hinged beam without any discretisation error for both locked and lock-free finite element model. In Table 1, the eigenvalue obtained from virtual work principle for the lock-free case is denoted as $\hat{\omega}^2$. In general, this should deviate from the exact solution ω^2 if a variationally incorrect

Table 1
Results of the exact solution, virtual work principle and the finite element solution (with locked and lock-free model) of the pure shear mode ($n = 0$) for a thin and thick simply supported beam discretised using two linear Timoshenko beam elements

			$L/d = 10$	$L/d = 100$
Exact solution (Eq. (7))		$\langle \varepsilon,\varepsilon \rangle$	4.25×10^{10}	4.25×10^{9}
		(u,u)	4.1667×10^{8}	4.1667×10^{5}
		$\omega^2 = \frac{\langle \varepsilon,\varepsilon \rangle}{(u,u)}$	102	102×10^{2}
Virtual work principle (Eq. (8))	$\bar{u}$, $\bar{\varepsilon}$ from the locked solution	$\langle \bar{\varepsilon},\varepsilon \rangle$	2.4537×10^{10}	2.4537×10^{8}
		$(\bar{u},u)$	2.4056×10^{8}	2.4056×10^{5}
		$\omega^2 = \frac{\langle \bar{\varepsilon},\varepsilon \rangle}{(\bar{u},u)}$	102	102×10^{2}
	$\bar{u}^*$, $\bar{\varepsilon}^*$ from the lock-free solution	$\langle \bar{\varepsilon}^*, \varepsilon \rangle$	2.4537×10^{10}	2.4537×10^{9}
		$(\bar{u}^*,u)$	2.4056×10^{8}	2.4056×10^{5}
		$\hat{\omega}^2 = \frac{\langle \bar{\varepsilon}^*,\varepsilon \rangle}{(\bar{u}^*,u)}$	102	102×10^{2}
Finite element model (Eq. (6))	$\bar{u}$, $\bar{\varepsilon}$ from the locked solution	$\langle \bar{\varepsilon}, \bar{\varepsilon} \rangle$	1.4167×10^{9}	1.4167×10^{9}
		$(\bar{u}, \bar{u})$	1.3889×10^{7}	1.3889×10^{5}
		$\bar{\omega}^2 = \frac{\langle \bar{\varepsilon}, \bar{\varepsilon} \rangle}{(\bar{u}, \bar{u})}$	102	102×10^{2}
	$\bar{u}^*$, $\bar{\varepsilon}^*$ from the lock-free solution	$\langle \bar{\varepsilon}^*, \bar{\varepsilon}^* \rangle$	1.4167×10^{9}	1.4167×10^{9}
		$(\bar{u}^*, \bar{u}^*)$	1.3889×10^{7}	1.3889×10^{5}
		$\bar{\omega}^{*2} = \frac{\langle \bar{\varepsilon}^*, \bar{\varepsilon}^* \rangle}{(\bar{u}^*, \bar{u}^*)}$	102	102×10^{2}

formulation had been used. We see from Table 1 that, for this particular case $\hat{\omega}^2 = \omega^2$, where ω^2 is the exact eigenvalue. We shall see in the next section that reduced integration, in general, violates the virtual work principle (Eq. (8)) and the frequency error hyperboloid rule (Eq. (13)). In Tables 1–3, $\bar{u}^*$ & $\bar{\varepsilon}^*$ are the approximate modal displacement and the corresponding modal strain vector from the lock-free finite element model.

7.1.2. *The coupled flexural and rotational vibration*

The results for the 1st mode of the bending and the shear spectra are summarised in Tables 2 and 3 for thick and thin beams. The 1st bending frequency obtained from a locked solution $\left(\bar{\omega}^2 = 2.7065\right)$ is much higher than the exact frequency ($\omega^2 = 1.5802 \times 10^{-2}$) due to locking. However, it can be noted that $\bar{\varepsilon}$ from the locked finite element solution still satisfies the virtual work principle (Eq. (8)) in both the cases of flexural and the shear spectra. Also we see that $\bar{\varepsilon}^*$ from the reduced integrated lock-free finite element model violates the virtual work principle (Tables 2 and 3). It is clear from Table 2 that, for lock-free finite element model, the virtual work principle predicts an eigenvalue which is different from the exact solution (i.e., $\hat{\omega}^2 = 1.9948 \times 10^{-2}$ is not equal to the $\omega^2 = 1.5802 \times 10^{-2}$). This violation of the virtual work principle is attributed to the variational incorrectness of the lock-free finite element model. It is very interesting to note that in the case of flexural spectra, the lock-free solution is more accurate than that of the locked solution because locking has been eliminated through reduced integration. However, in the case of shear dominated spectra, the locked solution is more accurate than the lock-free finite element solution. The increased accuracy of the flexural spectra by using reduced integration is actually due to cancellation of errors [15,17], where the error introduced by the reduced integration in the finite element model is in opposite sign of the error due to discretisation. The violation of the virtual work principle by the reduced integration finite element model will cause the loss of boundedness of the approximate eigenvalues with the exact eigenvalues. This shall be demonstrated using the

Table 2
Results of the exact solution, virtual work principle and the finite element solution (with locked and lock-free models) for the fundamental mode of the bending spectra ($n = 1$) for a thin and thick simply supported beam discretised using two linear Timoshenko beam elements

			$L/d = 10$	$L/d = 100$
Exact solution (Eq. (7))		$\langle \varepsilon,\varepsilon \rangle$	3.9817×10^3	4.0579
		(u,u)	2.5198×10^5	2.5002×10^4
		$\omega^2 = \dfrac{\langle \varepsilon,\varepsilon \rangle}{(u,u)}$	1.5802×10^{-2}	1.6230×10^{-4}
Virtual work principle (Eq. (8))	$\bar{u}$, $\bar{\varepsilon}$ from the locked solution	$\langle \bar{\varepsilon},\varepsilon \rangle$	3.2262×10^3	3.2892
		$(\bar{u},u)$	2.0417×10^5	2.0266×10^4
		$\omega^2 = \dfrac{\langle \bar{\varepsilon},\varepsilon \rangle}{(\bar{u},u)}$	1.5802×10^{-2}	1.6230×10^{-4}
	$\bar{u}^*$, $\bar{\varepsilon}^*$ from the lock-free solution	$\langle \bar{\varepsilon}^*, \varepsilon \rangle$	4.0826×10^3	4.1876
		$(\bar{u}^*,u)$	2.0466×10^5	2.0266×10^2
		$\hat{\omega}^2 = \dfrac{\langle \bar{\varepsilon}^*,\varepsilon \rangle}{(\bar{u}^*,u)}$	1.9948×10^{-2}	2.0663×10^{-4}
Finite element model (Eq. (6))	$\bar{u}$, $\bar{\varepsilon}$ from the locked solution	$\langle \bar{\varepsilon}, \bar{\varepsilon} \rangle$	4.5431×10^4	4.2529×10^3
		$(\bar{u}, \bar{u})$	1.6786×10^5	1.6667×10^4
		$\bar{\omega}^2 = \dfrac{\langle \bar{\varepsilon}, \bar{\varepsilon} \rangle}{(\bar{u}, \bar{u})}$	2.7065	2.5517
	$\bar{u}^*$, $\bar{\varepsilon}^*$ from the lock-free solution	$\langle \bar{\varepsilon}^*, \bar{\varepsilon}^* \rangle$	5.1710×10^3	5.3315
		$(\bar{u}^*, \bar{u}^*)$	1.6875×10^5	1.6668×10^4
		$\bar{\omega}^{*2} = \dfrac{\langle \bar{\varepsilon}^*, \bar{\varepsilon}^* \rangle}{(\bar{u}^*, \bar{u}^*)}$	3.0642×10^{-2}	3.1986×10^{-4}

Table 3
Results of the exact solution, virtual work principle and the finite element solution (with locked and lock-free models) for the 1st mode of the shear spectra ($n = 1$) for a thin and thick simply supported beam discretised using two linear Timoshenko beam elements

			$L/d = 10$	$L/d = 100$
Exact solution (Eq. (7))		$\langle \varepsilon,\varepsilon \rangle$	3.3351×10^9	3.1028×10^9
		(u,u)	3.1824×10^7	3.0411×10^8
		$\omega^2 = \dfrac{\langle \varepsilon, \varepsilon \rangle}{(u,u)}$	104.7970	102.0289×10^2
Virtual work principle (Eq. (8))	$\bar{u}$, $\bar{\varepsilon}$ from the locked solution	$\langle \bar{\varepsilon}, \varepsilon \rangle$	2.4547×10^9	4.6541×10^{10}
		$(\bar{u}, u)$	2.3423×10^7	4.5616×10^6
		$\omega^2 = \dfrac{\langle \bar{\varepsilon}, \varepsilon \rangle}{(\bar{u}, u)}$	104.7970	102.0281×10^2
	$\bar{u}^*$, $\bar{\varepsilon}^*$ from the lock-free solution	$\langle \bar{\varepsilon}^*, \varepsilon \rangle$	1.5607×10^9	3.6553×10^{10}
		$(\bar{u}^*, u)$	1.8815×10^7	4.5611×10^3
		$\hat{\omega}^2 = \dfrac{\langle \bar{\varepsilon}^*, \varepsilon \rangle}{(\bar{u}^*, u)}$	82.95	80.1408×10^2
Finite element model (Eq. (6))	$\bar{u}$, $\bar{\varepsilon}$ from the locked solution	$\langle \bar{\varepsilon}, \bar{\varepsilon} \rangle$	1.8393×10^9	7.0838×10^8
		$(\bar{u}, \bar{u})$	1.7493×10^7	6.9428×10^4
		$\bar{\omega}^2 = \dfrac{\langle \bar{\varepsilon}, \bar{\varepsilon} \rangle}{(\bar{u}, \bar{u})}$	105.14	102.0309×10^2
	$\bar{u}^*$, $\bar{\varepsilon}^*$ from the lock-free solution	$\langle \bar{\varepsilon}^*, \bar{\varepsilon}^* \rangle$	9.0215×10^8	5.3126×10^8
		$(\bar{u}^*, \bar{u}^*)$	1.1292×10^7	6.9415×10^4
		$\bar{\omega}^{*2} = \dfrac{\langle \bar{\varepsilon}^*, \bar{\varepsilon}^* \rangle}{(\bar{u}^*, \bar{u}^*)}$	79.89	76.5338×10^2

Table 4
Comparison of eigenvalues of the flexural and shear dominated spectra for a thin beam, $L/d = 100$, (Number of elements = 40)

Mode number	Exact solution		FEM solution			
	Bending	Shear	Locked solution		Lock-free solution	
			Bending	Shear	Bending	Shear
1	1.6255360E–4	1.0002796E4	5.855691E–4	1.0002797E4	1.623031E–4	9.9976546E3
2	2.6107312E–3	1.0011183E4	9.390151E–3	1.0011952E4	2.594674E–3	9.990602E3
3	1.3300617E–2	1.0025154E4	4.771556E–2	1.0025232E4	1.311723E–2	9.9787926E3
4	4.2410473E–2	1.0044701E4	1.515942E–1	1.0044956E4	4.137626E–2	9.9621436E3
5	1.047286E–1	1.0069809E4	3.725903E–1	1.0070436E4	1.007644E–1	9.9405376E3

sweep test in Section 7.3. The effect of reduced integration while using adaptive meshing is also discussed (Section 7.3).

7.2. Error convergence study for the eigenvalues

Table 4 summarises a comparison of the first five eigenvalues from the flexural and the shear dominated spectra for a thin beam. It can be noted that eigenvalues of the flexural spectra obtained by exact integration are higher than the exact solution. A one-point rule reduced integration technique eliminates shear locking and predicts more accurate eigenvalues than that of exact integration scheme. These lock-free eigenvalues are now less than the exact eigenvalues of the respective modes. It is well established that the eigenvalues obtained from a variationally correct consistent mass finite element model will always be upper bound to the corresponding exact eigenvalues [17]. In this case this rule has been violated, as reduced integration is an extra-variational technique, which violates the virtual work principle. It can be noted that the eigenvalues of the shear spectra obtained by an exact integration are more accurate than that obtained from a reduced integration technique. It can also be noted that, here too, a reduced integration technique predicts eigenvalues which are less than the exact eigenvalues. This lower boundedness nature of the eigenvalues with a reduced integration technique cannot be generalised, as we shall see later that the eigenvalues obtained using reduced integration scheme can be higher or lower than the exact eigenvalues.

Table 5 presents similar results for a thick beam. In this case, the approximate eigenvalues of the flexural spectra obtained by both exact and reduced integration techniques are above the exact solutions. It is now very clear that the finite element solution with a reduced integration technique employed to eliminate shear locking, is extra-variational and does not ensure the boundedness with respect to the corresponding exact solution. This is due to the fact that reduced integration is an extra-variational technique as it violates the frequency-error-hyperboloid formula presented in Eq. (13). However, as seen already, the variationally correct exact integration solution satisfies Eq. (13) exactly.

For the present study, $e = (\bar{\omega}^2/\omega^2 - 1)$ is used as error indicator for eigenvalues. Studies earlier have shown [17] that for a variationally correct formulation the value of e turns out to be a positive value, but no guarantee of this can be given for any extra-variational formulations. We first compare accuracies and rates of convergence for the locked and lock-free solutions for the thin beam ($L/d = 100$) as the mesh size is varied from $h = 250$ to 25 (Fig. 3). Fig. 3 presents the comparison between the accuracy and the convergence rate of the locked and lock-free solutions for the fundamental bending mode. It can be seen from Fig. 3 that both the locked and lock-free solutions are converging at $O(h^2)$. However the error in the locked solution is much more than the lock-free solution.

Table 5
Comparison of eigenvalues of the flexural and shear dominated spectra for a thick beam, $L/d = 10$, (number of elements = 40)

Mode number	Exact solution		FEM solution			
			Locked solution		Lock-free solution	
	Bending	Shear	Bending	Shear	Bending	Shear
1	1.5795636E–2	1.0278059E2	1.6223491E–2	1.027816E2	1.581958E–2	1.0273056E2
2	2.3411827E–1	1.1095143E2	2.4117007E–1	1.109674E2	2.354741E–1	1.1076706E2
3	1.0595801	1.2410791E2	1.0965707	1.2418757E2	1.0725456	1.2374681E2
4	2.930737	1.4181147E2	3.0515435	1.4205985E2	2.9902583	1.4129188E2
5	6.1979551	1.6371174E2	6.5008694	1.6431179E2	6.381806	1.6312915E2

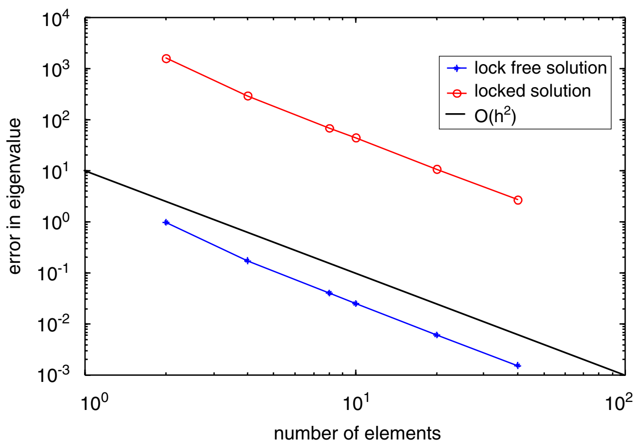

Fig. 3. Comparison of accuracy and convergence rate for the fundamental bending mode for both the locked and lock-free case.

The recent trend in literature is to study the rate of convergence of error by plotting e against the mode number n for a fixed but very large number of elements N (say $N = 40$) on a *log–log* scale [13–15]. As we have seen above, if any extra-variational trick is introduced in the formulation, the error e may turn out to be consistently negative (we shall see such a case in shear spectra resulting from reduced integration), or it may even be inconsistent in the sign; in such cases the absolute value of e is used for obtaining the convergence graph.

Fig. 4 shows a convergence plot for approximate eigenvalues obtained for a thin beam using exactly integrated stiffness matrix. Due to shear locking, as predicted, the basic bending dominated spectra shows almost a zero convergence rate as shown by a horizontal line instead of $O(h^2)$ that is predicted. This is because all the flexural modes are affected in the same way due to the spurious stiffening as the rigidity has been modified by the factor $I^*/I = 1 + kGAl^2/12EI$. The shear dominated spectra does not show any locking behaviour, and it converges at $O(h^4)$. For a better comparison of the rate of convergence of the computational results a reference line with slope 4 (i.e. $O(h^4)$) is drawn in the same plot.

Similar results for a thick beam ($L/d = 10$) are presented in Fig. 5. Again the bending dominated flexural spectra shows over-stiffened results with very poor convergence (almost zero). Here too, the shear dominated

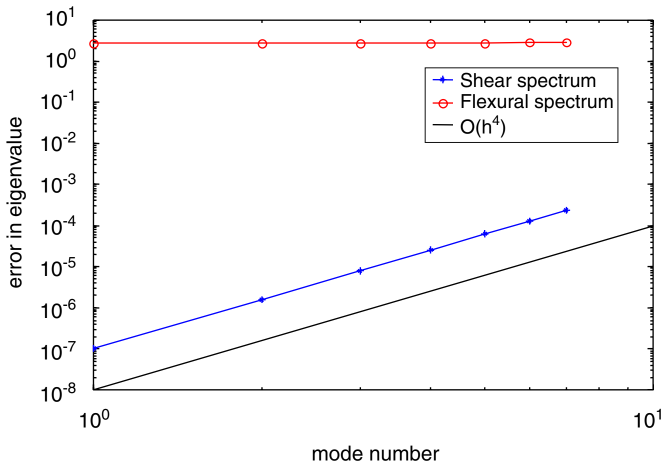

Fig. 4. Variation of error with mode number compared with Timoshenko theory for a thin beam using linear Timoshenko beam element (locked solution), $L/d = 100$, 40 elements.

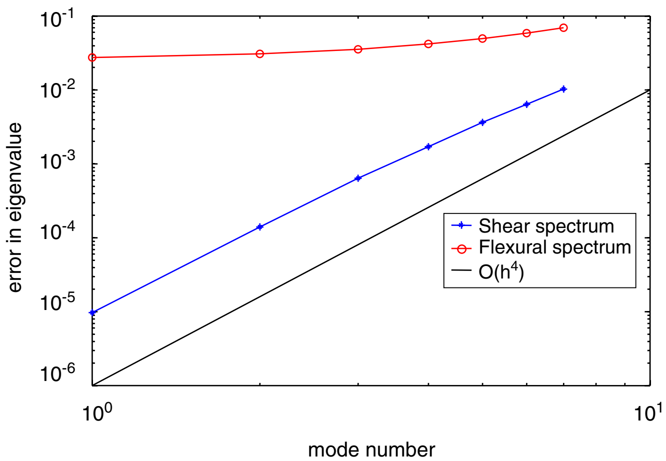

Fig. 5. Variation of error with mode number compared with Timoshenko theory for a thick beam using linear Timoshenko beam element (locked solution), $L/d = 10$, 40 elements.

spectra does not show any locking. The above two experiments confirm that locking is seen only for the bending dominated spectra but not for the shear spectra. This confirms an earlier prediction [9] based on field consistency arguments that locking is due to the artificial stiffening of the bending action due to inconsistent terms from the shear strain interpolation while the shear stiffness remains unaffected.

To avoid shear locking, the shear energy terms are integrated one order less than that required for an equivalent exact integration [4–9]. The results for this case are presented in Fig. 6 for a thin beam. The bending dominated spectra now converges at $O(h^2)$. This is in fact the optimal rate expected for the two noded Timoshenko beam element. But the order of convergence of the shear dominated spectra is now reduced from $O(h^4)$ to $O(h^2)$ as predicted in the previous section. It is very interesting that, here the approximate eigenvalue for the shear dominated spectra is now less than the exact solution. So, the absolute value of the error e is used to obtain Fig. 6.

Fig. 7 shows similar results for a thick beam ($L/d = 10$). Again as predicted, the bending dominated flexural spectra converges nearly at $O(h^2)$. For the shear dominated spectra, the eigenvalues of the first 6 modes converge with $O(h^2)$. But for the higher modes the approximate solutions are totally away from the exact, and show inconsistency in the boundedness nature. For this case, the 1st seven modes show lower bound and 8th & 9th modes show upper bound with respect to the respective exact solutions. Hence Fig. 7 is obtained by

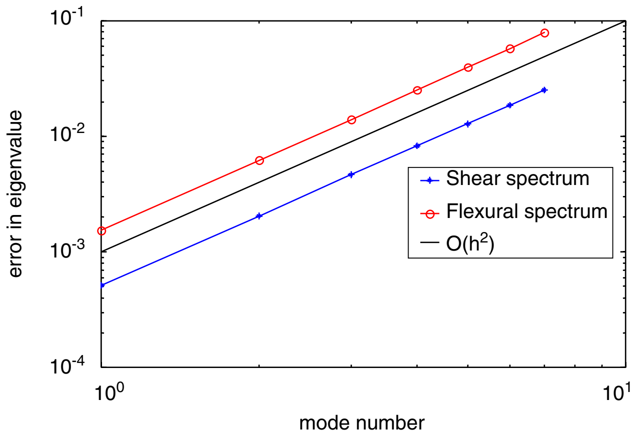

Fig. 6. Variation of error with mode number compared with Timoshenko theory for a thin beam using linear Timoshenko beam element (lock-free solution), $L/d = 100$, 40 elements.

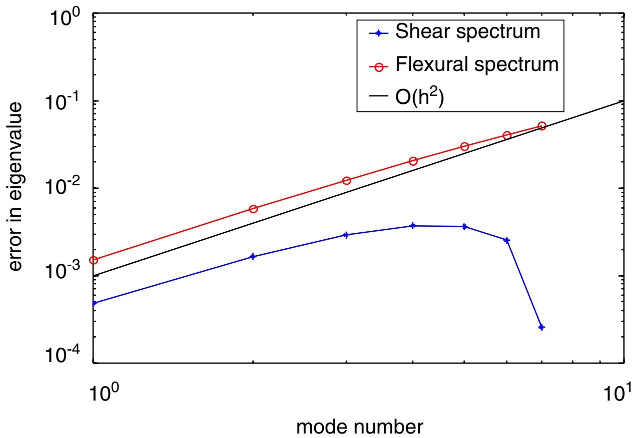

Fig. 7. Variation of error with mode number compared with Timoshenko theory for a thick beam using linear Timoshenko beam element (lock-free solution), $L/d = 10$, 40 elements.

considering only 1st 7 modes. This inconsistency in the boundedness nature can be attributed to the extra-variational nature of reduced integration. This is demonstrated below using a two element moving node "sweep test".

7.3. The sweep test

The sweep test result for a thin beam is presented in Figs. 8 and 9 for the first eigenvalues from the bending dominated flexural spectra and the shear dominated spectra respectively, with locked and lock-free solutions. The locked solution, as it is variationally correct, is upper bound to the exact solution for both flexural and shear spectra. But as one introduces the "*variational crime*" of reduced integration [16] in the shear energy terms the upper bound nature of the shear spectra is lost (Fig. 9). For this particular case the flexural spectra is still the upper bound to the exact solution; in general, this cannot be ensured always in any extra variational finite element formulation.

Note that the error in the approximate eigenvalues of both spectra obtained in this two-element moving node test is a minimum, when the 2nd node is placed exactly at the centre of the beam. This is expected, because the energy error will be least for this discretisation as it is equi-partitioned. It is observed that the error in the eigenvalues of flexural mode with reduced integration is much less than that of a locked solution.

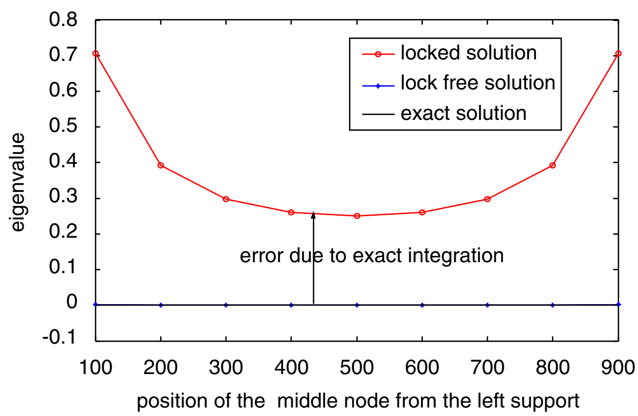

Fig. 8. Variation of eigenvalues of the bending dominated flexural spectra with change in position of middle node for a thin beam, $L/d = 100$, 2 elements.

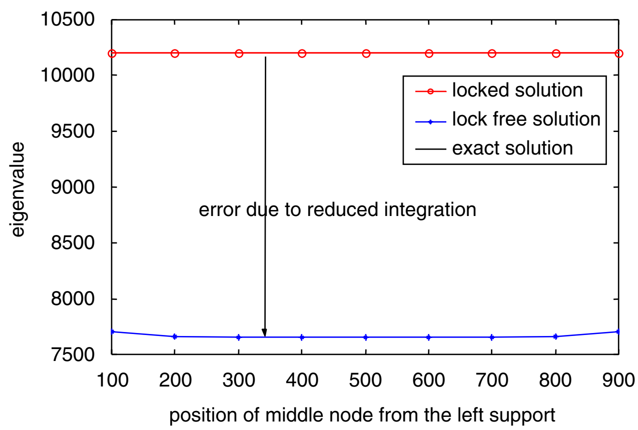

Fig. 9. Variation of eigenvalues of the shear dominated spectra with change in position of middle node for a thin beam, $L/d = 100$, 2 elements.

Conversely in the case of the shear mode, the error is more in a lock-free solution than that of a locked solution. Also the lock-free solution is below the exact solution. In a strict sense, the error introduced in the approximate solution due to the "*variational crime*" of reduced integration should have been compared with that of the approximate solution obtained from the variationally correct formulation, in this case the locked solution. The total error introduced due to reduced integration is marked in the respective figures.

Results for a thick beam are presented in Figs. 10 and 11. In this case the computed eigenvalues of shear spectra with reduced integration even crosses the exact solution. This signifies that as the mesh becomes asymmetric, the discretisation error is opposite in sign to that introduced by variational incorrectness, and with sufficiently large mesh distortion, the two errors can exactly compensate each other. Using this discretisation the finite element model can predict the exact eigenvalues even with two elements! However, there is no variational basis to find out such a super-optimal finite element discretisation.

It is now very clear that reduced integration technique is extra variational in nature and produces erroneous results in the shear dominated spectra, though the convergence of bending dominated flexural spectra is very good, consistent with the order expected for a linear element.

Most of the general purpose software packages use adaptive meshing to get the optimal mesh discretisation of the finite element models. It is clear from the above experiments that the adaptive meshing may give wrong

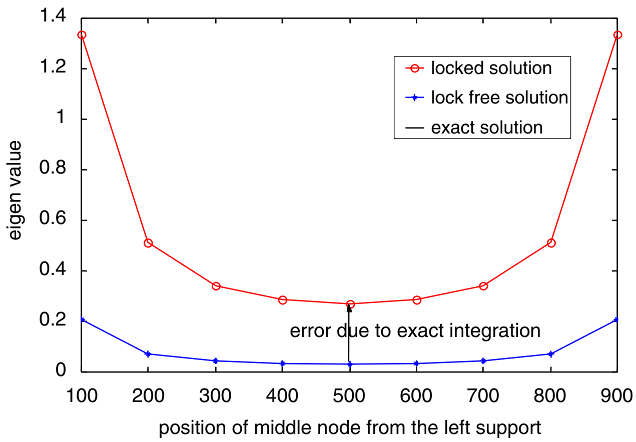

Fig. 10. Variation of eigenvalues of the bending dominated flexural spectra with change in position of middle node for a thick beam, $L/d = 10$, 2 elements.

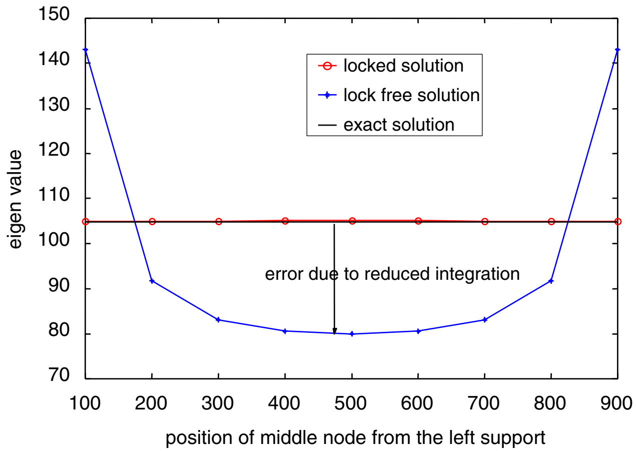

Fig. 11. Variation of eigenvalues of the shear dominated spectra with change in position of middle node for a thick beam, $L/d = 10$, 2 elements.

results in case of variationally incorrect finite element formulations, such as, reduced integration. From the examples above, one can note that, in the case of reduced integration, the optimum solutions (where the slope of "egg cup" profile is zero) are not the best solutions; in fact it even predicts the exact solution for a particular position of 2nd node where the error due to mesh distortion compensates for the error due to the variational incorrectness introduced by reduced integration. It is also seen that for complicated problems, the finite element models with extra variational formulations can produce multi-optimum points; which brings further complications in the optimisation of adaptive meshing algorithms.

8. Conclusion

The finite element elastodynamics of 1D beams using the two-noded Timoshenko beam element has been studied carefully from the error analysis aspects, using the energy-error rule and the projection theorems derived from the variational (weak form) approach of Strang and Fix [16]. The relatively different behaviour of the flexural and the shear spectra is rationalised. The variationally correct element locks. It is a common practise to remove the locking behaviour of the element by using reduced integration for the shear stiffness matrix. The present paper demonstrates that reduced integration violates virtual work principle (Eq. (8)) and frequency-error-hyperboloid rule (Eq. (13)), and hence, is an extra-variational procedure [16] and gives inaccurate results for eigenvalues of shear spectra. This is also vindicated when the errors from the finite element formulation are examined, as the upper-bound nature of the frequencies, especially of the second spectrum, is no longer conserved. This paper also discusses the implications of extra variational techniques such as reduced integration while using the adaptive meshing for optimal mesh discretisation.

Appendix A

A.1. *Stiffness-inner product*

If $\{a\}$ and $\{b\}$ are vectors each of r-rows, and $[D]$ is a positive definite square rigidity matrix of size $r \times r$, then their *stiffness-inner product* is defined as

$$\langle a, b\rangle = \sum_{\mathrm{ele}=1}^{N^e} \int_{\mathrm{ele}} \{a\}^{\mathrm{T}}[D]\{b\}\,\mathrm{d}x \tag{17a}$$

and the *stiffness-norm squared* value of the vector $\{a\}$ is given as

$$\|a\|^2 = \langle a, a\rangle, \tag{17b}$$

where N^e is the number of elements.

A.2. *Inertia-inner product*

If $\{c\}$ and $\{d\}$ are vectors each of s-rows, and $[\rho]$ is a positive definite square inertia density matrix of size $s \times s$, then their *inertia-inner product* is defined as

$$(c, d) = \sum_{\mathrm{ele}=1}^{N^e} \int_{\mathrm{ele}} \{c\}^{\mathrm{T}}[\rho]\{d\}\mathrm{d}x \tag{18a}$$

and the *inertia-norm squared* value of the vector $\{c\}$ is given as

$$|c|^2 = (c, c). \tag{18b}$$

Note that these two inner products are global in character.

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