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Short Communication

Fundamental frequency of clamped plates with circularly periodic boundaries

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Abstract

A boundary perturbation method is developed to determine the fundamental frequency of vibrating plates. The method is then applied to wavy, star shape and polygonal plates with clamped boundary conditions. Approximate analytical solutions of the fundamental frequency are obtained with an accuracy of $O(\varepsilon^4)$, where ε is the deviation from the unit circle.

1. Introduction

The governing equation for the transverse vibration of a thin plate is given by

$$\nabla^4 W - k^4 W = 0, \tag{1}$$

where $k^2 = \omega L^2 \sqrt{\rho/D}$, ω is the natural frequency, ρ is the density, D is the flexural rigidity, and L is some length scale. The fundamental frequency, which is the smallest eigenvalue of Eq. (1), plays a crucial role when designing plates in applied sciences. The eigenvalue problem has a general analytical solution in a circular domain in terms of a linear combination of the Bessel functions [1]. If the domain is rectangle, Navier's double series solution and Levy's single series solution is possible for certain boundary conditions [1]. But for other domains, numerical methods are necessary [2–5].

The purpose of the present work is to use an analytic perturbation method to solve for the fundamental frequencies of wavy, star, and polygonal plates with clamped boundary conditions.

2. General perturbation method for clamped plates

Consider a thin, nearly circular plate with constant thickness. Let L be the average radius and the boundary is given by $r = 1 + \varepsilon f(\theta)$, where $f(\theta)$ is the boundary function of zero mean and ε is the small amplitude of the boundary. Then we perturb the solution $W(r, \theta)$ and the fundamental frequency k about the circular state

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as follows:

$$W(r,\theta) = W_0(r) + \varepsilon W_1(r,\theta) + \varepsilon^2 W_2(r,\theta) + O(\varepsilon^3), \tag{2}$$

$$k^4 = k_0^4\left[1 + \varepsilon^2 b + O(\varepsilon^4)\right], \tag{3}$$

where $\varepsilon^2 b$ is the correction to the square of the fundamental frequency and $b = (k_1/k_0)^4$. Expanding $W(1 + \varepsilon f(\theta), \theta)$ into Taylor series and applying the perturbation (2) gives

$$W(1 + \varepsilon f(\theta), \theta) = W_0(1) + \varepsilon[W_1(1,\theta) - \Phi_1(1,\theta)] + \varepsilon^2[W_2(1,\theta) - \Phi_2(1,\theta)] + \cdots, \tag{4}$$

where $\Phi_1(1,\theta)$ and $\Phi_2(1,\theta)$ are given in Appendix. Similarly, the normal derivative at the boundary can be expanded as

$$\left.\frac{\partial W}{\partial n}\right|_{r=1+\varepsilon f(\theta)} = W_{0_r} + \varepsilon\left[W_{1_r}(1,\theta) - \Psi_1^c(1,\theta)\right] + \varepsilon^2\left[W_{2_r}(1,\theta) - \Psi_2^c(1,\theta)\right] + \cdots, \tag{5}$$

where $\Psi_1^c(1,\theta)$ and $\Psi_2^c(1,\theta)$ are also given in Appendix. Eqs. (4) and (5), being zero, give the perturbed clamped boundary conditions.

Substituting (2) and (3) into Eq. (1), we obtain the following sequential boundary value problems:

$$\begin{cases} \nabla^4 W_0(r) - k_0^4 W_0(r) = 0, \\ \nabla^4 W_1(r,\theta) - k_0^4 W_1(r,\theta) = 0, \\ \nabla^4 W_2(r,\theta) - k_0^4 W_2(r,\theta) = b k_0^4 W_0(r), \\ \vdots \end{cases} \tag{6}$$

Solutions of sequential Eqs. (6) with clamped boundary conditions (4) and (5) of specific shapes are given below.

3. Wavy boundary plates

Consider wavy circular boundary by taking $f(\theta) = \cos(M\theta)$ in the previous section, then $r = 1 + \varepsilon\cos(M\theta)$, where $\varepsilon \ll 1$ is the small amplitude and $M \geqslant 2$ is the number of circumferential waves. As an example, we will consider the $M = 6$ case.

The zeroeth order equation with corresponding zeroeth order homogenous clamped boundary conditions

$$\begin{aligned} &\nabla^4 W_0(r) - k_0^4 W_0(r) = 0, \\ &W_0(1) = W_{0_r}(1) = 0 \end{aligned} \tag{7}$$

which corresponds to a clamped circular plate, has the following solution (axisymmetric case in which no nodal diameter occurs)

$$W_0(r,\theta) = J_0(k_0 r) + \alpha_0 I_0(k_0 r), \tag{8}$$

where $\alpha_0 = J_1(k_0)/I_1(k_0)$ and J_n, I_n are Bessel functions. The fundamental frequency of the clamped circular plate is $k_0 = 3.1937$ which is the first root of the characteristic equation $J_0(k_0)I_1(k_0) + J_1(k_0)I_0(k_0) = 0$.

The first-order equation with corresponding first-order clamped boundary conditions are

$$\begin{aligned} &\nabla^4 W_1(r,\theta) - k_0^4 W_1(r,\theta) = 0, \\ &W_1(1,\theta) = 0, \quad W_{1_r}(1,\theta) = \Psi_1^c(1,\theta). \end{aligned} \tag{9}$$

The general solution of (9) is given by $W_1(r,\theta) = \sum_{n=1}^{\infty} W_1^{nM}(r)\cos(nM\theta)$, where $W_1^{nM}(r) = C_{11}^n J_{nM}(k_0 r) + C_{12}^n I_{nM}(k_0 r)$. The boundary conditions of (9) suggest that the first-order equation has a solution

$$W_1(r,\theta) = \left[C_{11}^1 J_M(k_0 r) + C_{12}^1 I_M(k_0 r)\right]\cos(M\theta), \tag{10}$$

where

$$C^1_{11} = -\frac{2I_M(k_0)}{H} W_{0_{rr}}(1),$$

$$C^1_{12} = \frac{2J_M(k_0)}{H} W_{0_{rr}}(1)$$

and

$$H = k_0\{-J_M(k_0)[I_{M-1}(k_0) + I_{M+1}(k_0)] + I_M(k_0)[J_{M-1}(k_0) + J_{M+1}(k_0)]\}.$$

The second-order equation with corresponding second-order clamped boundary conditions are

$$\begin{aligned} &\nabla^4 W_2(r,\theta) - k_0^4 W_2(r,\theta) = bk_0^4 W_0(r), \\ &W_2(1,\theta) = \Phi_2^c(r,\theta), \quad W_{2_r}(1,\theta) = \Psi_2^c(1,\theta). \end{aligned} \tag{11}$$

The boundary conditions of (11) consist of two parts:

$$\begin{aligned} \Phi_2^c(r,\theta) &= \varphi_0^c(r) + \varphi_2^c(r)\cos(2M\theta), \\ \Psi_2^c(r,\theta) &= \xi_0^c(r) + \xi_2^c(r)\cos(2M\theta), \end{aligned} \tag{12}$$

where $\varphi_0^c(r) = W_{0_{rr}}(r)/4$ and $\xi_0^c(r) = M^2 W_1^M(r)/2 - W_{1_{rr}}^M(r)/2 - W_{1_{rr}}^M(r)/4$. The second-order boundary conditions suggest that we have a solution of the type $W_2(r,\theta) = U(r) + V(r)\cos(2M\theta)$. The contribution to the fundamental frequency comes from the θ independent terms of $\nabla^4 U(r,\theta) - k_0^4 U(r,\theta) = bk_0^4 W_0(r)$ which has the following general solution:

$$U(r) = B_1 J_0(k_0 r) + B_2 I_0(k_0 r) - \frac{bk_0 r}{4}(J_1(k_0 r) - \alpha_0 I_1(k_0 r)). \tag{13}$$

Imposing the boundary conditions $U(1) = \varphi_0(1)$, $U_r(1) = \xi_1^c(1)$ into Eq. (13) and solvability condition into the resulting linear system of equations, we obtain a unique solution of b,

$$b = \frac{I_1(k_0)\varphi_0^c(1) - I_0(k_0)\xi_0^c(1)}{I_0(k_0)F_c'(1) - I_1(k_0)F_c(1)}, \tag{14}$$

where

$$F_c(r) = \frac{k_0 r}{4}(J_1(k_0 r) - \alpha_0 I_1(k_0 r)).$$

The product $\varepsilon^2 b$ is the first correction to the square of the fundamental frequency of a clamped wavy boundary plate. Table 1 below lists the values of the fundamental frequencies k with various ε and M. Notice that for $\varepsilon = 0$, the fundamental frequency of clamped circular plate is recovered.

4. Polygonal plates

Let a be the normalized radius of the inscribing circle of a regular polygon with M sides. Then in polar coordinates (r,θ) one side is given by

$$r = \frac{a}{\cos\theta}, \quad -\beta \leqslant \theta \leqslant \beta, \tag{15}$$

Table 1
Fundamental frequency k of clamped wavy boundary plate

$M\backslash\varepsilon$	0	0.02	0.04	0.06	0.08	0.1
5	3.19622	3.20182	3.21846	3.24563	3.28256	3.32828
6	3.19622	3.20311	3.22352	3.25671	3.30152	3.35659
7	3.19622	3.20439	3.22854	3.26761	3.32007	3.38406
8	3.19622	3.20567	3.23351	3.27838	3.33827	3.41081
12	3.19622	3.21073	3.25313	3.32031	3.40802	3.51167

where β is the half-central angle π/M. We determine a such that the mean radius of the polygon is 1 by setting

$$a = \frac{1}{1/\beta \int_0^\beta \frac{1}{\cos\theta}\,\mathrm{d}\theta} = \frac{\beta}{\ln\left[\tan\left(\frac{\pi}{2} + \frac{\beta}{2}\right)\right]}. \tag{16}$$

Expanding Eq. (15) in a Fourier series, the boundary is given by

$$r = 1 + f(\theta) = 1 + \sum_{n=1}^{\infty} c_n \cos(nM\theta), \tag{17}$$

where

$$c_n = \frac{2a}{\beta}\int_0^\beta \frac{\cos(nM\theta)}{\cos\theta}\,\mathrm{d}\theta. \tag{18}$$

The Fourier coefficients c_n are less than 0.1 and alternate in sign and decrease rapidly with n, especially if M is large. For tabulated values of c_n see Wang [6]. Thus the boundary perturbation in Section 2 applies to $r = 1 + f(\theta)$ with f being $O(\varepsilon)$ understood. The more terms retained in the Fourier series (17) will result in a better approximated polygon. Fig. 1 shows the approximations of a hexagon using truncated series.

The perturbation process in Section 3 is applied to the boundary function $f(\theta)$ in (17) of the polygon. The zeroeth order $O(1)$ solution is given by (8). The first-order $O(\varepsilon)$ solution is

$$W_1(r,\theta) = \sum_{n=1}^{\infty} W_1^{nM}(r)\cos(nM\theta), \tag{19}$$

where $W_1^{nM}(r) = C_{11}^n J_{nM}(k_0 r) + C_{12}^n I_{nM}(k_0 r)$, and the coefficients are

$$C_{11}^n = \frac{-c_n I_{nM}(k_0) W_{0_{rr}}(1)}{J_{nM}(k_0) I'_{nM}(k_0) - J'_{nM}(k_0) I_{nM}(k_0)},$$
$$C_{12}^n = \frac{-c_n J_{nM}(k_0) W_{0_{rr}}(1)}{J_{nM}(k_0) I'_{nM}(k_0) - J'_{nM}(k_0) I_{nM}(k_0)}. \tag{20}$$

The non-homogenous part of the second-order $O(\varepsilon^2)$ Eq. (11), $\nabla^4 U - k_0^4 U = bk_0^4 W_0$, has the general solution (13) and its boundary conditions are the θ-independent parts of $\Phi_2^c(r,\theta)$ and $\Psi_2^c(r,\theta)$. Then the boundary conditions for $U(r)$ become $U(1) = \varphi_2^c(1)$, $U_r(1) = \xi_2^c(1)$, where

$$\varphi_2^c(r) = \frac{1}{4} W_{0_{rr}}(r) \sum_{n=1}^{\infty} c_n^2, \tag{21}$$

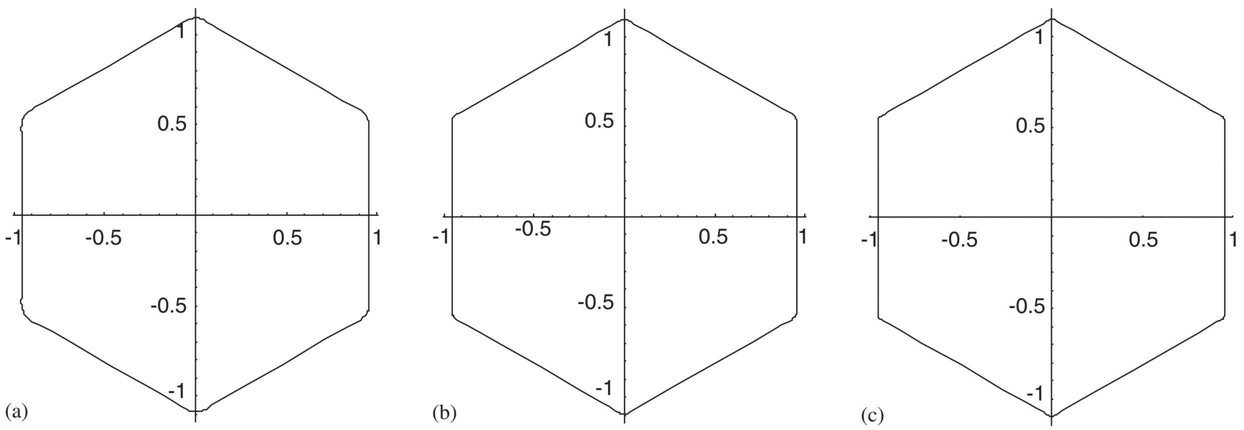

Fig. 1. Hexagon ($M = 6$) shapes using finite number of terms: (a) $n = N = 5$, (b) $n = N = 10$ and (c) $n = N = 12$.

$$\xi_0^c(r) = \frac{1}{2}M^2\sum_{n=1}^{\infty} n^2 c_n W_1^{nM}(r) - \frac{1}{2}\sum_{n=1}^{\infty} c_n W_{1_{rr}}^{nM}(r) - \frac{1}{4}W_{0_{rr}}(r)\sum_{n=1}^{\infty} c_n^2. \tag{22}$$

The frequency correction $\varepsilon^2 b$ can be determined by imposing the above boundary conditions and employing the solvability condition for the resulting augmented system of equations. Thus

$$\begin{vmatrix} I_0(k_0) & \varphi_0^c(1) + bF_c(1) \\ I_1(k_0) & \xi_0^c(1) + bF_c'(1) \end{vmatrix} = 0, \tag{23}$$

where $F_c(r) = k_0 r(J_1(k_0 r) - \alpha_0 I_1(k_0 r))/4$, and hence

$$b = \frac{I_1(k_0)\varphi_0^c(1) - I_0(k_0)\xi_0^c(1)}{I_0(k_0)F_c'(1) - I_1(k_0)F_c(1)}. \tag{24}$$

The fundamental frequency k of a clamped polygonal plate for $M = 6$ (hexagon) with $N = 12$ is found to be 3.26809. In practice the corners may not be mathematically sharp and a finite N would be desirable. From Fig. 1 the boundary function $f(\theta)$ gives an acceptable approximation to an hexagon for $N = 12$. Table 2 shows the convergence of the frequency approximation to various clamped polygonal plates inscribed by a circle of radius a as N increases.

All the published values of the fundamental frequencies are normalized with respect to either inscribing or circumscribing radius of a circle for a polygonal plate. The present result however is normalized with respect to the normalized radius of the averaging circle. In order to be able to compare out result, the frequencies in Table 2 are recalculated with respect to the normalized radius of an inscribed circle and the frequency values are tabulated in Table 3. Table 4 compares the present fundamental frequencies (fundamental eigenvalues) with those obtained by other authors. The present result agrees well with those obtained by Irie et al. [3,8] and Walkinshaw and Kennedy [7]. Irie et al. [8] used conformal mapping to obtain fundamental frequencies. Other authors in Table 4 obtained frequencies by numerical methods. However, the present work gives an analytical approximation method of finding fundamental frequencies of plates. The order of the error in our frequency results is $O(\varepsilon^4)$.

Conformal mapping method was used by Gutierrez et al. [11] to obtain the fundamental frequency of clamped and simply supported regular polygonal plates normalized by the side of the polygons. In the case of the hexagon they found that the fundamental frequency of 3.58189. Our perturbation method gives the frequency of 3.62929 normalized with respect to the side of the hexagon.

Table 2
Fundamental frequency k of regular polygonal clamped plates (using normalized averaging circle)

$M\backslash N$	3	6	12	24	48	96	192
5	3.31667	3.321	3.32235	3.32273	3.32283	3.32286	3.32286
6	3.26493	3.26715	3.26784	3.26803	3.26808	3.26809	3.26809
7	3.23915	3.24045	3.24085	3.24096	3.24099	3.24099	3.24099
8	3.22485	3.22568	3.22593	3.226	3.22602	3.22602	3.22602
12	3.20466	3.20488	3.20495	3.20497	3.20497	3.20498	3.20498

Table 3
Fundamental frequency k of regular polygonal clamped plates (using normalized inscribed circle)

$M\backslash N$	3	6	12	24	48	96	192
5	3.04764	3.04912	3.04943	3.04948	3.04949	3.04949	3.04949
6	3.09128	3.09215	3.09234	3.09237	3.09238	3.09238	3.09238
7	3.11721	3.11778	3.1179	3.11793	3.11793	3.11793	3.11793
8	3.13423	3.13462	3.13471	3.13473	3.13473	3.13473	3.13473
12	3.16652	3.16665	3.16668	3.16669	3.16669	3.16669	3.16669

Table 4
Comparison of fundamental frequency k of regular clamped plates

M	5	6	7	8	12
Present	3.04949	3.09238	3.11793	3.13473	3.16669
Irie et al. [8]	3.062	3.097	3.120	3.136	—
Shahady et al. [9]	3.075	3.104	3.125	3.138	—
Yu [10]	3.144	3.155	3.165	3.171	—
Irie et al. [3]	3.068	3.106	3.128	3.145	—
Walkinshaw and Kennedy [7]	3.061	3.098	3.120	3.137	—

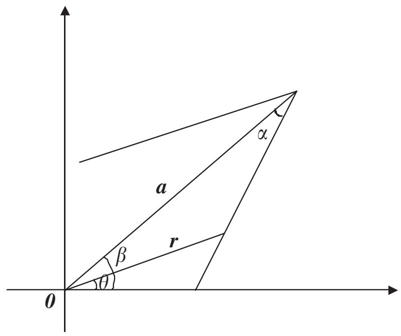

Fig. 2. A tip of a star in polar coordinates.

5. Star shape plates

Let a star have M' tips, and let the tip angle be 2α as shown in Fig. 2. Notice that $\alpha = (\pi/s) - \beta$, where $1 < s < M'$ gives a star and $s = 2$ results in a regular polygon. Then in polar coordinates (r, θ)

$$r = a\frac{d}{\cos\theta - \cot(\alpha+\beta)\sin\theta}, \quad -\beta \leqslant \theta \leqslant \beta, \tag{25}$$

where $d = \cos\beta - \cot(\alpha+\beta)\sin\beta$ and β is the half-central angle π/M'. We determine a such that the mean radius of the star is 1 by

$$a = \left(\frac{1}{\beta}\int_0^\beta \frac{d}{\cos\theta - \cot(\alpha+\beta)\sin\theta}\,\mathrm{d}\theta\right)^{-1}. \tag{26}$$

Expanding Eq. (25) in a Fourier series, the boundary is given by

$$r = 1 + f(\theta) = 1 + \sum_{n=1}^{\infty} c_n \cos(nM'\theta), \tag{27}$$

where

$$c_n = \frac{2a}{\beta}\int_0^\beta \left(\frac{d}{\cos\theta - \cot(\alpha+\beta)\sin\theta}\right)\cos(nM'\theta)\,\mathrm{d}\theta. \tag{28}$$

Table 5 shows that for $M' \geqslant 5$ and for $\alpha = (\pi/3) - \beta$ all c_n are less than 0.1. Also, c_n alternate in sign and the sequence corresponding to each sign decrease rapidly with n, especially if M' is large.

In practice, the infinite sum is truncated to N terms. Fig. 3 shows the approximations of a star with six tips using truncated series. More terms are retained in the Fourier series Eq. (27) will result in a better approximated star.

Table 5
Coefficients of regular star shape plates with M' tips, where $s = 2M'/(M'-2)$

M	A	$\{c_n\}$
5	1.54595	{−0.301180, 0.075944, −0.049344, 0.022382, −0.018935, 0.010377, −0.009866,...}
6	1.36416	{−0.220557, 0.0439395, −0.031449, 0.012127, −0.011705, 0.005513, −0.006033, ...}
7	1.27094	{−0.173268, 0.028760, −0.022901, 0.007680, −0.008413, 0.0034619, −0.004318, ...}
8	1.21459	{−0.142272, 0.020324, −0.017960, 0.005326, −0.006555, 0.002390, −0.003357, ...}
9	1.17702	{−0.120452, 0.015140, −0.014758, 0.003921, −0.005365, 0.001755, −0.002745, ···}
10	1.15029	{−0.104297, 0.011722, −0.012519, 0.003012, −0.004540, 0.001345, −0.002321, ...}
12	1.11495	{−0.082043, 0.007630, −0.009599, 0.001942, −0.003472, 0.000866, −0.001773, ...}
20	1.05833	{−0.043866, 0.002438, −0.004959, 0.000613, −0.001787, 0.000272, −0.000912, ...}

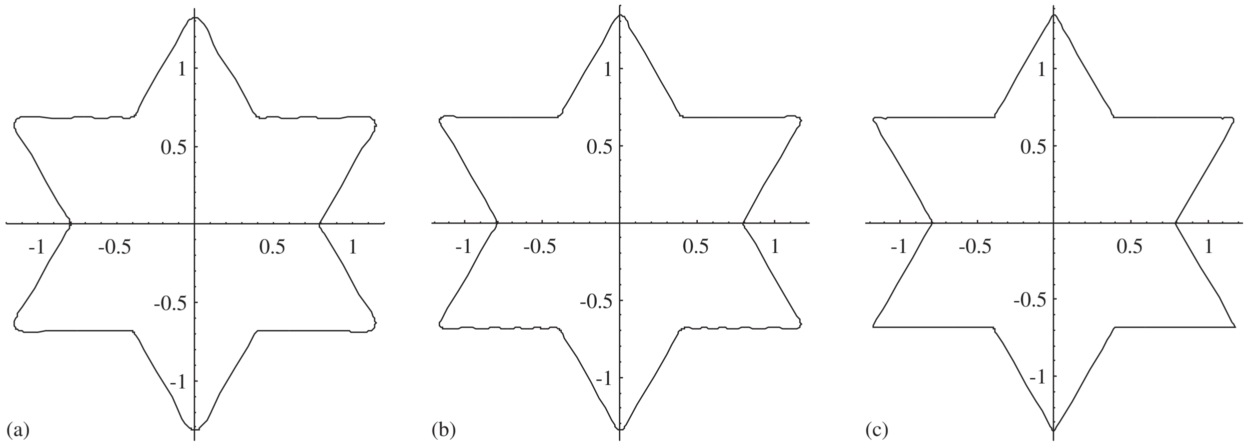

Fig. 3. Star ($M = 6$, $s = 3$) shape using finite number of terms: (a) $n = N = 5$, (b) $n = N = 10$ and (c) $n = N = 12$.

The perturbation process in Section 3 is applied to the boundary function $f(\theta)$ in Eq. (27) of the star. The zeroeth order $O(1)$ solution is given by Eq. (8). The first-order $O(\varepsilon)$ solution is given by Eq. (19) with star shape plate coefficients given in Table 5. Thus, the frequency correction b can be determined by Eq. (24) with respect to these coefficients. Table 6 shows the frequency approximation to star shape plates with various number of tips M' and tip angles 2α, where $\alpha = (\pi/s) - \beta$ for $1 < s < M'$ for $N = 24$. Small tip angle (i.e., large $s < M'$) makes clamped star plate more rigid and therefore it results in a larger frequency value. Note also that for $s = 2$ (the last row in Table 6) we have fundamental frequency values of regular polygons which agree with the result given in Table 2.

6. Conclusions

A boundary perturbation method is developed to extract the fundamental eigenvalue of the biharmonic equation. The method is then applied to wavy, polygonal, and star shape plates with clamped boundary conditions. For simplicity we started with wavy boundary plates and generalized the boundary function from polygon to star. Perturbed results of fundamental frequencies of clamped polygonal plates are found to be in good agreement with those that are available in the literature. Also, approximate analytical solutions of the biharmonic problem and formulations of the fundamental frequency for general plates are obtained. We remark that this method can also be used for other boundary conditions as well.

Table 6
Fundamental frequency k of regular star shaped plates

$s\backslash M'$	5	6	7	8	12
7	—	—	—	8.17093	5.2750
6	—	—	7.48178	6.13904	4.70888
5	—	6.73955	5.56478	5.03411	4.21922
4	5.92408	4.93754	4.50228	4.24753	3.7919
3	4.2350	3.9195	3.74554	3.63588	3.43405
2	3.32273	3.26803	3.24096	3.2260	3.20497

Appendix

The clamped boundary conditions are given by the following functions:

$$\Phi_1^c(1,\theta) = -f(\theta)W_{0_r}(1,\theta), \tag{29}$$

$$\Phi_2^c(1,\theta) = -f(\theta)W_{1_r}(1,\theta) - \tfrac{1}{2}f^2(\theta)W_{0_{rr}}(1,\theta), \tag{30}$$

$$\Psi_1^c(1,\theta) = f'(\theta)W_{0_\theta}(1,\theta) - f(\theta)W_{0_{rr}}(1,\theta), \tag{31}$$

$$\begin{aligned}\Psi_2^c(1,\theta) = & -2f(\theta)f'(\theta)W_{0_\theta}(1,\theta) + f'(\theta)W_{0_\theta}(1,\theta) - f(\theta)W_{1_{rr}}(1,\theta)\\ & + f(\theta)f'(\theta)W_{0_{r\theta}}(1,\theta) - \tfrac{1}{2}f^2(\theta)W_{0_{rrr}}(1,\theta).\end{aligned} \tag{32}$$

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