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Optimal tracking control with zero steady-state error for time-delay systems with sinusoidal disturbances

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Abstract

The problem of optimal tracking control with zero steady-state error for linear time-delay systems with sinusoidal disturbances is considered. Based on the internal model principle, a disturbance compensator is constructed such that the system with external sinusoidal disturbances is transformed into an augmented system without disturbances. By introducing a sensitivity parameter and expanding power series around it, the optimal tracking control problem can be simplified into the problem of solving an infinite sum of linear optimal control series without time-delay and disturbance. The obtained optimal tracking control law with zero steady-state error consists of accurate linear state feedback terms and a time-delay compensating term, which is an infinite sum of an adjoint vector series. The accurate linear terms can be obtained by solving a *Riccati* matrix equation and a *Sylvester* equation, respectively. The compensation term can be approximately obtained through a recursive algorithm. A numerical simulation shows that the algorithm is effective and easily implemented, and the designed tracking controller is robust with respect to the sinusoidal disturbances.

1. Introduction

Many practical control systems located in hostile marine environment, such as a fully submerged hydrofoil craft [1], off-shore platform [2,3], ships [4], etc., undergo excessive vibrations due to wave loads. The vibrations of wave-induced loads acting on the control systems can be considered as external sinusoidal disturbances. The influence with respect to waves must be taken into account. Especially in tracking control systems, in order to obtain a good tracking performance, it is necessary to find the effective active vibration control strategy. There are many good approaches to this problem, for instance, internal model based control [5] and adaptive tracking control [6]. These approaches can achieve disturbance rejection and global asymptotic tracking with zero steady-state error, but neither of them took the optimality problem into account. Sinusoidal disturbance is a typical disturbance in the vibration control systems. The sinusoidal disturbance rejection for servo systems has long intrigued many control theorists and engineers. Tang [7] and Lindquist et al. [8]

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proposed feedforward and feedback optimal damping controllers based on quadratic average performance index for linear continuous and discrete systems with sinusoidal disturbances. However, the feedforward and feedback optimal damping controllers cannot guarantee the closed-loop systems with zero steady-state error.

In real industrial processes, the phenomenon of time delays is quite common. The optimal control problems for time-delay systems with quadratic performance index generally lead to a two-point boundary value (TPBV) problem with both time-delay and time-advance terms, which is very difficult to be solved precisely. So solving an approximate optimal control law is one of the important aims of researchers. At present, many better results in the approximate approach of optimal control for time-delay and/or nonlinear systems have been obtained [9–11], but quite few results have been available to optimal tracking control with zero steady-state error for time-delay systems with sinusoidal disturbances. Therefore, the studies of the optimal tracking control with zero steady-state error for time-delay systems with sinusoidal disturbances are of quite significance.

In this paper, we consider the optimal tracking control with zero steady-state error for linear time-delay systems with sinusoidal disturbances. The disturbance rejection and optimal tracking controller are designed, respectively. A sinusoidal disturbance compensator is first constructed based on the internal model principle. Then the plant model with external disturbances is transformed into an augmented system without disturbances. For the augmented system, we choose an infinite-time horizon quadratic cost functional. Therefore, solving the problem of disturbance rejection for zero steady-state tracking error becomes designing an optimal tracking controller for the augmented system. By introducing a sensitivity parameter ε, the variables of the systems are extended to *Maclaurin* series at $\varepsilon = 0$. The original optimal tracking control problem is translated into a series of TPBV problems without delay and advance terms. Then by solving the TPBV problem sequences recursively, we obtain the optimal tracking control law consisting of linear state feedback terms and a compensation term. The linear terms can be uniquely obtained by solving a *Riccati* matrix equation and a *Sylvester* matrix equation. The compensation term can be obtained by a recursion formula of adjoint vectors. By intercepting a finite sum of the series, we obtain an approximate optimal control law that minimizes the cost functional.

The organization of the paper is as follows. In Section 2, the problem is precisely formulated and the basic assumptions are stated. In Section 3, a disturbance compensator is constructed based on the internal model principle. Section 4 designs an optimal tracking controller with zero steady-state error for time-delay systems by using a sensitivity approach. Section 5 shows simulation results and Section 6 concludes the work.

2. Problem statement

Consider the following linear time-delay system with sinusoidal disturbances

$$\begin{aligned} &\dot{x}(t) = Ax(t) + A_1x(t-\tau) + Bu(t) + Dv(t), \qquad t>0,\\ &x(t) = \phi(t), \qquad -\tau \leqslant t \leqslant 0,\\ &y(t) = Cx(t), \end{aligned} \tag{1}$$

where $x \in R^n$, $u \in R^r$, $v \in R^r$, and $y \in R^p$ are the state, the control input, the disturbance, and the output vectors, respectively; A, B, C, and D are constant matrices of appropriate dimensions. $\phi(t)$ is a given continuous initial state function. τ is a positive time-delay. Assume that the pair (A, B) is completely controllable, the pair (A, C) is completely observable, and

$$\text{Rank } B = \text{Rank}(B \quad D) = r. \tag{2}$$

Disturbance vector v can be given by

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_r \end{bmatrix} = \begin{bmatrix} \alpha_1 \sin(\omega_1 t + \varphi_1) \\ \alpha_2 \sin(\omega_2 t + \varphi_2) \\ \vdots \\ \alpha_r \sin(\omega_r t + \varphi_r) \end{bmatrix}, \tag{3}$$

where the frequency ω_i is known constant, the amplitude α_i and the phase φ_i may be unknown. Assume that the reference input (desired output) $\tilde{y}$ is given by the following exosystem:

$$\begin{aligned} \dot{z}(t) &= Fz(t), \\ \tilde{y}(t) &= Hz(t), \end{aligned} \tag{4}$$

where $z \in R^m$, $\tilde{y} \in R^p$; F and H are constant matrices of appropriate dimensions. We assume that

Assumption 1. Eq. (4) is stable, but unnecessary asymptotically stable.

Assumption 2. The pair (F, H) is completely observable, and $\text{Rank}(H) = p$.

The aim is to find a control law such that the output tracking error

$$e(t) = \tilde{y}(t) - y(t), \tag{5}$$

satisfies

$$e(\infty) = \lim_{t\to\infty} (\tilde{y}(t) - y(t)) = 0,$$

in an optimal fashion. Since Eq. (1) is affected by the external sinusoidal disturbances, it is obvious that the control vector u will not tend to zero in the zero steady-state tracking error control system. The traditional infinite-time horizon quadratic cost functional associated with Eq. (1) is not convergent.

3. Design of disturbance compensator

In this section, a disturbance compensator based on the internal model principle will be constructed. The system with disturbances is transformed into an augmented system without disturbances. In this augmented system, we first define a function $\bar{u}(t)$ as a virtual "control vector" of $u(t)$. Then we choose an infinite-time horizon quadratic cost functional with respect to tracking error and control law of the augmented system. Therefore, the problem becomes designing an optimal tracking controller for the augmented system.

From Eq. (2), there exists the unique invertible matrix $M \in R^{r\times r}$ such that $D = BM$ holds. Therefore, Eq. (1) may be rewritten as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_1 x(t-\tau) + B(u(t) + w(t)), \qquad t > 0, \\ x(t) &= \phi(t), \qquad -\tau \leqslant t \leqslant 0, \\ y(t) &= Cx(t), \end{aligned} \tag{6}$$

where $w(t) = Mv(t)$. Note that from Eq. (3)

$$\ddot{v}(t) = -\Omega v(t), \tag{7}$$

where

$$\Omega = \text{Diag}\{\omega_1^2, \omega_2^2, \ldots, \omega_r^2\}.$$

Hence

$$\ddot{w}(t) = -M\Omega v(t) = -M\Omega M^{-1} w(t). \tag{8}$$

Let

$$\begin{aligned} \bar{x}_1(t) &= x(t), \\ \bar{x}_2(t) &= u(t) + w(t), \\ \bar{x}_3(t) &= \dot{\bar{x}}_2(t) = \dot{u}(t) + \dot{w}(t), \end{aligned} \tag{9}$$

then

$$\begin{aligned} \dot{\bar{x}}_2(t) &= \bar{x}_3(t), \\ \dot{\bar{x}}_3(t) &= -M\Omega M^{-1}\bar{x}_2(t) + [\ddot{u}(t) + M\Omega M^{-1}u(t)]. \end{aligned} \tag{10}$$

Let

$$\bar{x}(t) = \begin{bmatrix} x(t) \\ \bar{x}_2(t) \\ \bar{x}_3(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ u(t) + w(t) \\ \dot{u}(t) + \dot{w}(t) \end{bmatrix},$$
$$\bar{u}(t) = \ddot{u}(t) + M\Omega M^{-1}u(t). \tag{11}$$

By Eqs. (6), (10), and (11), we can obtain the $(n+2r)$-dimensional augmented system without disturbances

$$\begin{aligned} &\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{A}_1\bar{x}(t-\tau) + \bar{B}\bar{u}(t), \qquad t>0, \\ &\bar{x}(t) = \bar{\phi}(t), \qquad -\tau \leqslant t \leqslant 0, \\ &\bar{y}(t) = \bar{C}\bar{x}(t), \end{aligned} \tag{12}$$

where

$$\bar{A} = \begin{bmatrix} A & B & 0 \\ 0 & 0 & I \\ 0 & -M\Omega M^{-1} & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}, \quad \bar{A}_1 = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$\bar{C} = \begin{bmatrix} C^{\mathrm{T}} \\ 0 \\ 0 \end{bmatrix}^{\mathrm{T}}, \quad \bar{\phi}(t) = \begin{bmatrix} \phi(t) \\ \bar{x}_2(0) \\ \bar{x}_3(0) \end{bmatrix}, \tag{13}$$

and I is an identity matrix.

We choose the following infinite-time horizon quadratic cost functional associated with the augmented system in Eq. (12)

$$J = \int_0^\infty \left[e^{\mathrm{T}}(t)Qe(t) + \bar{u}^{\mathrm{T}}(t)R\bar{u}(t)\right] \mathrm{d}t, \tag{14}$$

where Q and R are positive-definite matrices, respectively.

Thus, the original optimal tracking control problem with zero steady-state error becomes finding the optimal tracking "control law" $\bar{u}^*(t)$ of the augmented system to minimize the cost functional equation (14).

4. Design of optimal tracking controller with zero steady-state error

Because the pair (A, B) is controllable and the pair (A, C) is observable, it can be proved that the pair $(\bar{A}, \bar{B})$ is controllable and the pair $(\bar{A}, \bar{C})$ is observable. In fact, it is obvious that

$$\begin{aligned} &\mathrm{Rank}\left[\bar{B} \quad \bar{A}\bar{B} \quad \dots \quad \bar{A}^{n+2r-1}\bar{B}\right] = n+2r, \\ &\mathrm{Rank}\left[\bar{C}^{\mathrm{T}} \quad \bar{A}^{\mathrm{T}}\bar{C}^{\mathrm{T}} \quad \dots \quad \left(\bar{A}^{\mathrm{T}}\right)^{n+2r-1}\bar{C}^{\mathrm{T}}\right]^{\mathrm{T}} = n+2r. \end{aligned}$$

By the optimal control theory, the optimal output tracking control law with zero steady-state error for Eq. (12) associated with the cost functional in Eq. (14) can be expressed as

$$\bar{u}^*(t) = -R^{-1}\bar{B}^{\mathrm{T}}\lambda(t), \qquad t>0, \tag{15}$$

where $\lambda(t)$ is the solution of the following TPBV problem:

$$\begin{aligned} \dot{\bar{x}}(t) &= \bar{A}\bar{x}(t) + \bar{A}_1\bar{x}(t-\tau) - S\lambda(t), \\ -\dot{\lambda}(t) &= \bar{C}^{\mathrm{T}}Q\bar{C}\bar{x}(t) - \bar{C}^{\mathrm{T}}QHz(t) + \bar{A}^{\mathrm{T}}\lambda(t) + \bar{A}_1^{\mathrm{T}}\lambda(t+\tau), \quad t>0, \\ \bar{x}(t) &= \bar{\phi}(t), \qquad -\tau \leqslant t \leqslant 0, \\ \lambda(\infty) &= 0, \end{aligned} \tag{16}$$

where $S = \bar{B}R^{-1}\bar{B}^{\mathrm{T}}$. Note that the TPBV problem in Eq. (16) has both time-delay and time-advance terms. Obtaining the exact analytical solution to this problem is, in general, extremely difficult. We will propose a sensitivity approach to simplify Eq. (16) and get an approximate optimal control law of Eq. (12) with the cost functional in Eq. (14). Introducing a sensitivity parameter ε, we construct a new TPBV problem with ε as follows:

$$\begin{aligned} \dot{\bar{x}}(t,\varepsilon) &= \bar{A}\bar{x}(t,\varepsilon) + \bar{A}_1\bar{x}(t-\tau,\varepsilon) - S\lambda(t,\varepsilon), \\ -\dot{\lambda}(t,\varepsilon) &= \bar{C}^{\mathrm{T}}Q\bar{C}\bar{x}(t,\varepsilon) - \bar{C}^{\mathrm{T}}QHz(t) + \bar{A}^{\mathrm{T}}\lambda(t,\varepsilon) + \bar{A}_1^{\mathrm{T}}\lambda(t+\tau,\varepsilon), \qquad t>0, \\ \bar{x}(t,\varepsilon) &= \bar{\phi}(t), \quad -\tau \leqslant t \leqslant 0, \\ \lambda(\infty,\varepsilon) &= 0, \end{aligned} \tag{17}$$

and a new optimal control law in the form

$$\bar{u}(t,\varepsilon) = -R^{-1}\bar{B}^{\mathrm{T}}\lambda(t,\varepsilon), \qquad t>0, \tag{18}$$

where $0 \leqslant \varepsilon \leqslant 1$ is a scalar sensitivity parameter irrelative to t.

Assume that $\bar{u}(t,\varepsilon)$, $\bar{x}(t,\varepsilon)$, $\lambda(t,\varepsilon)$ are infinitely differentiable with respect to ε at $\varepsilon = 0$, and their *Maclaurin* series in ε can be described as

$$\bar{u}(t,\varepsilon) = \sum_{i=0}^{\infty} \frac{\varepsilon^i}{i!}\bar{u}^{(i)}(t), \quad \bar{x}(t,\varepsilon) = \sum_{i=0}^{\infty} \frac{\varepsilon^i}{i!}\bar{x}^{(i)}(t), \quad \lambda(t,\varepsilon) = \sum_{i=0}^{\infty} \frac{\varepsilon^i}{i!}\lambda^{(i)}(t), \tag{19}$$

where superscript (i) denotes the ith-order derivative with respect to ε at $\varepsilon = 0$ in Eq. (19). In the following discussion, we assume that the series sum in Eq. (19) is convergent at $\varepsilon = 1$. Note that when $\varepsilon = 1$, the TPBV problem of Eq. (17) is equivalent to that of Eq. (16). Therefore, the optimal control law in Eq. (19) can be rewritten in the form

$$\bar{u}^{*}(t) = \bar{u}(t,1) = \sum_{i=0}^{\infty} \frac{1}{i!}\bar{u}^{(i)}(t). \tag{20}$$

The solution of this optimal tracking control problem is given by the following theorem.

Theorem 1. *Consider the optimal output tracking control problem for the augmented system described by Eqs.* (12), (4), *and* (14). *The optimal output tracking control law with zero steady-state error can be given by*

$$\bar{u}^{*}(t) = -R^{-1}\bar{B}^{\mathrm{T}}\left[P\bar{x}(t) + P_1 z(t) + \sum_{i=1}^{\infty}\frac{1}{i!}g_i(t)\right], \tag{21}$$

where P is the unique positive-definite solution to the following Riccati *matrix equation:*

$$\bar{A}^{\mathrm{T}}P + P\bar{A} - PSP + \bar{C}^{\mathrm{T}}Q\bar{C} = 0, \tag{22}$$

P_1 *is the unique solution to the following* Sylvester *matrix equation:*

$$(\bar{A}^{\mathrm{T}} - PS)P_1 + P_1F = \bar{C}^{\mathrm{T}}QH, \tag{23}$$

$g_i(t)$ is given by the following ith adjoint differential equation:

$$
\begin{aligned}
g_0(t) &= 0,\\
\dot{g}_i(t) &= -(\bar{A}^{\mathrm{T}} - PS)g_i(t) - iP\bar{A}_1\bar{x}^{(i-1)}(t-\tau)\\
&\quad - i\bar{A}_1^{\mathrm{T}}[P\bar{x}^{(i-1)}(t+\tau) + g_{i-1}(t+\tau)],\\
g_i(\infty) &= 0, \qquad i = 1,2,\ldots,
\end{aligned}
\tag{24}
$$

$\bar{x}^{(i)}(t)$ is given by the following ith differential equation:

$$
\begin{aligned}
&\dot{\bar{x}}^{(0)}(t) = (\bar{A} - SP)\bar{x}^{(0)}(t) - SP_1 z(t), \qquad t>0,\\
&\bar{x}^{(0)}(t) = \bar{\phi}(t), \qquad -\tau \leqslant t \leqslant 0,\\
&\dot{\bar{x}}^{(i)}(t) = (\bar{A} - SP)\bar{x}^{(i)} + i\bar{A}_1\bar{x}^{(i-1)}(t-\tau) - Sg_i(t), \quad t>0,\\
&\bar{x}^{(i)}(t) = 0, \qquad -\tau \leqslant t \leqslant 0.
\end{aligned}
\tag{25}
$$

Proof. Substituting Eq. (19) into Eq. (17) and comparing the coefficients of the same order terms with respect to ε, we obtain

$$
\begin{aligned}
\dot{\bar{x}}^{(0)}(t) &= \bar{A}\bar{x}^{(0)}(t) - S\lambda^{(0)}(t),\\
-\dot{\lambda}^{(0)}(t) &= \bar{C}^{\mathrm{T}}Q\bar{C}\bar{x}^{(0)}(t) - \bar{C}^{\mathrm{T}}QHz(t) + \bar{A}^{\mathrm{T}}\lambda^{(0)}(t),\\
\bar{x}^{(0)}(0) &= \bar{\phi}(0),\\
\lambda^{(0)}(\infty) &= 0,
\end{aligned}
\tag{26}
$$

and

$$
\begin{aligned}
\dot{\bar{x}}^{(i)}(t) &= \bar{A}\bar{x}^{(i)}(t) - S\lambda^{(i)}(t) + i\bar{A}_1 x^{(i-1)}(t-\tau),\\
-\dot{\lambda}^{(i)}(t) &= \bar{C}^{\mathrm{T}}Q\bar{C}\bar{x}^{(i)}(t) + \bar{A}^{\mathrm{T}}\lambda^{(i)}(t) + i\bar{A}_1^{\mathrm{T}}\lambda^{(i-1)}(t+\tau),\\
\bar{x}^{(i)}(0) &= 0,\\
\lambda^{(i)}(\infty) &= 0.
\end{aligned}
\tag{27}
$$

Substituting Eq. (19) into Eq. (18), $\bar{u}^{(i)}(t)$ can be expressed as

$$\bar{u}^{(i)}(t) = -R^{-1}\bar{B}^{\mathrm{T}}\lambda^{(i)}(t), \qquad i = 0,1,2,\ldots. \tag{28}$$

Now we analyze the solution of $\bar{u}^{(i)}(t)$. When $i=0$, let

$$\lambda^{(0)}(t) = P\bar{x}^{(0)}(t) + P_1 z(t). \tag{29}$$

From Eqs. (29) and (26), we can obtain the *Riccati* equation in Eq. (22) and the *Sylvester* equation in Eq. (23). Under the given assumption, P is the unique positive-definite solution to Eq. (22). In order to prove the existence and uniqueness of the solution to Eq. (23), we introduce the following lemma.

Lemma 1. (Lancaster et al. [12]) *The* Sylvester *equation with respect to $X \in R^{n\times m}$*

$$JX + XN = G \tag{30}$$

has a unique solution, if and only if

$$\lambda_i(J) + \lambda_j(N) \neq 0, \qquad i = 1,2,\ldots,n, \qquad j = 1,2,\ldots,m, \tag{31}$$

where $J \in R^{n\times n}$ and $N \in R^{m\times m}$ are known matrices.

According to the optimal control theory of linear systems, the following inequality holds:

$$\mathrm{Re}(\lambda_i(\bar{A} - SP)) < 0. \tag{32}$$

By Assumption 1, the following inequality holds:

$$\mathrm{Re}(\lambda_j(F)) \leqslant 0. \tag{33}$$

Hence

$$\lambda_i(\bar{A}^{\mathrm{T}} - PS) + \lambda_j(F) \neq 0, \quad i = 1, 2, \ldots, n, \; j = 1, 2, \ldots, m. \tag{34}$$

According to Lemma 1, the solution P_1 of Eq. (23) is existent and unique. Then

$$\bar{u}^{(0)}(t) = -R^{-1}\bar{B}^{\mathrm{T}}(P\bar{x}^{(0)}(t) + P_1 z(t)). \tag{35}$$

$\bar{u}^{(i)}(t)$ $(i = 1, 2, \ldots)$ is given by the following recursive algorithm. Let

$$\lambda^{(i)}(t) = P\bar{x}^{(i)}(t) + g_i(t). \tag{36}$$

Taking the derivative of Eq. (36) and substituting the first equation in Eq. (27) into it, we obtain

$$\begin{aligned} \dot{\lambda}^{(i)}(t) &= P\dot{\bar{x}}^{(i)}(t) + \dot{g}_i(t) \\ &= (P\bar{A} - PSP)\bar{x}^{(i)}(t) - PSg_i(t) + iP\bar{A}_1\bar{x}^{(i-1)}(t-\tau) + \dot{g}_i(t), \quad i = 1, 2, \ldots. \end{aligned} \tag{37}$$

By substituting Eq. (36) into the second equation in Eq. (27) we have

$$-\dot{\lambda}^{(i)}(t) = (\bar{C}^{\mathrm{T}}Q\bar{C} + \bar{A}^{\mathrm{T}}P)\bar{x}^{(i)}(t) + \bar{A}^{\mathrm{T}}g_i(t) + \mathrm{i}\bar{A}_1^{\mathrm{T}}\left[P\bar{x}^{(i-1)}(t+\tau) + g_{i-1}(t+\tau)\right], \quad i = 1, 2, \ldots. \tag{38}$$

From Eqs. (37) and (38) we obtain Eq. (24).

By Eq. (32), for any of t, $g_i(t)$ can be described as

$$g_i(t) = \mathrm{i}\int_t^{\infty} \mathrm{e}^{(\bar{A}^{\mathrm{T}}-PS)(r-t)}\left[P\bar{A}_1\bar{x}^{(i-1)}(r-\tau) + \bar{A}_1^{\mathrm{T}}P\bar{x}^{(i-1)}(r+\tau) + \bar{A}_1^{\mathrm{T}}g_{i-1}(r+\tau)\right]\mathrm{d}r, \quad i = 1, 2, \ldots, \tag{39}$$

where $\bar{x}^{(i-1)}$ and g_{i-1} are known terms. From Eqs. (26), (39) and (27), $g_i(t)$ and $\bar{x}^{(i)}(t)$ can be calculated by the recursive algorithm. Substituting Eq. (36) into Eq. (28), we obtain

$$\bar{u}^{(i)}(t) = -R^{-1}\bar{B}^{\mathrm{T}}\left(P\bar{x}^{(i)}(t) + g^{(i)}(t)\right), \qquad i = 1, 2, \ldots. \tag{40}$$

From Eqs. (35), (40), (18), (19) and (20), we obtain the optimal tracking control law expressed as Eq. (21). □

Remark 1. The infinite series $\sum_{i=1}^{\infty}(1/i!)g_i(t)$ in Eq. (21) is almost impossible to be solved precisely. In practical engineering, we can obtain an approximate optimal control law by replacing ∞ with a positive integer N. The Nth suboptimal control law is obtained as follows:

$$\bar{u}_N(t) = -R^{-1}\bar{B}^{\mathrm{T}}\left[P\bar{x}(t) + P_1 z(t) + \sum_{i=1}^{N}\frac{1}{i!}g_i(t)\right]. \tag{41}$$

We can select N according to the control precision of performance index. We give a designed algorithm of $\bar{u}_N(t)$ as follows.

Algorithm 1. *Step* 1: Obtain the desire output $\tilde{y}(t)$ from Eq. (4), the value of P and P_1 from Eqs. (22) and (23) respectively. Get the 0th-order control law $\bar{u}_0(t)$ from Eq. (41). Give a small enough number $\delta > 0$.

Step 2: Substitute $\bar{u}_0(t)$ into Eq. (12), and then obtain $y_0(t)$. Calculate the value of $e_0(t)$ from Eq. (5) and J_0 from Eq. (14). Let $i = 1$.

Step 3: Obtain the $g_i(t)$ from Eq. (39) and $\bar{x}^{(i)}(t)$ from the third equation in Eq. (25), respectively.

Step 4: Obtain the ith control law $\bar{u}_i(t)$ from Eq. (41). Substitute $\bar{u}_i(t)$ into Eq. (12), and then obtain $y_i(t)$. Calculate the value of $e_i(t)$ from Eq. (5) and J_i from Eq. (14), respectively.

Step 5: If $|(J_{i-1} - J_i)/J_i| < \delta$, then let $N = i$. Output $\bar{u}_N(t)$ and stop. Else let $i = i+1$ and go to step 3.

Remark 2. In the process of constructing disturbance compensator and the optimal tracking controller, we only consider the frequencies of disturbances, no amplitudes and phases. So in this paper, the optimal tracking control law is robust and always optimal for all choices of parameters except frequencies.

Remark 3. From Eq. (41), we can see the first two terms are accurate solutions and only the last term is an approximation. If $g_i(t)$ attenuates fast, the tracking control law in Eq. (41) will approach the optimal tracking control law at less iteration times N.

Remark 4. Denote that

$$P = \begin{bmatrix} * & * & * \\ P_2 & P_3 & P_4 \end{bmatrix},$$

where * is the unconcerned term, P_2, P_3, and P_4 are matrices of appropriate dimensions. From Eqs. (11) and (13), Eq. (41) can be expressed in the form

$$\bar{u}_N(t) = -R^{-1}\left[P_2x(t) + P_3\bar{x}_2(t) + P_4\bar{x}_3(t) + \bar{B}^{\mathrm{T}}P_1z(t) + \bar{B}^{\mathrm{T}}\sum_{i=1}^{N}\frac{1}{i!}g_i(t)\right]. \tag{42}$$

We can see from Eq. (42) that the 5th term compensates the effect of time-delay. Obviously, if there isn't a time-delay term in Eq. (1), i.e., $A_1 = 0$, we can conclude that $g_i(t)\equiv 0$.

Remark 5. The suboptimal control law $\bar{u}_N(t)$ in Eq. (42) contains the unknown state variable $z(t)$ of the exosystem in Eq. (4), which is physically unrealizable. In practical engineering, a reference input observer can be introduced to make it physically realizable.

We now construct a reduced-order observer for the reference input's state. It is well known that for the full rank matrix H in Eq. (4), there must exist a constant matrix $L \in R^{(m-p)\times m}$ such that the matrix $[\,H^{\mathrm{T}} \quad L^{\mathrm{T}}\,] \in R^{m\times m}$ is nonsingular. Let

$$T = \begin{bmatrix} H \\ L \end{bmatrix}^{-1} = \begin{bmatrix} T_1 & T_2 \end{bmatrix}, \qquad T^{-1}FT = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}, \tag{43}$$

where $T_1 \in R^{m\times p}$, $T_2 \in R^{m\times(m-p)}$, $F_{11} \in R^{p\times p}$, $F_{12} \in R^{p\times(m-p)}$, $F_{12} \in R^{(m-p)\times p}$ and $F_{22} \in R^{(m-p)\times(m-p)}$ are constant matrices. In order to construct a reference input observer, we make the equivalent linear transformation $z = T\tilde{z}$. Denote that $\tilde{z}^{\mathrm{T}} = [\,\tilde{z}_1^{\mathrm{T}} \quad \tilde{z}_2^{\mathrm{T}}\,]$, where $\tilde{z}_1 \in R^p$, $\tilde{z}_2 \in R^{m-p}$. An equivalent system of exosystem in Eq. (4) is obtained as follows:

$$\begin{aligned} \dot{\tilde{z}}_1(t) &= F_{11}\tilde{z}_1(t) + F_{12}\tilde{z}_2(t), \\ \dot{\tilde{z}}_2(t) &= F_{21}\tilde{z}_1(t) + F_{22}\tilde{z}_2(t), \\ \tilde{y}(t) &= \tilde{z}_1(t). \end{aligned} \tag{44}$$

In Eq. (44), $\tilde{z}_1(t)$ is just the reference input $\tilde{y}(t)$. We need only construct a reduced-order observer with respect to $\tilde{z}_2(t)$. Noting that $HT = [\,I_p \quad 0\,]$ and the pair (F, H) is completely observable, obviously the pair (F_{22}, F_{12}) is also completely observable. Construct the reduced-order observer as follows:

$$\begin{aligned} \dot{r}(t) &= \hat{F}r(t) + \hat{H}\tilde{y}(t), \\ \hat{z}(t) &= r(t) + K\tilde{y}(t), \end{aligned} \tag{45}$$

where $r \in R^{m-p}$ is a constructed variable; $\hat{z}(t)$ is the observing value of $\tilde{z}_2(t)$; $\hat{F} = F_{22} - KF_{12}$; $\hat{H} = F_{22}K - KF_{12}K + F_{21} - KF_{11}$; K is an undetermined coefficient matrix. In order to guarantee the speediness and nicety of observer in Eq. (45), we can select matrix K such that all the eigenvalues of matrix $\hat{F}$ are assigned to appointed places.

Replacing $z(t)$ with $[\tilde{y}^{\mathrm{T}}(t) \quad \hat{z}^{\mathrm{T}}(t)]^{\mathrm{T}}$ in Eq. (42) and substituting Eq. (45) into Eq. (42), the suboptimal tracking control law in Eq. (21) can be expressed as

$$\begin{aligned} &\dot{r}(t) = \hat{F}r(t) + \hat{H}\tilde{y}(t), \\ &\bar{u}_N(t) = -K_1x(t) - K_2\bar{x}_2(t) - K_3\bar{x}_3(t) - K_4r(t) - K_5\tilde{y}(t) - K_6\sum_{i=1}^{N}\frac{1}{i!}g^{(i)}(t), \end{aligned} \tag{46}$$

where

$$\begin{aligned} &K_1 = R^{-1}P_2, \quad K_2 = R^{-1}P_3, \quad K_3 = R^{-1}P_4, \\ &K_4 = R^{-1}\bar{B}^{\mathrm{T}}P_1T_2, \quad K_5 = R^{-1}\bar{B}^{\mathrm{T}}P_1(T_1 + T_2K), \quad K_6 = R^{-1}\bar{B}^{\mathrm{T}}. \end{aligned} \tag{47}$$

Remark 6. The suboptimal "control law" $\bar{u}_N(t)$ obtained from Eq. (46) is the suboptimal control law of the augmented system in Eq. (12). It is necessary to solve the suboptimal control law $u_N(t)$ of the original system in Eq. (1). From Eqs. (8), (11) and (46), we have

$$\begin{aligned} \dot{r}(t) &= \hat{F}r(t) + \hat{H}\tilde{y}(t), \\ \ddot{u}_N(t) + M\Omega M^{-1}u_N(t) &= -K_1x(t) - K_2(u_N(t) + Mv(t)) - K_3(\dot{u}_N(t) + M\dot{v}(t)) - K_4r(t) \\ &\quad - K_5\tilde{y}(t) - K_6\sum_{i=1}^{N}\frac{1}{i!}g^{(i)}(t). \end{aligned} \tag{48}$$

Let

$$\begin{aligned} &\xi_1(t) = r(t), \quad \xi_2(t) = u_N(t), \quad \xi_3(t) = \dot{u}_N(t) + K_3Mv(t), \\ &\xi^{\mathrm{T}}(t) = \begin{bmatrix} \xi_1^{\mathrm{T}}(t) & \xi_2^{\mathrm{T}}(t) & \xi_3^{\mathrm{T}}(t) \end{bmatrix}. \end{aligned} \tag{49}$$

Then we obtain a physically realizable dynamic suboptimal tracking control law of the original system in Eq. (1)

$$\begin{aligned} \dot{\xi}(t) = &\begin{bmatrix} \hat{F} & 0 & 0 \\ 0 & 0 & I \\ -K_4 & -K_2 - M\Omega M^{-1} & -K_3 \end{bmatrix}\xi(t) + \begin{bmatrix} \hat{H} \\ 0 \\ -K_5 \end{bmatrix}\tilde{y}(t) \\ &- \begin{bmatrix} 0 \\ K_3 \\ K_2 - K_3^2 \end{bmatrix}Mv(t) - \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}\left[K_1x(t) + K_6\sum_{i=1}^{N}\frac{1}{i!}g^{(i)}(t)\right], \\ u_N(t) = &\ \xi_2(t). \end{aligned} \tag{50}$$

5. A simulation example

Consider a second-order linear system with sinusoidal disturbances described by Eq. (1), where

$$\begin{aligned} &A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.3 & 0.4 \\ 1 & 0.2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 50 \end{bmatrix}, \\ &D = \begin{bmatrix} 0 \\ 50 \end{bmatrix}, \quad C = \begin{bmatrix} 20 & 0 \end{bmatrix}, \\ &\phi(t) = 0, \quad v(t) = 3\sin(1.5t), \quad \tau = 1. \end{aligned}$$

The desired output is given by

$$\tilde{y}(t) = 0.1\ \sin(t),$$

and it can be described by Eq. (4), where

$$F = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad z(0) = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}.$$

Choose the quadratic cost functional described by Eq. (14), where

$$Q = 10, \quad R = 0.1.$$

The cost functional values at different iteration times are listed in Table 1 and the simulation results are presented in Fig. 1. From Table 1, we can see that $J_1 > J_2 > J_3$, that is, the cost functional values decrease as iteration times increase and tend to a deterministic optimal value J^* ultimately. Letting the standard precision $\delta = 0.01$, then we have $|(J_2 - J_3)/J_3| = 0.002 < \delta$. So $\bar{u}_3$ is considered as the approximate

Table 1
Performance index values and control precisions at different iteration times

i	1	2	3
J_i	68.4722	65.9685	65.8164
$\|(J_i - J_{i-1})/J_i\|$	/	0.037	0.002

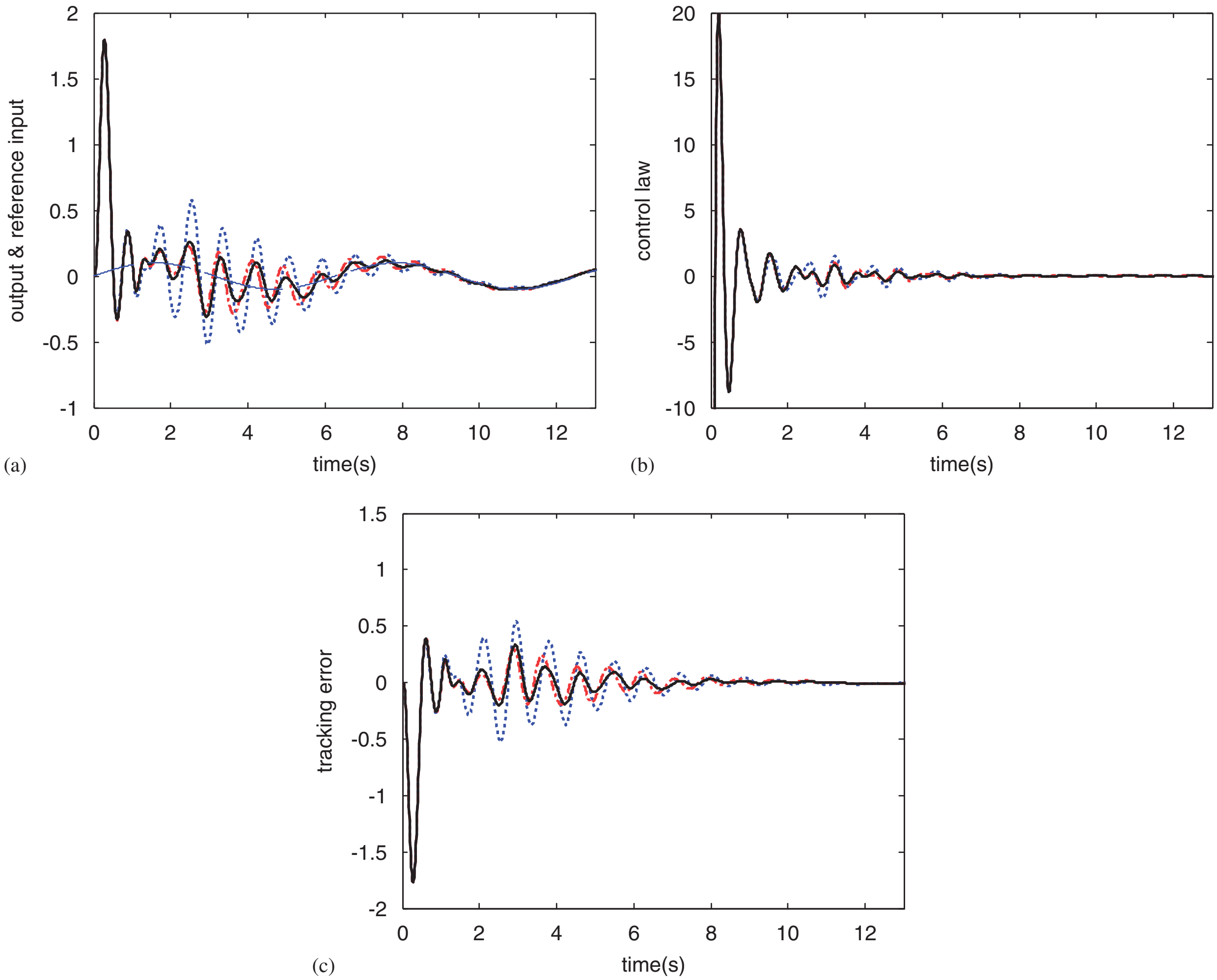

Fig. 1. Simulation curves of the system when $i = 1, 2, 3$: (a) Curves of output $y(t)$ and reference input $\tilde{y}(t)$, (b) curves of control law $\bar{u}^*(t)$, and (c) curves of tracking error $e(t)$. ------- $i = 1$, –·–·–·– $i = 2$, —— $i = 3$, – – – $\tilde{y}(t)$.

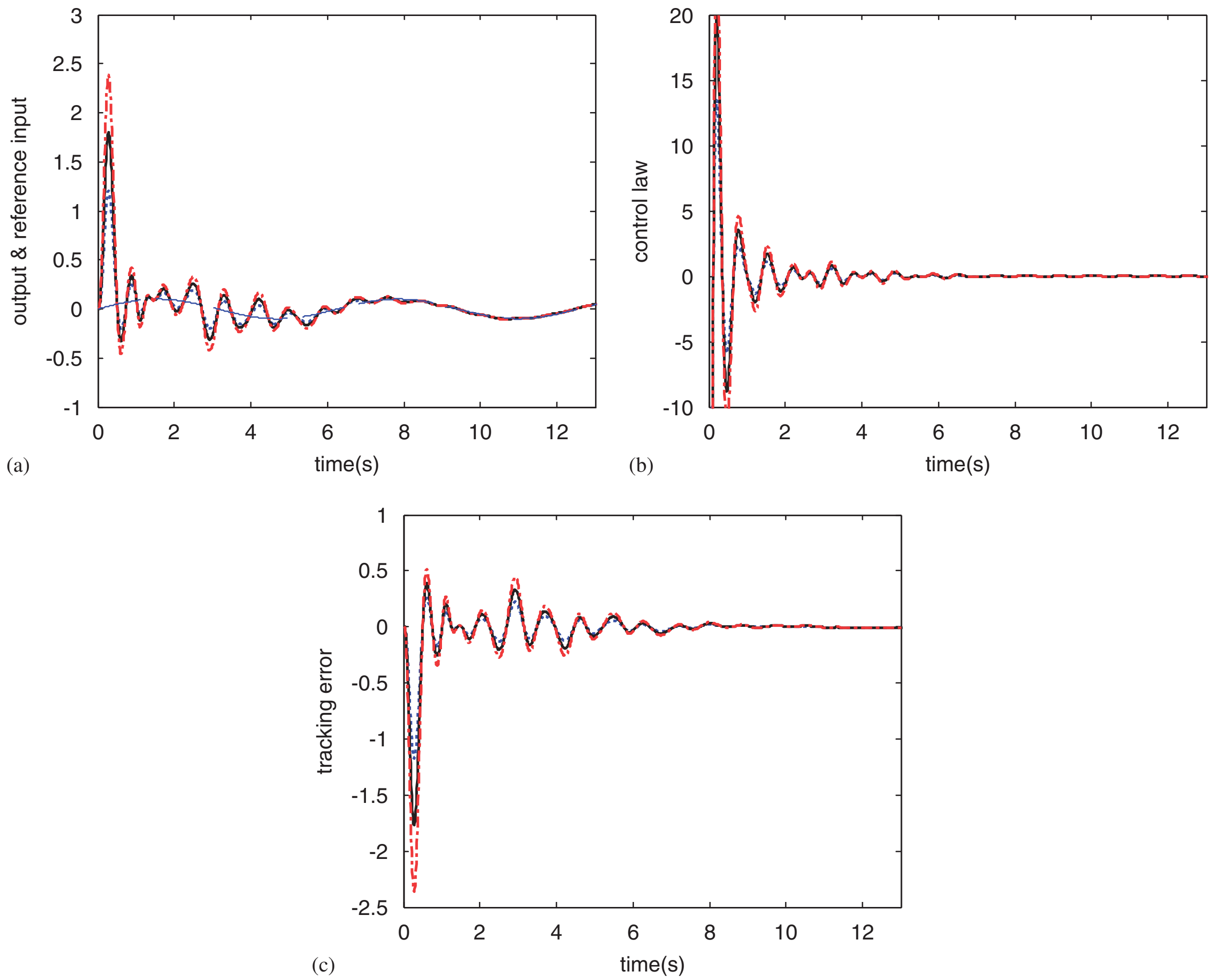

Fig. 2. Simulation curves of the system when $\alpha = 2,3,4$: (a) Curves of output $y(t)$ and reference input $\tilde{y}(t)$, (b) curves of control law $\bar{u}^*(t)$, and (c) curves of tracking error $e(t)$. ------- $\alpha = 2$, —— $\alpha = 3$, –·–·–·– $\alpha = 4$, – – – $\tilde{y}(t)$.

virtual optimal control law. The simulation results demonstrate the effectiveness of the proposed approach in this paper.

In order to verify the robustness of designed controller, we choose different amplitudes of sinusoidal disturbance. The simulation curves of $y(t)$, $e(t)$ and $\bar{u}_N(t)$ when $\alpha = 2, 3, 4$ are presented in Fig. 2. From Fig. 2, we can see that designed approximate optimal control law can achieve disturbance rejection with zero steady-state tracking error for all choices of such parameters except frequencies. So the designed tracking controller in this paper is more robust with respect to the disturbances.

6. Conclusions

In this paper, the proposed optimal tracking control law with zero steady-state error has been robust and achieved the aim of the sinusoidal disturbances rejection as long as the closed-loop system is stable. For a class of time-delay systems, a sensitivity approach has been developed to avoid solving the TPBV problem with both time-delay and time-advance terms directly. The proposed algorithm has better convergence properties. In contrast to feedforward control, the approach proposed in this paper is more reliable, less expensive and easy to implement.

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References

[1] S.-H. Kim, H. Yamato, On the design of a longitudinal motion control system of a fully-submerged hydrofoil craft based on the optimal preview servo system, *Ocean Engineering* 31 (2004) 1637–1653.
[2] W. Wang, G.-Y. Tang, Feedback and feedforward optimal control for offshore jacket platforms, *China Ocean Engineering* 18 (2004) 515–526.
[3] H. Ma, G.-Y. Tang, Y.-D. Zhao, Feedforward and feedback optimal control for offshore structures subjected to irregular wave forces, *Ocean Engineering* 33 (2006) 1105–1117.
[4] J.V. Perunovic, J.J. Jensen, Wave loads on ships sailing in restricted water depth, *Marine Structures* 16 (2003) 469–485.
[5] R. Rebarber, G. Weiss, Internal model based tracking and disturbance rejection for stable well-posed systems, *Automatica* 39 (2003) 1555–1569.
[6] A. Ilchmann, E.P. Ryan, On tacking and disturbance rejection by adaptive control, *Systems & Control Letters* 52 (2004) 137–147.
[7] G.-Y. Tang, Feedforward and feedback optimal control for linear systems with sinusoidal disturbances, *High Technology Letters* 7 (2001) 16–19.
[8] A. Lindquist, V.A. Yakubovich, Universal regulators for optimal tracking in discrete-time systems affected by harmonic disturbances, *IEEE Transactions on Automatic Control* 44 (1999) 1688–1704.
[9] G.-Y. Tang, H.-H. Wang, Suboptimal control for discrete linear systems with time-delay: a non-delay conversion approach, *Acta Automatica Sinica* 31 (2005) 419–426.
[10] G.-Y. Tang, Suboptimal control for nonlinear systems: a successive approximation approach, *Systems and Control Letters* 54 (5) (2005) 429–434.
[11] T. Çimen, S.P. Banks, Nonlinear optimal tracking control with application to super-tankers for autopilot design, *Automatica* 40 (2004) 1845–1863.
[12] P. Lancaster, L. Lerer, M. Tismenetsky, Factored forms for solutions of $AX-XB=C$ and $X-AXB=C$ in companion matrices, *Linear Algebra and its Applications* 62 (1984) 19–49.