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Short Communication

Solution of a mixed parity nonlinear oscillator: Harmonic balance

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Abstract

The method of harmonic balance is used to calculate first-order approximations to the periodic solutions of a mixed parity nonlinear oscillator. First, the amplitude in the negative direction is expressed in terms of the amplitude in the positive direction. Then the two auxiliary equations, where the restoring force functions are odd, are solved by using a first harmonic term (without a constant). Therefore, we obtain the two approximate solutions to the mixed parity nonlinear oscillator. One is expressed in terms of the exact amplitude in the negative direction, the other in terms of the approximate amplitude. These solutions are more accurate than the second approximate solution obtained by the Lindstedt–Poincaré method for large amplitudes.

1. Introduction

A differential equation modeling many of the important features of nonlinear oscillations is [1]

$$\frac{d^2x}{d\bar{t}^2} + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 = 0, \tag{1}$$

where $(\alpha_1, \alpha_2, \alpha_3)$ are non-negative parameters. The method of harmonic balance is a useful technique [2,3] for solving nonlinear oscillatory problems. It has been found to work well when all terms in the dependent variable have odd parity [4]. Nayfeh and Mook [1, pp. 59–61] have cautioned against use of the method when terms of mixed parity are involved, pointing out that for full consistency, a second harmonic term (as well as a constant) must be taken into account in the solution expression. The main purpose of this communication is to use the first-order harmonic balance method to determine analytical approximations to the periodic solutions of mixed parity nonlinear oscillators by resorting to two auxiliary equations. This work represents a companion to previous work on the quadratic nonlinear oscillator [5]:

$$\ddot{x} + x + \varepsilon x^2 = 0. \tag{2}$$

For convenience, defining $x = \sqrt{\alpha_1/\alpha_3}\,y$ and $\bar{t} = t/\sqrt{\alpha_1}$, Eq. (1) is reduced to the following equation:

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$$\ddot{y} + y + \varepsilon y^2 + y^3 = 0, \tag{3}$$

where $\varepsilon = \alpha_2/\sqrt{\alpha_1\alpha_3}$ and overdots denote differentiation with respect to time, t. Let the initial conditions be

$$y(0) = A > 0, \quad \dot{y}(0) = 0. \tag{4}$$

By the use of scaling [6, pp. 398–400] and the Lindstedt–Poincaré perturbation method, Mickens [2, pp. 68–71] obtained the second approximation for Eqs. (3) and (4):

$$\begin{aligned} y_M = y_M(\theta, A) &= A\cos\theta + \frac{\varepsilon A^2}{6}(-3 + 2\cos\theta + \cos 2\theta) \\ &+ \frac{A^3}{3}\left[-\varepsilon^2 + \left(\frac{174\varepsilon^2 - 27}{288}\right)\cos\theta + \frac{\varepsilon^2}{3}\cos 2\theta + \left(\frac{2\varepsilon^2 + 3}{32}\right)\cos 3\theta\right], \end{aligned} \tag{5}$$

where

$$\theta = \omega_M t, \quad \omega_M = 1 + (9 - 10\varepsilon^2)A^2/24, \quad 0 < A \ll 1. \tag{6a, b, c}$$

(Eqs. (5) and (6) can be obtained by letting $\alpha = \varepsilon$ and $\beta = 1$ in Eqs. (2.105) and (2.106) of Ref. [2].) The corresponding approximate period of the oscillation is

$$T_M = 2\pi/\omega_M = 2\pi[1 + (9 - 10\varepsilon^2)A^2/24]^{-1}. \tag{7}$$

We will see that the first approximations obtained in this paper are more accurate than Mickens's results for large amplitudes.

2. The amplitude B in the negative direction

Eq. (3) can be rewritten as

$$\dot{y}\,\mathrm{d}\dot{y} + (y + \varepsilon y^2 + y^3)\,\mathrm{d}y = 0. \tag{8}$$

Integrating of this equation gives the first integral

$$\frac{\dot{y}^2}{2} + \frac{y^2}{2} + \frac{\varepsilon y^3}{3} + \frac{y^4}{4} = h, \tag{9}$$

where h is a constant of integration. The behavior of mixed parity nonlinear oscillators is different for positive and negative directions. Assume that the system oscillates between asymmetric limits $[-B, A]$ ($B > 0$). Noting that when $y = A$ and $y = -B$, the corresponding $\dot{y} = 0$, we have from Eq. (9)

$$\frac{B^2}{2} - \frac{\varepsilon B^3}{3} + \frac{B^4}{4} = \frac{A^2}{2} + \frac{\varepsilon A^3}{3} + \frac{A^4}{4}. \tag{10}$$

Solving for B with MATLAB gives the following exact solution:

$$B_e = \frac{A}{3} + \frac{4\varepsilon}{9} + \frac{C^{1/3}}{9} - \left(1 - \frac{8\varepsilon^2}{27} + \frac{2\varepsilon A}{9} + \frac{A^2}{3}\right)\frac{6}{C^{1/3}}, \tag{11}$$

where

$$\begin{aligned} C = {} & 270A^2(\varepsilon + A) + 18A(27 - 4\varepsilon^2) + 4\varepsilon(16\varepsilon^2 - 81) + 54[54(1 + \varepsilon A^3 + \varepsilon A^5) + 3\varepsilon^2(3A^4 - 4) \\ & + 108A^2(A^2 - \varepsilon^2) + 9A^2(15 + 3A^4) - 8\varepsilon A(9 + \varepsilon^2 A^2) + 16\varepsilon^3 A(1 + \varepsilon A)]^{1/2}. \end{aligned} \tag{12}$$

We now seek an approximate expression for B. Actually, the smaller ε is, the more "cubic" Eq. (3) is. In this case, B is close to A even though A is not small. Therefore, we have

$$B = A + \Delta B, \tag{13}$$

where $\Delta B \to 0$ when $\varepsilon \to 0$. Let

$$h(B, \varepsilon) = \frac{B^2}{2} - \frac{\varepsilon B^3}{3} + \frac{B^4}{4}, \quad h_B(B, \varepsilon) = \frac{\partial h}{\partial B}. \tag{14a, b}$$

Table 1
Comparison of the approximate amplitudes with the exact amplitude for $\varepsilon = 1$

A	B_e	B_M	B_a
0.1	0.1071	0.1071	0.1073
0.2	0.2299	0.2302	0.2317
0.4	0.5229	0.5351	0.5404
0.5	0.6846	0.7222	0.7222
0.6	0.8464	0.9360	0.9158
0.8	1.1511	1.4542	1.3079
1.0	1.4257	2.1111	1.6667
2.0	2.5789	8.2222	2.8889
5.0	5.6488	77.2222	5.7937
10.0	10.6618	521.1111	10.7326
100.0	100.6666	4.5121×10^5	100.6733

Substituting Eq. (13) into Eq. (14a) results in

$$h(B,\varepsilon) \approx h(A,\varepsilon) + h_B(A,\varepsilon)\,\Delta B = \frac{A^2}{2} - \frac{\varepsilon A^3}{3} + \frac{A^4}{4} + (A - \varepsilon A^2 + A^3)\,\Delta B. \tag{15}$$

From Eqs. (10) and (15), we obtain

$$\Delta B = \frac{2\varepsilon A^2}{3(1 - \varepsilon A + A^2)}. \tag{16}$$

Thus, the approximate expression for B is

$$B_a = B = A + \Delta B = A + \frac{2\varepsilon A^2}{3(1 - \varepsilon A + A^2)}. \tag{17}$$

We also have an approximate expression from Eq. (5)

$$B_M = \left|y_M(\pi, A)\right| = A\left[1 + \frac{2\varepsilon A}{3}\left(1 + \frac{2\varepsilon A}{3}\right)\right]. \tag{18}$$

For comparison, the exact amplitude B_e and the approximate amplitudes computed by Eqs. (17) and (18), respectively, are listed for $\varepsilon = 1$ in Table 1. Table 1 shows that B_M is somewhat more accurate than B_a if $A \leqslant 0.4$, but when $A \geqslant 0.6$ B_a is more accurate than B_M.

3. Solutions of the two auxiliary equations

Based on the discussion in Ref. [5], we first consider the following auxiliary equation:

$$\ddot{y} + y + \varepsilon y^2\,\mathrm{sgn}(y) + y^3 = 0, \quad y(0) = A, \quad \dot{y}(0) = 0, \tag{19}$$

where $\mathrm{sgn}(y)$ is the sign function, equal to $+1$ if $y > 0$, 0 if $y = 0$ and -1 if $y < 0$. Let ω_A be the angular frequency of Eq. (19). Then substituting

$$y = A\cos\omega_A t = A\cos\theta \tag{20}$$

into Eq. (19) and taking into account that [5]

$$(A\cos\theta)^2\,\mathrm{sgn}(A\cos\theta) = |A\cos\theta|A\cos\theta = \frac{8A^2}{3\pi}\cos\theta + \text{higher order harmonics}, \tag{21}$$

we obtain

$$\left(-\omega_A^2 + 1 + \frac{8\varepsilon A}{3\pi} + \frac{3A^2}{4}\right)A\cos\theta + \text{higher order harmonics} = 0. \tag{22}$$

Setting the coefficient of $\cos\,\omega_A t$ equal to zero and solving for ω_A yields

$$\omega_A = \sqrt{1 + \frac{8\varepsilon A}{3\pi} + \frac{3A^2}{4}}. \tag{23}$$

Therefore, a first approximation to the periodic solution of Eq. (19) is

$$y = A\cos\sqrt{1 + \frac{8\varepsilon A}{3\pi} + \frac{3A^2}{4}}t. \tag{24}$$

The corresponding approximate period of the oscillation is

$$T_A = \frac{2\pi}{\omega_A} = 2\pi\left(1 + \frac{8\varepsilon A}{3\pi} + \frac{3A^2}{4}\right)^{-1/2}. \tag{25}$$

Now we consider the second auxiliary equation:

$$\ddot{y} + y - \varepsilon y^2\,\mathrm{sgn}(y) + y^3 = 0, \quad y(0) = B, \quad \dot{y}(0) = 0. \tag{26}$$

The first approximation to Eq. (26) is assumed to be

$$y = B\cos\omega_B t = B\cos\theta, \tag{27}$$

where ω_B is the angular frequency of Eq. (26). Substituting Eq. (27) into Eq. (26) and noting the relation given in Eq. (21), we have

$$\left(-\omega_B^2 + 1 - \frac{8\varepsilon B}{3\pi} + \frac{3B^2}{4}\right)B\cos\theta + \text{higher order harmonics} = 0. \tag{28}$$

Appling the harmonic balance technique gives

$$\omega_B = \sqrt{1 - \frac{8\varepsilon B}{3\pi} + \frac{3B^2}{4}}. \tag{29}$$

Then from Eq. (27) we obtain

$$y = B\cos\sqrt{1 - \frac{8\varepsilon B}{3\pi} + \frac{3B^2}{4}}t. \tag{30}$$

The corresponding approximate period of the oscillation to Eq. (26) is

$$T_B = \frac{2\pi}{\omega_B} = 2\pi\left(1 - \frac{8\varepsilon B}{3\pi} + \frac{3B^2}{4}\right)^{-1/2}. \tag{31}$$

4. Results and discussion

The first approximate period T_1 and the corresponding periodic solution $y_1(t)$ to Eqs. (3) and (4) are, respectively,

$$T_1 = \frac{T_A + T_B}{2} = \pi\left[\left(1 + \frac{8\varepsilon A}{3\pi} + \frac{3A^2}{4}\right)^{-1/2} + \left(1 - \frac{8\varepsilon B}{3\pi} + \frac{3B^2}{4}\right)^{-1/2}\right], \tag{32}$$

$$y_1(t) = A\cos\omega_A t, \quad 0 \leqslant t \leqslant \frac{T_A}{4}, \tag{33a}$$

$$y_1(t) = B\cos\omega_B\left(t - \frac{T_A}{4} + \frac{T_B}{4}\right), \quad \frac{T_A}{4} \leqslant t \leqslant \frac{T_A}{4} + \frac{T_B}{2}, \tag{33b}$$

Table 2
Comparison of approximate periods with the corresponding exact period to Eqs. (3) and (4) for $\varepsilon = 1$

A	T_e	T_M (%error)	T_{1Be} (%error)	T_{1Ba} (%error)
0.1	6.28559	6.28580 (0.0035)	6.28526 (−0.0052)	6.28554 (−0.00007)
0.2	6.28794	6.29367 (0.0911)	6.28673 (−0.0193)	6.28865 (0.0113)
0.4	6.20413	6.32535 (1.9539)	6.20119 (−0.0475)	6.20331 (−0.0132)
0.5	6.05759	6.34932 (4.8159)	6.05370 (−0.0642)	6.03580 (−0.3597)
0.6	5.83368	6.37887 (9.3455)	5.82692 (−0.1160)	5.75941 (−1.2732)
0.8	5.27286	6.45533 (22.4256)	5.25371 (−0.3631)	5.03897 (−4.4357)
1.0	4.72140	6.55637 (38.8649)	4.68905 (−0.6853)	4.38081 (−7.2137)
2.0	2.97577	7.53982 (153.3738)	2.92793 (−1.6076)	2.74902 (−7.6201)
5.0	1.36965	−150.79645 (—)	1.34144 (−2.0595)	1.32275 (−3.4245)
10.0	0.71463	−1.98416 (—)	0.69932 (−2.1420)	0.69686 (−2.4868)
100.0	0.073913	−0.015116 (—)	0.072308 (−2.1720)	0.072305 (−2.1752)

$$y_1(t) = A\cos\omega_A\left(t + \frac{T_A}{2} - \frac{T_B}{2}\right), \quad \frac{T_A}{4} + \frac{T_B}{2} \leqslant t \leqslant T_1. \tag{33c}$$

If $B = B_e$ (B_a) in Eqs. (32) and (33), then we will use T_{1Be} (T_{1Ba}) and y_{1Be} (y_{1Ba}) to denote T_1 and y_1, respectively.

The exact period T_e to Eqs. (3) and (4) is

$$T_e = \int_0^A \frac{2\,\mathrm{d}x}{\sqrt{A^2 - x^2 + \frac{2}{3}\varepsilon(A^3 - x^3) + \frac{1}{2}(A^4 - x^4)}} + \int_0^B \frac{2\,\mathrm{d}x}{\sqrt{B^2 - x^2 - \frac{2}{3}\varepsilon(B^3 - x^3) + \frac{1}{2}(B^4 - x^4)}}, \tag{34}$$

where B is given, in terms of A, in Eq. (11).

For comparison, the exact period T_e obtained by integrating Eq. (34) and the approximate periods T_M (Eq. (7)), T_{1Be} ($B = B_e$ in Eq. (32)) and T_{1Ba} ($B = B_a$ in Eq. (32)) are listed in Table 2 for $\varepsilon = 1$. The percentage errors are defined as $100[T_M(T_{1Be}, T_{1Ba}) - T_e]/T_e$. Table 2 indicates that there is not much difference between the exact period and the approximate periods if $A \leqslant 0.2$. But T_{1Be} and T_{1Ba} are more accurate than T_M when $A \geqslant 0.4$. T_M is not valid for large amplitudes.

Comparisons between the numerical solution y_{Num} of Eqs. (3) and (4) with the approximate solutions y_M (Eq. (5)), y_{1Be} and y_{1Ba} are shown in Figs. 1 and 2 for the time in one exact period ($\varepsilon = 1$). Fig. 1 shows that y_{1Be} and y_{1Ba} are more accurate than y_M. When $A \geqslant 1 y_M$ is not valid. Therefore, y_M does not appear in Fig. 2.

5. Conclusions

A mixed parity nonlinear oscillator modeled by Eqs. (3) and (4) has been attacked by the first-order harmonic balance method. First, the amplitude B in the negative direction is expressed in terms of the amplitude A. Then the method is applied to the two auxiliary equations (19) and (26), where the restoring force functions are odd. The first approximate periods T_{1Be} and T_{1Ba} are more accurate than the second approximate period T_M obtained by the Lindstedt–Poincaré method when $A \geqslant 0.2$, and the approximate solutions y_{1Be} and y_{1Ba} are more accurate than the second approximate solution y_M for large amplitudes. Furthermore, if we only use the approximate amplitude B_a, the first-order harmonic balance is much simpler than the second-order perturbation. In fact, if we obtain the approximate solutions to Eq. (19), then we do not need to actually solve Eq. (26). Letting $\varepsilon = -\varepsilon$ and $A = B$ in Eqs. (23), (24) and (25) gives Eqs. (29), (30) and (31) immediately. Obviously, the approach in this paper can be applied to other types of mixed parity nonlinear oscillators.

Although our result complements the analysis of Mickens [2, pp. 68–71] for small amplitudes, there is clearly room for improvement of the accuracy of the approximate solutions y_{1Be} and y_{1Ba}, especially when $A \geqslant 1$. It is very difficult to use the method of harmonic balance to construct higher-order analytical approximations because it requires analytical solutions of sets of complicated nonlinear algebraic equations.

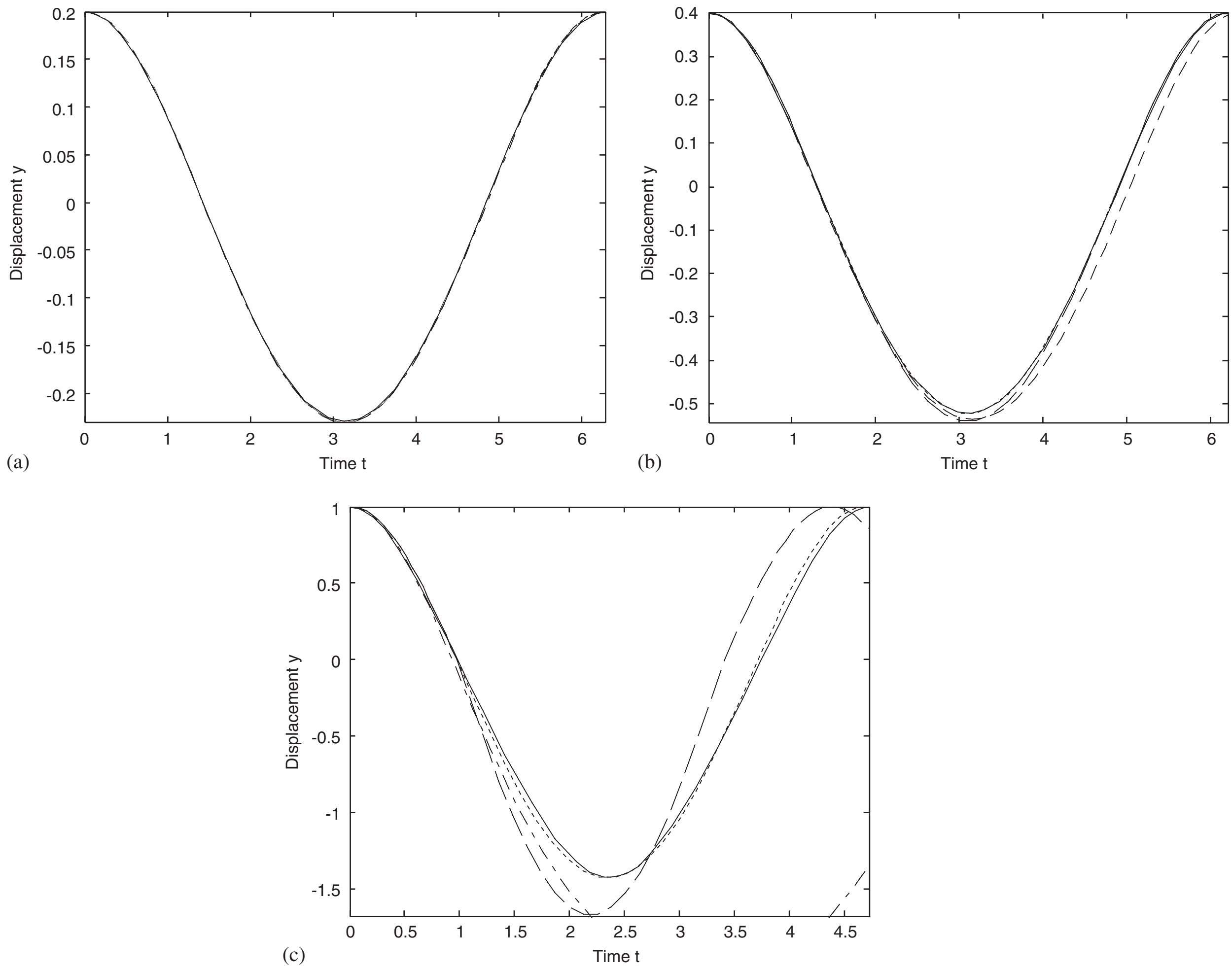

Fig. 1. Comparison of the approximate solutions y_M (dash-dot curve), y_{1Be} (dotted curve) and y_{1Ba} (dashed curve) with the numerical solution y_{Num} (solid curve) for $\varepsilon = 1$: (a) $A = 0.2$; (b) $A = 0.4$; (c) $A = 1.0$.

Thus, in this paper we restrict our investigation by the first harmonic only. We will apply a modified iteration procedure [7] to Eqs. (3) and (4) for second-order approximations in another paper.

The mixed parity nonlinear differential equation (1) occurs quite widely in a variety of engineering applications such as the nonlinear free vibrations of laminated plates [2,8–10]. If $\alpha_2 < 0$ and $\alpha_3 < 0$, then we may let $\alpha_2 = -\beta_2$ and $\alpha_3 = -\beta_3$. Thus, defining $x = \sqrt{\alpha_1/\beta_3}y$ and $\bar{t} = t/\sqrt{\alpha_1}$, Eq. (1) is reduced to

$$\ddot{y} + y - \varepsilon y^2 - y^3 = 0, \tag{35}$$

where $\varepsilon = \beta_2/\sqrt{\alpha_1\beta_3}$. Obviously, the present method can also be used to deal with this equation.

The main difference between this paper and Ref. [5] is that here we present a method for approximate expressions for B. When the restoring force function is a combination of quadratic, cubic and quintic terms, there is no exact expression for B. But we can still use the method in this paper to obtain approximate solutions for B.

Because the behavior of quadratic or mixed parity oscillators is different for positive and negative directions, quadratic and mixed parity oscillators are more complicated than cubic oscillators. One limitation of the results developed in this paper is the requirement that the initial conditions should be $y(0) = A > 0$ and $\dot{y}(0) = 0$. It needs further research to solve Eq. (3) with arbitrary initial conditions by using the present method.

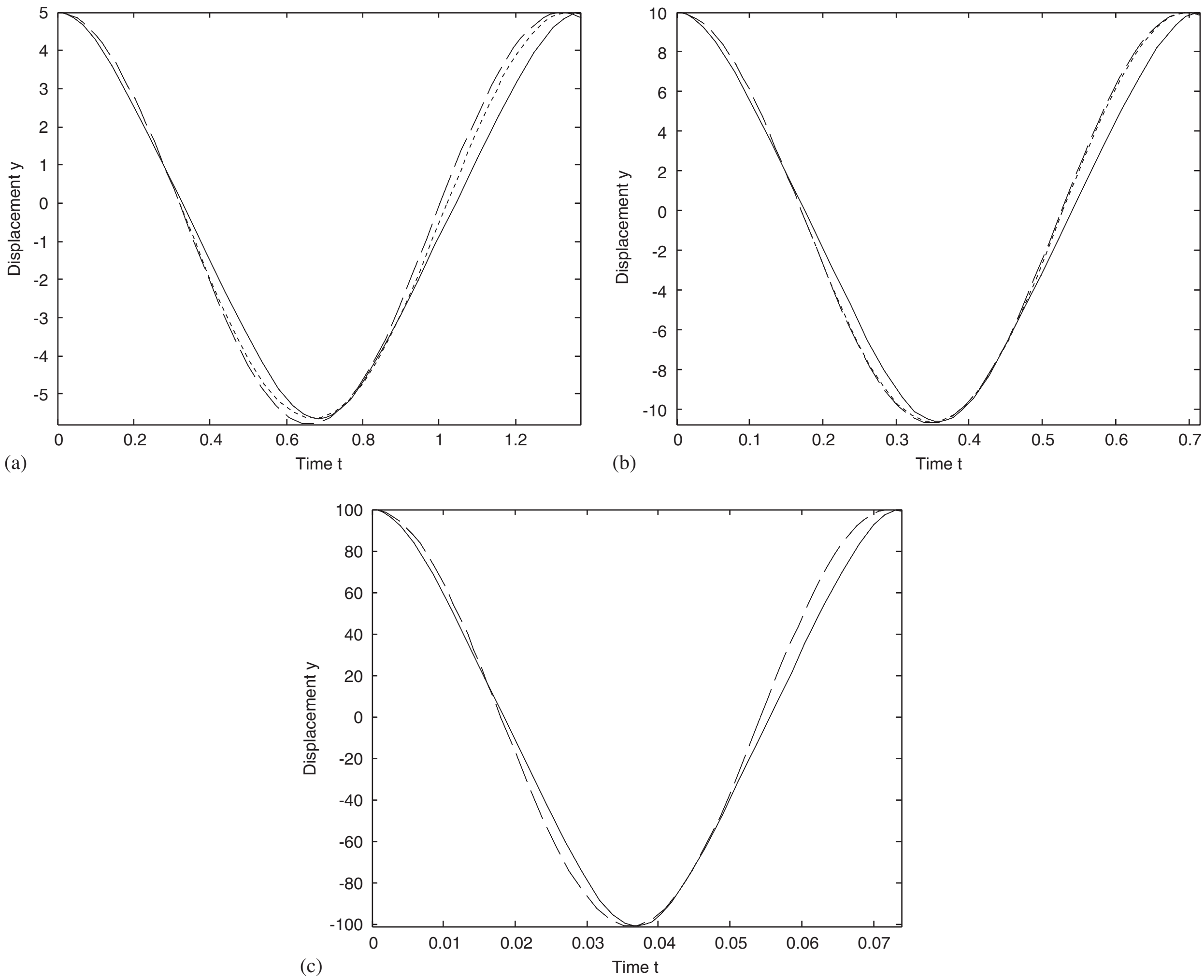

Fig. 2. Comparison of the approximate solutions y_{1Be} (dotted curve) and y_{1Ba} (dashed curve) with the numerical solution y_{Num} (solid curve) for $\varepsilon = 1$: (a) $A = 5.0$; (b) $A = 10.0$; (c) $A = 100.0$.

In this paper, the approximate solution to Eqs. (3) and (4) is obtained with the aid of the two auxiliary equations (19) and (26). In general, consider the nonlinear oscillator modeled by

$$\ddot{y} + f_{\mathrm{odd}}(y) + f_{\mathrm{even}}(y) = 0, \quad y(0) = A > 0, \quad \dot{y}(0) = 0, \tag{36}$$

where

$$f_{\mathrm{odd}}(-y) = -f_{\mathrm{odd}}(y), \quad f_{\mathrm{even}}(-y) = f_{\mathrm{even}}(y). \tag{37a, b}$$

When $y \geqslant 0$, Eq. (36) is equivalent to

$$\ddot{y} + f_{\mathrm{odd}}(y) + \mathrm{sgn}(y) f_{\mathrm{even}}(y) = 0, \quad y(0) = A > 0, \quad \dot{y}(0) = 0, \tag{38}$$

in which the restoring force function $f(y) = f_{\mathrm{odd}}(y) + \mathrm{sgn}(y) f_{\mathrm{even}}(y)$ is odd. If $y < 0$, then substituting $y = -\bar{y}$ ($\bar{y} > 0$) into Eq. (36) gives

$$\ddot{\bar{y}} + f_{\mathrm{odd}}(\bar{y}) - f_{\mathrm{even}}(\bar{y}) = 0, \quad \bar{y}(0) = B > 0, \quad \dot{\bar{y}}(0) = 0. \tag{39}$$

For $\bar{y} > 0$, Eq. (39) is also equivalent to

$$\ddot{\bar{y}} + f_{\mathrm{odd}}(\bar{y}) - \mathrm{sgn}(\bar{y}) f_{\mathrm{even}}(\bar{y}) = 0, \quad \bar{y}(0) = B > 0, \quad \dot{\bar{y}}(0) = 0, \tag{40}$$

where the restoring force function $f(\bar{y}) = f_{\text{odd}}(\bar{y}) - \text{sgn}(\bar{y}) f_{\text{even}}(\bar{y})$ is also odd. Like Eq. (10), the relation between A and B is described by the following equation:

$$\int_0^A f_{\text{odd}}(y)\,\mathrm{d}y + \int_0^A f_{\text{even}}(y)\,\mathrm{d}y = \int_0^{-B} f_{\text{odd}}(y)\,\mathrm{d}y + \int_0^{-B} f_{\text{even}}(y)\,\mathrm{d}y = \int_0^{B} f_{\text{odd}}(y)\,\mathrm{d}y - \int_0^{B} f_{\text{even}}(y)\,\mathrm{d}y. \quad (41)$$

Obviously, the two auxiliary equations of Eq. (36) are Eqs. (38) and (40).

Acknowledgements

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