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Short Communication

A classical iteration procedure valid for certain strongly nonlinear oscillators

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Abstract

A classical iteration procedure is applied to nonlinear oscillations of conservative single-degree-of-freedom systems with odd nonlinearity. With the procedure, the analytical approximate frequency and the corresponding periodic solution, valid for small as well as large amplitudes of oscillation, can be obtained. Two examples are given to illustrate the accuracy and effectiveness of the method. Another advantage of this classical iteration approach is that it also works if the linear part of restoring force is zero.

1. Introduction

The most common methods for constructing approximate analytical solutions to the nonlinear oscillator equation are the perturbation methods. These methods include the Lindstedt–Poincaré method [1–4], the method of Krylov–Bogoliubov–Mitropolshy [1] and the method of multiple scales [1–4], which apply to weakly nonlinear cases only. The method of harmonic balance [1,5] is capable of producing analytical approximations to the frequency and periodic solution of nonlinear oscillations, valid even for rather large values of oscillation amplitude. However, it is usually difficult to give high-order analytical approximations to the solution by applying the method. The main purpose of this paper is to show that a classical iteration procedure [6] is valid for strongly nonlinear oscillations of conservative single-degree-of-freedom systems.

2. Solution method

Consider a nonlinear oscillator modeled by

$$\ddot{x} + f(x) = 0, \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{1}$$

where overdots denote differentiation with respect to time t, and $f(x)$ satisfies the condition

$$f(-x) = -f(x). \tag{2}$$

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Supposing that the natural frequency of Eq. (1) is ω, which is unknown to be further determined, Eq. (1) can be rewritten as [6–10]

$$\ddot{x} + \omega^2 x = \omega^2 x - f(x) =: g(x), \quad x(0) = A, \quad \dot{x}(0) = 0. \tag{3}$$

The linearized equation of Eq. (1) is

$$\ddot{x} + \omega^2 x = 0, \quad x(0) = A, \quad \dot{x}(0) = 0. \tag{4}$$

Comparing Eq. (1) with Eq. (4), we see that even though $f(x)$ is not "small", the function $g(x) = \omega^2 x - f(x)$ is "small". Then the left-hand side of Eq. (3) is linear and the term $g(x)$ on the right-hand side is a "small" function. That is why we prefer Eq. (3) to Eq. (1).

The iteration scheme is [7]

$$\ddot{x}_{k+1} + \omega^2 x_{k+1} = g(x_k), \quad x_k(0) = A, \quad \dot{x}_k(0) = 0, \quad k = 0, 1, 2, \cdots, \tag{5}$$

where the input or starting function is

$$x_0(t) = A \cos \omega t. \tag{6}$$

This procedure can be performed to any value of k desired. However, for most problems the calculations can be stopped at $k = 2$ [7]. Timoshenko et al. [6] have applied this technique to the Duffing equation, but they only gave the first iteration result. In the next section, the details of this classical procedure will be illustrated by applying it to two examples.

3. Examples

Example 1. Consider the Duffing equation

$$\ddot{x} + \omega_0^2 x + \varepsilon x^3 = 0, \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{7}$$

where $\omega_0 > 0$ is a constant and $\varepsilon > 0$ is a parameter. For this example, $f(x) = \omega_0^2 x + \varepsilon x^3$. Eq. (5) gives

$$\ddot{x}_{k+1} + \omega^2 x_{k+1} = (\omega^2 - \omega_0^2) x_k - \varepsilon x_k^3. \tag{8}$$

The first iteration of starting function given by Eq. (6) leads to

$$\ddot{x}_1 + \omega^2 x_1 = \left(\omega^2 - \omega_0^2 - \frac{3}{4}\varepsilon A^2\right) A \cos \omega t - \frac{1}{4}\varepsilon A^3 \cos 3\omega t. \tag{9}$$

The requirement of no secular terms in $x_1(t)$ implies that

$$\omega = \omega_1 = \sqrt{\omega_0^2 + 3\varepsilon A^2/4}. \tag{10}$$

Eq. (10) gives the first approximate frequency of Eq. (7) and the corresponding approximate periodic solution is

$$x_1(t) = A \cos \omega t + \frac{\varepsilon A^3}{32\omega^2}(\cos 3\omega t - \cos \omega t), \tag{11}$$

where the frequency ω is given by Eq. (10).

For the second iteration, we have

$$\ddot{x}_2 + \omega^2 x_2 = (\omega^2 - \omega_0^2) x_1 - \varepsilon x_1^3. \tag{12}$$

The substitution of $x_1(t)$ into the right-hand side of Eq. (12) results in

$$\ddot{x}_2 + \omega^2 x_2 = \left[(\omega^2 - \omega_0^2)\left(1 - \frac{\varepsilon A^2}{32\omega^2}\right) - \frac{3\varepsilon A^2}{4}\left(1 - \frac{\varepsilon A^2}{16\omega^2}\right)\right] A\cos\omega t + \frac{\varepsilon A^3}{4}\left[\frac{1}{8\omega^2}(\omega^2 - \omega_0^2) - \left(1 + \frac{3\varepsilon A^2}{32\omega^2}\right)\right]\cos 3\omega t - \frac{3\varepsilon^2 A^5}{128\omega^2}\cos 5\omega t + O(\varepsilon^3). \tag{13}$$

No secular term requires that

$$(\omega^2 - \omega_0^2)\left(1 - \frac{\varepsilon A^2}{32\omega^2}\right) - \frac{3\varepsilon A^2}{4}\left(1 - \frac{\varepsilon A^2}{16\omega^2}\right) = 0 \tag{14}$$

or

$$\omega^4 - \left(\omega_0^2 + \frac{25}{32}\varepsilon A^2\right)\omega^2 + \frac{\omega_0^2 \varepsilon A^2}{32} + \frac{3\varepsilon^2 A^4}{64} = 0. \tag{15}$$

Solving Eq. (15) for ω yields

$$\omega = \omega_2 = \frac{1}{8}\sqrt{32\omega_0^2 + 25\varepsilon A^2 + \sqrt{1024\omega_0^4 + 1472\omega_0^2\varepsilon A^2 + 433\varepsilon^2 A^4}}. \tag{16}$$

Eq. (16) expresses the second approximate frequency of Eq. (7) and corresponding approximate periodic solution is

$$x_2(t) = A\cos\omega t - \frac{\varepsilon A^3}{32\omega^2}\left[\frac{1}{8\omega^2}(\omega^2 - \omega_0^2) - \left(1 + \frac{3\varepsilon A^2}{32\omega^2}\right)\right](\cos 3\omega t - \cos\omega t) + \frac{\varepsilon^2 A^5}{1024\omega^4}(\cos 5\omega t - \cos\omega t), \tag{17}$$

where ω is given by Eq. (16).

For comparison, we let $\omega_0 = 1$. Then Eqs. (16) and (17) become

$$\omega = \omega_2 = \frac{1}{8}\sqrt{32 + 25\varepsilon A^2 + \sqrt{1024 + 1472\varepsilon A^2 + 433\varepsilon^2 A^4}} \tag{18}$$

and

$$x_2(t) = A\cos\omega t + \left[\frac{7\varepsilon A^3}{256\omega^2} + \frac{4\varepsilon A^3 + 3\varepsilon^2 A^5}{1024\omega^4}\right](\cos 3\omega t - \cos\omega t) + \frac{\varepsilon^2 A^5}{1024\omega^4}(\cos 5\omega t - \cos\omega t), \tag{19}$$

respectively, which are in agreement with the results in Ref. [8]. The approximate frequency ω given by Eq. (18) is valid for both small and large values of εA^2 [8]. The exact periodic solution to Eq. (7) (for $\omega_0 = 1$) is [11]

$$x_e(t) = Acn(wt, k), \tag{20}$$

where $cn(wt,k)$ is the cosine Jacobian elliptic function, $w = \sqrt{1 + \varepsilon A^2}$, and $k = \sqrt{1/2\left(1 - \left(1/\left(1 + \varepsilon A^2\right)\right)\right)}$. A comparison of periodic solutions $x_e(t)$, $x_1(t)$ (for $\omega_0 = 1$) and $x_2(t)$ (given by Eq. (19)) is presented in Fig. 1 for: $\varepsilon = 1$, $A = 10$; $\varepsilon = 10$, $A = 100$; $\varepsilon = 100$, $A = 1000$; respectively. Fig. 1 indicates that $x_1(t)$ and $x_2(t)$ are close to $x_e(t)$ and $x_2(t)$ is more accurate than $x_1(t)$.

Example 2. Consider the nonlinear differential equation [10,12,13]

$$\ddot{x} + x^{1/3} = 0, \quad x(0) = A, \quad \dot{x}(0) = 0. \tag{21}$$

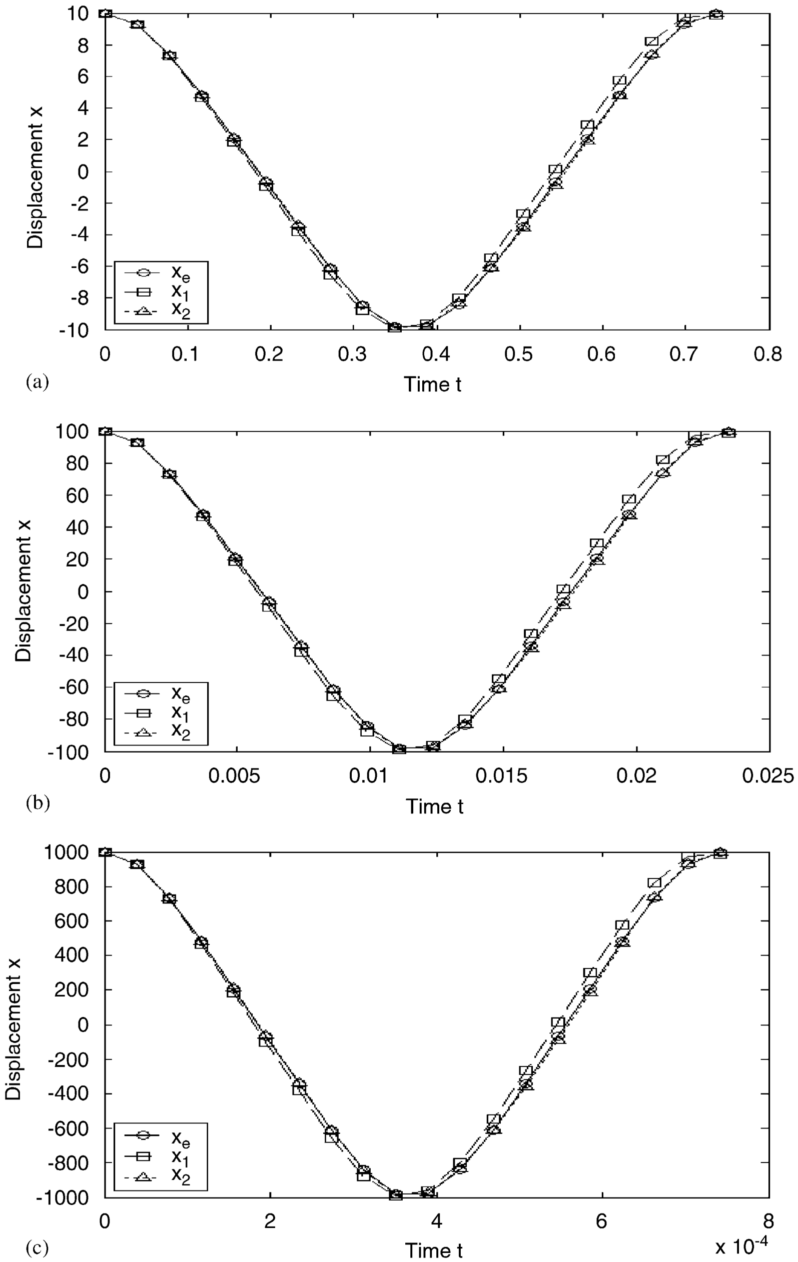

Fig. 1. Comparison of the approximate solutions with the exact solutions to Eq. (7) for: (a) $\varepsilon = 1$, $A = 10$; (b) $\varepsilon = 10$, $A = 100$; and (c) $\varepsilon = 100$, $A = 1000$.

For this example, $f(x) = x^{1/3}$, and Eq. (5) gives

$$\ddot{x}_{k+1} + \omega^2 x_{k+1} = \omega^2 x_k - x_k^{1/3}. \tag{22}$$

The first iteration of the starting function $x_0(t) = A\cos\omega t$ results in

$$\ddot{x}_1 + \omega^2 x_1 = \omega^2 A\cos\omega t - A^{1/3}(\cos\omega t)^{1/3}, \quad x_1(0) = A, \quad \dot{x}_1(0) = 0. \tag{23}$$

Using the relation [10]

$$(\cos\theta)^{1/3} = b_1 \cos\theta + b_3 \cos 3\theta + \cdots, \tag{24}$$

where $b_1 = 1.15960$, $b_3 = -0.231919$, etc., Eq. (23) can be rewritten as

$$\ddot{x}_1 + \omega^2 x_1 = (\omega^2 A - b_1 A^{1/3})\cos \omega t - b_3 A^{1/3} \cos 3\omega t + \text{(higher order harmonics)} = 0. \tag{25}$$

The requirement of no secular terms in $x_1(t)$ implies that

$$\omega = \frac{\sqrt{b_1}}{A^{1/3}} = \frac{1.07685}{A^{1/3}}. \tag{26}$$

The corresponding approximate periodic solution becomes

$$x_1(t) = A\cos \omega t + \frac{b_3 A^{1/3}}{8\omega^2}(\cos 3\omega t - \cos \omega t) = A\cos \omega t - \frac{0.02899 A^{1/3}}{\omega^2}(\cos 3\omega t - \cos \omega t). \tag{27}$$

The exact frequency of the periodic motion of Eq. (21) is [14]

$$\omega_e = \frac{\sqrt{\pi}\Gamma(1/4)}{2\sqrt{6}\Gamma(3/4)A^{1/3}} = \frac{1.07045}{A^{1/3}}, \tag{28}$$

where $\Gamma(n)$ is the Gamma function. Therefore, for any values of A, it can be easily proved that the maximal relative error of the approximate frequency given by Eq. (26) is less than 0.6%. Even for large value of A, $x_1(t)$ given in Eq. (27) is nearly identical to the numerical solution [10].

4. Conclusions

A classical iteration procedure has been used to solve nonlinear oscillations of conservative single-degree-of-freedom systems with odd nonlinearity. With the procedure, the analytical approximate frequency and the corresponding periodic solution, valid for small as well as large amplitudes of oscillation, can be obtained. The details of the method have been illustrated by two examples. The iteration procedure can be carried on if solutions of higher degree of accuracy are required. But when $f(x) = x^{(2n+1)/(2m+1)}$ ($m, n = 0, 1, 2, \ldots, n<m$), the second iteration is not convenient. The nonlinear problem given in Eq. (21) cannot be attacked by usual perturbation techniques except the method of harmonic balance.

Another advantage of this classical iteration procedure is that it works even when the linear part of restoring force is zero, as in Example 2. For Example 1, if $\omega_0 = 0$, Eqs. (8)–(17) are still valid. The same situation is discussed in detail in Ref. [15].

In Ref. [8], Lim et al. presented the following iteration scheme for Eq. (1):

$$\ddot{x}_{k+1} + \omega^2 x_{k+1} = g(x_{k-1}) + g(x_{k-1})(x_k - x_{k-1}), \quad k = 0, 1, 2, \ldots. \tag{29}$$

Obviously, Eq. (5) is simpler than Eq. (29), but both of them give the same result for the Duffing equation. The possibility of further generalizing this classical iteration procedure will be investigated for the case where the restoring force $f(x)$ is a general nonlinear function of x.

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