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Short Communication

Nonlinear free transverse vibration of an axially moving beam: Comparison of two models

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Abstract

Nonlinear free transverse vibration of an axially moving beam is investigated. A partial-differential equation governing the transverse vibration is derived from the Newton's second law. Under the assumption that the tension of beam can be replaced by the averaged tension over the beam, the partial-differential reduces to a widely used integro-partial-differential equation for nonlinear free transverse vibration. The method of multiple scales is applied directly to two equations to evaluate nonlinear natural frequencies. Numerical examples are presented to demonstrate the analytical results and to highlight the difference between two models. Two models yield the essentially same results for the weak nonlinearity, the small axial speed and the low mode, while the difference between two models increases with the nonlinear term, the axial speed, and the order of mode.

1. Introduction

Axially moving beams can represent many engineering devices. Understanding transverse vibrations of axially moving beams is important for the design of the devices. The transverse motion of an axially moving beam can be regarded as free vibration if both external excitations and parametric excitation are not taken into consideration. Barakat [1] and Simpson [2], respectively studied the unstressed moving beam, while their models did not account for the effect of tension. Mote [3] first investigated the effect of tension in an axially moving beam and computed numerically the first three frequencies and modes for simply supported boundary conditions. Wickert and Mote [4] presented a complex modal method for axially moving continua including beams where natural frequencies and modes associated with free vibration serve as a basis for analysis. Öz and Pakdemirli [5] and Öz [6] calculated the natural frequencies in the cases of pinned–pinned ends and clamped–clamped ends, respectively. Özkaya and Öz applied artificial neural networks to determine the natural frequencies of axially moving beams [7]. Öz [8] computed natural frequencies of an axially moving beam in contact with a small stationary mass under pinned–pinned or clamped–clamped boundary conditions.

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Kong and Parker [9] combined perturbation techniques for algebraic equations and the phase closure principle to derive closed-form approximate natural frequencies of an axially moving beam with small flexural stiffness. All these free vibration analysis referred adopted the linear mode. Thurman and Mote [10] derived a nonlinear mode for coupled longitudinal and transverse vibration of axially moving beams from Hamilton's principle, and developed a perturbation method to calculate the nonlinear period. Wickert [11] supposed that the influence of longitudinal inertia could be neglected (the so-called "quasi-static stretch" assumption) to establish a decoupled transverse equation of motion, an integro-partial-differential equation, and modified the asymptotic method of Krylov, Bogoliubov, and Mitropolsky to analyze the effect of nonlinearity on the fundamental frequency. Based on the same governing equation, Pellicano and Zirilli [12] presented a boundary layer analysis for transverse vibration of an axially moving beam with vanishing flexural stiffness and small nonlinearities. Pellicano and Vestroni [13] used the Galerkin method to discretize the governing equation to study the stability and bifurcation. Without using the "quasi-static stretch" assumption, Chen and Zu [14] studied energetics and defined a conserved functional for moving beams undergoing nonlinear free vibration.

In the present study, a nonlinear partial-differential equation governing transverse motion of an axially moving beam is derived from the Newtonian second law. The equation reduces to the integro-partial-differential equation in Refs. [11–13] if the axial tension is replaced by its averaged value over the entire beam. The method of multiple scales is applied directly to the partial-differential equation and the integro-partial-differential equation to evaluate nonlinear natural frequencies. Numerical examples are presented to demonstrate the analytical results and to highlight the difference between two models.

2. Equations of motion

A uniform axially moving beam, with linear density ρ_l, cross-sectional area A, cross-sectional area moment of inertial I and initial tension P_0, travels at the constant axial transport speed c between two boundaries separated by distance L. The material of the beam is linear elastic, defined by Hooke's law:

$$\sigma(X,T) = E\varepsilon_L(X,T), \tag{1}$$

where $\sigma(X,T)$ and $\varepsilon_L(X,T)$ are respectively the axial disturbed stress and strain at the longitudinal coordinate X and time T, and E is the elastic modulus. Consider only the bending vibration described by the transverse displacement $U(X,T)$. The Lagrangian strain

$$\varepsilon_L(X,T) = \frac{1}{2}\left[\frac{\partial U(X,T)}{\partial X}\right]^2 \tag{2}$$

is used to account for geometric nonlinearity due to small but finite stretching of the beam. The Newton second law of motion yields

$$\rho_l\left(\frac{\partial^2 U}{\partial T^2} + 2c\frac{\partial^2 U}{\partial X \partial T} + c^2\frac{\partial^2 U}{\partial X^2}\right) = \frac{\partial}{\partial X}\left[(P_0 + A\sigma)\frac{\partial U}{\partial X}\right] - \frac{\partial^2}{\partial X^2}\left(EI\frac{\partial^2 U}{\partial X^2}\right), \tag{3}$$

where EI is the flexural rigidity of the beam.

If the spatial variation of the tension is rather small, then one can use the averaged value of the disturbed tension $1/L\int_0^L A\sigma\,\mathrm{d}x$ to replace the exact value $A\sigma$. In this case, Eq. (3) becomes

$$\rho_l\left(\frac{\partial^2 U}{\partial T^2} + 2c\frac{\partial^2 U}{\partial X \partial T} + c^2\frac{\partial^2 U}{\partial X^2}\right) = \frac{\partial}{\partial X}\left[\left(P_0 + \frac{1}{L}\int_0^L A\sigma \mathrm{d}x\right)\frac{\partial U}{\partial X}\right] + \frac{\partial^2}{\partial X^2}\left(EI\frac{\partial^2 U}{\partial X^2}\right). \tag{4}$$

Substitution of Eqs. (1) and (2) into Eq. (3) or (4) respectively leads to the dynamic models of nonlinear free transverse vibration of an axially moving beam

$$\rho_l\left(\frac{\partial^2 U}{\partial T^2} + 2c\frac{\partial^2 U}{\partial X \partial T} + c^2\frac{\partial^2 U}{\partial X^2}\right) - P_0\frac{\partial^2 U}{\partial X^2} + EI\frac{\partial^4 U}{\partial X^4} = \frac{3}{2}EA\frac{\partial^2 U}{\partial X^2}\left(\frac{\partial U}{\partial X}\right)^2, \tag{5}$$

$$\rho_l\left(\frac{\partial^2 U}{\partial T^2} + 2c\frac{\partial^2 U}{\partial X \partial T} + c^2\frac{\partial^2 U}{\partial X^2}\right) - P_0\frac{\partial^2 U}{\partial X^2} + EI\frac{\partial^4 U}{\partial X^4} = \frac{EA}{2L}\frac{\partial^2 U}{\partial X^2}\int_0^L\left(\frac{\partial U}{\partial X}\right)^2 \mathrm{d}x. \tag{6}$$

Introduce the transformation

$$x=\frac{X}{L},\ u=\frac{U}{\sqrt{\varepsilon}L},\ t=\frac{T}{L}\sqrt{\frac{P}{A\rho_l}},\ \gamma=c\sqrt{\frac{A\rho}{P_0}},\ k_f^2=\frac{EI}{P_0L^2},\ k_1^2=\frac{EA}{P_0},\tag{7}$$

where the small real number ε is a booking device. Substation of Eq. (7) into Eqs. (5) and (6) respectively yields the dimensionless form

$$\frac{\partial^2 u}{\partial t^2}+2\gamma\frac{\partial^2 u}{\partial t\partial x}+(\gamma^2-1)\frac{\partial^2 u}{\partial x^2}+k_f^2\frac{\partial^4 u}{\partial x^4}=\frac{3}{2}\varepsilon k_1^2\frac{\partial^2 u}{\partial x^2}\left(\frac{\partial u}{\partial x}\right)^2,\tag{8}$$

$$\frac{\partial^2 u}{\partial t^2}+2\gamma\frac{\partial^2 u}{\partial t\partial x}+(\gamma^2-1)\frac{\partial^2 u}{\partial x^2}+k_f^2\frac{\partial^4 u}{\partial x^4}=\frac{1}{2}\varepsilon k_1^2\frac{\partial^2 u}{\partial x^2}\int_0^1\left(\frac{\partial u}{\partial x}\right)^2\mathrm{d}x.\tag{9}$$

Obviously, Eq. (8) is a nonlinear partial-differential equation, while Eq. (9) is an integro-partial-differential equation.

Eq. (8) can be derived from the governing equation for coupled longitudinal and transverse vibration under the assumption that $u^4 \ll u^2$ (Eqs. (15) and (16) in Ref. [10]) by considering the transverse vibration only and setting all longitudinal variables to zero. Eq. (9) has been obtained through uncoupling the governing equation for coupled longitudinal and transverse vibration under the "quasi-static stretch" assumption (Eq. (30) in Ref. [11]). The assumption means the dynamic tension component to be a function of time alone. In traditional derivation, Eq. (9) seems more exact than Eq. (8) because it is the transverse equation of motion in which the longitudinal displacement field is taken into account. However, the derivation here indicates that Eq. (8) can be reduced to Eq. (9) based on the "quasi-static stretch" assumption.

In present study, the simply support conditions are considered. Therefore, the boundary conditions in dimensionless form are:

$$u(0,t)=u(1,t)=0,\quad \left.\frac{\partial^2 u}{\partial x^2}\right|_{x=0}=\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=1}=0.\tag{10}$$

3. Analysis via the method of multiple scales

The method of multiple scales will be directly employed to solve Eq. (11). A first-order uniform approximation is sought in the form

$$u(x,t;\varepsilon)=u_0(x,T_0,T_1)+\varepsilon u_1(x,T_0,T_1)+\ldots,\tag{11}$$

where $T_0=t$ and $T_1=\varepsilon t$ are the fast and slow time scale, respectively. Substitution of Eq. (11) and the following relationship:

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial T_0}+\varepsilon\frac{\partial}{\partial T_1}+\cdots\tag{12}$$

into Eq. (8) and grouping of terms of like-order yield

$$\frac{\partial^2 u_0}{\partial T_0^2}+2\gamma\frac{\partial^2 u_0}{\partial T_0\partial x}+(\gamma^2-1)\frac{\partial^2 u_0}{\partial x^2}+k_f^2\frac{\partial^4 u_0}{\partial x^4}=0,\tag{13}$$

$$\frac{\partial^2 u_1}{\partial T_0^2}+2\gamma\frac{\partial^2 u_1}{\partial T_0\partial x}+(\gamma^2-1)\frac{\partial^2 u_1}{\partial x^2}+k_f^2\frac{\partial^4 u_1}{\partial x^4}=-2\frac{\partial^2 u_0}{\partial T_0\partial T_1}-2\gamma\frac{\partial^2 u_0}{\partial T_1\partial x}+\frac{3}{2}k_1^2\frac{\partial^2 u_0}{\partial x^2}\left(\frac{\partial u_0}{\partial x}\right)^2.\tag{14}$$

The solution to Eq. (13) has been given by Wickert and Mote [4] as

$$u_0=\sum_{n=0,\pm1,\ldots}\left\lfloor\phi_n(x)A_n(T_1)\mathrm{e}^{\mathrm{i}\omega_nT_0}+\bar{\phi}_n(x)\bar{A}_n(T_1)\mathrm{e}^{-\mathrm{i}\omega_nT_0}\right\rfloor,\tag{15}$$

where ω_n and ϕ_n are respectively the nth natural frequency and complex mode function of the corresponding linear homogeneous system. Under the boundary conditions (10), the mode function is [5]

$$\begin{aligned}\phi_n(x) = {} & e^{i\beta_{1n}x} - \frac{(\beta_{4n}^2-\beta_{1n}^2)(e^{i\beta_{3n}}-e^{i\beta_{1n}})}{(\beta_{4n}^2-\beta_{2n}^2)(e^{i\beta_{3n}}-e^{i\beta_{2n}})}e^{i\beta_{2n}x} - \frac{(\beta_{4n}^2-\beta_{1n}^2)(e^{i\beta_{2n}}-e^{i\beta_{1n}})}{(\beta_{4n}^2-\beta_{3n}^2)(e^{i\beta_{2n}}-e^{i\beta_{3n}})}e^{i\beta_{3n}x} \\ & - \left[1 - \frac{(\beta_{4n}^2-\beta_{1n}^2)(e^{i\beta_{3n}}-e^{i\beta_{1n}})}{(\beta_{4n}^2-\beta_{2n}^2)(e^{i\beta_{3n}}-e^{i\beta_{2n}})} - \frac{(\beta_{4n}^2-\beta_{1n}^2)(e^{i\beta_{2n}}-e^{i\beta_{1n}})}{(\beta_{4n}^2-\beta_{3n}^2)(e^{i\beta_{2n}}-e^{i\beta_{3n}})}\right]e^{i\beta_{4n}x}, \end{aligned} \tag{16}$$

where β_{jn} ($j = 1,2,3,4$) are four roots of the following fourth-order algebraic equation:

$$k_f^4\beta_{jn}^4 + (1-\gamma^2)\beta_{jn}^2 - 2\omega_n\beta_{jn} - \omega_n^2 = 0. \tag{17}$$

In the case without internal resonance, when one calculates the nth nonlinear frequency, it does not lose generality for u_0 to include only the nth mode vibration:

$$u_0 = \phi_n(x)A_n(T_1)e^{i\omega_n T_0} + \bar{\phi}_n(x)\bar{A}_n(T_1)e^{-i\omega_n T_0}. \tag{18}$$

Substitution of Eq. (18) into Eq. (14) gives

$$\begin{aligned} & \frac{\partial^2 u_1}{\partial T_0^2} + 2\gamma\frac{\partial^2 u_1}{\partial T_0\partial x} + (\gamma^2-1)\frac{\partial^2 u_1}{\partial x^2} + k_f^2\frac{\partial^4 u_1}{\partial x^4} \\ & = -2\gamma(i\omega_n\phi_n + \gamma\phi_n')\frac{dA_n}{dT_1}e^{i\omega_n T_0} + \frac{3}{2}k_1^2A_n^2\bar{A}_n\phi_n'\left(\phi_n'\bar{\phi}_n'' + 2\phi_n''\bar{\phi}_n'\right)e^{i\omega_n T_0} + cc + NST, \end{aligned} \tag{19}$$

where cc represents complex conjugate to the previous term, and NST denotes the terms that will not bring secular terms into the solution.

Eq. (19) has a bounded solution only if a solvability condition holds. The solvability condition demands that the right side of Eq. (19) be orthogonal to every solution of the homogeneous problem. That is

$$\left\langle \phi_n, -2\gamma(i\omega_n\phi_n + \gamma\phi_n')\frac{dA_n}{dT_1}e^{i\omega_n T_0} + \frac{3}{2}k_1^2A_n^2\bar{A}_n\phi_n'(\phi_n'\bar{\phi}_n'' + 2\phi_n''\bar{\phi}_n')\,e^{i\omega_n T_0}\right\rangle = 0, \tag{20}$$

where the inner production is defined by

$$\langle f, g\rangle = \int_0^1 \bar{f}g\,dx. \tag{21}$$

Eq. (20) can be cast into the form

$$\frac{dA_n}{dT_1} - \kappa_n A_n^2\bar{A}_n = 0, \tag{22}$$

where

$$\kappa_n = \frac{3\int_0^1 \bar{\phi}_n\bar{\phi}_n''\bar{\phi}_n'^2\,dx + 6\int_0^1 \bar{\phi}_n\bar{\phi}_n''\bar{\phi}_n'\bar{\phi}_n'\,dx}{4\left(i\omega_n\int_0^1 \bar{\phi}_n\phi_n\,dx + \gamma\int_0^1 \bar{\phi}_n\bar{\phi}_n'\,dx\right)}k_1^2. \tag{23}$$

Express the solution to Eq. (22) in polar form

$$A_n = \alpha_n e^{i\beta_n}. \tag{24}$$

In Eq. (24), α_n and β_n, are respectively the amplitude and the phase angle of the nonlinear free vibration. Substituting Eq. (24) into Eq. (22) and separating the resulting equation into real and imaginary parts gives

$$\frac{d\alpha_n}{dT_1} = 0, \quad \alpha_n\frac{d\beta_n}{dT_1} = \frac{1}{4}\kappa_n^I\alpha_n^3, \tag{25}$$

where ${\kappa_n}^I$ is the imaginary part of κ_n. Integrating Eq. (25) yields

$$\alpha_n = a_{0n}, \quad \beta_n = \tfrac{1}{4}\kappa_n^I a_{0n}^2 T_1 + b_{0n}, \tag{26}$$

where a_{0n} and b_{0n} are constants. Inserting Eq. (26) into Eq. (24) and then inserting the resulting equation into Eq. (18) gives the nth frequency of nonlinear free vibration:

$$\omega_n^{\mathrm{NL}} = \omega_n + \tfrac{1}{4}\varepsilon\kappa_n^I a_{0n}^2. \tag{27}$$

Similarly, one can calculate the corresponding coefficient κ_n for Eq. (9) as

$$\kappa_n = \frac{\int_0^1 \bar{\phi}_n\bar{\phi}_n''\,\mathrm{d}x \int_0^1 \bar{\phi}'^2_n\,\mathrm{d}x + 2\int_0^1 \phi_n'\bar{\phi}_n'\,\mathrm{d}x \int_0^1 \bar{\phi}_n\phi_n''\,\mathrm{d}x}{4\left(\mathrm{i}\omega_n\int_0^1 \bar{\phi}_n\phi_n\,\mathrm{d}x + \gamma\int_0^1 \bar{\phi}_n\phi_n'\,\mathrm{d}x\right)} k_1^2. \tag{28}$$

In fact, such expression has been obtained in Ref. [15].

4. Results and comparisons

Unlike axially moving strings, axially moving beams have no explicit expression of their linear natural frequencies ω_n, and coefficients β_{jn} ($j = 1,2,3,4$) in the expression of their linear mode (16) cannot be determined analytically. In this section, numerical calculations are performed to investigate the nonlinear effects modeled by Eq. (8) or (9). In all calculations, $k_f = 1.0$ and $k_1 = 2.0$. The linear critical axial speeds for the first two modes are, respectively $\gamma_{1\mathrm{cr}} = 2.7045$ and $\gamma_{2\mathrm{cr}} = 5.2728$.

In Eqs. (8) and (9), the magnitude of nonlinear term depends on the value of nonlinear coefficient k_1^2. However, Eq. (27) indicates that the nonlinear characteristic of the vibration is represented by ${\kappa_n}^I$, while there is no nonlinear effect when ${\kappa_n}^I = 0$. From Eqs. (23) to (28), the nonlinear characteristic ${\kappa_n}^I$ is proportional to k_1^2, as k_1^2 is a real number. This fact is physically obvious. However, Eqs. (23) and (28) also indicate that the nonlinear characteristic ${\kappa_n}^I$ varies with the axial speed γ. Fig. 1 shows the change of the nonlinear characteristic ${\kappa_n}^I$ varies with the axial speed γ, where the dashed line and the solid line represent the results evaluated from Eqs. (23) and (28), respectively. For both models, the nonlinear characteristic increases with the growth of the axial speed, and it increases dramatically for the speed approaching the critical speed. Besides, the higher order mode has the larger nonlinear characteristic. From Eq. (27), it can be concluded that the difference between the nonlinear frequency and linear natural frequency increases with the axial speed and the order of the mode. For the same parameters, the nonlinear characteristic of Eq. (8) is larger than that of Eq. (9). Therefore, averaging the tension along the beam makes the nonlinearity weaker.

Based on Eq. (27), the frequency of nonlinear free vibration can be numerically calculated. Figs. 2 and 3 show the relationship between nonlinear frequencies and amplitudes at different axial speed for $\varepsilon = 0.005$ and 0.05, respectively. In these figures, the dashed line and the solid line represent the results for Eqs. (8) and (9), respectively. The numerical simulations indicate that two models are qualitatively same, while there exist quantitative differences. Both models predict the same trends of the nonlinear free vibration frequencies varying with the initial amplitudes, the axial speed, and the nonlinear term. Such trends were obtained in Ref. [11] via the asymptotic method of Krylov, Bogoliubov and Mitropolsky for Eq. (9). The frequencies

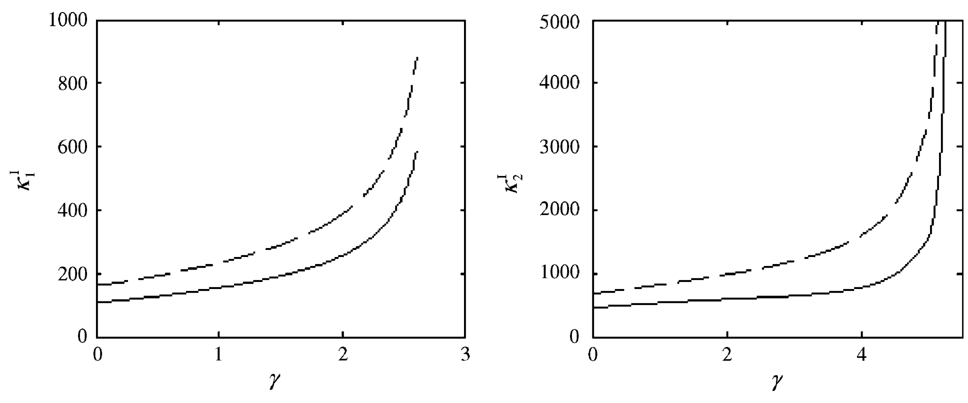

Fig. 1. Nonlinear characteristics varying with the axial speed: (a) the first mode; (b) the second mode. ----- for Eq. (23), — for Eq. (28).

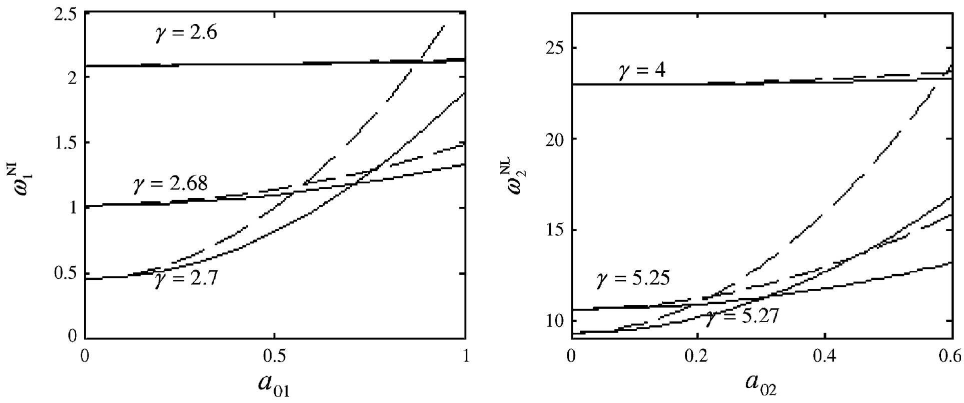

Fig. 2. The relationship between frequencies and amplitudes at different axial speed ($\varepsilon = 0.005$): (a) the first mode; (b) the second mode. ----- for Eq. (23), — for Eq. (28).

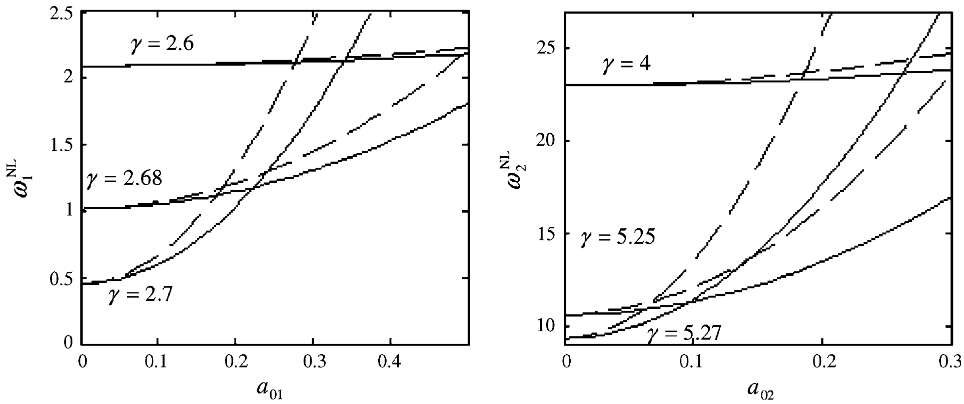

Fig. 3. The relationship between frequencies and amplitudes at different axial speed ($\varepsilon = 0.05$): (a) the first mode; (b) the second mode. ----- for Eq. (23), — for Eq. (28).

increase with the initial amplitudes. When the initial amplitude is zero, Eq. (27) yields the frequencies of the conresponding linear system, which is same for both the nonlinear model. The larger axial speed leads to the smaller frequencies and the rapider increase of the frequencies with the initial amplitudes. The larger nonlinear term results in the rapider increase of the frequencies with the initial amplitudes, and the increase becomes substantial when the axial speed is close to the critical one. Besides, the effects of the initial amplitudes, the axial speed, and the nonlinear term is more significant in the higher order mode. Quantitative, the two models yield the almost same results for weak nonlinearity, small axial speed and low mode, while the difference of the two models increases with the nonlinear term, the axial speed, and the order of the mode, especially, it increases rapidly when the axially speed is near the critical speed.

5. Conclusions

Nonlinear free vibration of an axially moving beam is investigated in the paper. A governing equation of transverse vibration is derived from Newton's second law. Under the assumption that the tension of beam can be replaced by the averaged tension over the beam, the equation reduces a widely used model. The nonlinear frequencies of two models are obtained via the method of multiple scales modes. Numerical calculations show that the models have the same tendencies to change with related parameters, and the two models give the essentially same results for weak nonlinearity, small axial speed and low mode. However, the difference between two models increases when the nonlinear term strengthens, the axial speed grows, or the order of mode becomes large.

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References

[1] R. Barakat, Transverse vibrations of a moving thin rod, *The Journal of the Acoustical Society of America* 43 (1968) 533–539.
[2] A. Simpson, Transverse modes and frequencies of beams translating between fixed end supports, *Journal of Mechanical Engineering Science* 15 (1973) 159–164.
[3] C.D. Mote Jr., Dynamic stability of an axially moving band, *Journal of the Franklin Institute* 285 (1968) 329–346.
[4] J.A. Wickert, C.D. Mote Jr., Classical vibration analysis of axially moving continua, *ASME Journal of Applied Mechanics* 57 (1990) 738–744.
[5] H.R. Öz, M. Pakdemirli, Vibrations of an axially moving beam with time dependent velocity, *Journal of Sound Vibration* 227 (1999) 239–257.
[6] H.R. Öz, On the vibrations of an axially traveling beam on fixed supports with variable velocity, *Journal of Sound Vibration* 239 (2001) 556–564.
[7] E. Özkaya, H.R. Öz, Determination of natural frequencies and stability regions of axially moving beams using artificial neural networks method, *Journal of Sound Vibration* 254 (2002) 782–789.
[8] H.R. Öz, Natural frequencies of axially travelling tensioned beams in contact with a stationary mass, *Journal of Sound Vibration* 259 (2003) 445–456.
[9] L. Kong, R.G. Parker, Approximate eigensolutions of axially moving beams with small flexural stiffness, *Journal of Sound and Vibration* 276 (2004) 459–469.
[10] A.L. Thurman, C.D. Mote Jr., Free, periodic, nonlinear oscillation of an axially moving strip, *ASME Journal of Applied Mechanics* 36 (1969) 83–91.
[11] J.A. Wickert, Non-linear vibration of a traveling tensioned beam, *International Journal of Non-linear Mechanics* 27 (1991) 503–517.
[12] F. Pellicano, F. Zirilli, Boundary layers and non-linear vibrations in an axially moving beam, *International Journal of Non-linear Mechanics* 32 (1997) 691–711.
[13] F. Pellicano, F. Vestroni, Nonlinear dynamics and bifurcations of an axially moving beam, *ASME Journal of Vibration and Acoustics* 122 (2000) 21–30.
[14] L.Q. Chen, J.W. Zu, Energetics and conserved functional of moving materials undergoing transverse nonlinear vibration, *ASME Journal of Vibration and Acoustics* 126 (2004) 452–455.
[15] H.R. Öz, M. Pakdemirli, H. Boyacı, Non-linear vibrations and stability of an axially moving beam with time dependent velocity, *International Journal of Non-Linear Mechanics* 36 (2001) 107–115.