

Finite element analysis of vibrating linear systems with fractional derivative viscoelastic models

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Received 30 July 2004; received in revised form 10 March 2006; accepted 17 July 2006

Available online 29 September 2006

Abstract

Fractional derivative rheological models are known to be very effective in describing the viscoelastic behaviour of materials, especially of polymers, and when applied to dynamic problems the resulting equations of motion, after a fractional state-space expansion, can still be studied in terms of modal analysis. But the growth in matrix dimensions carried by this expansion is in general so fast to make the calculations exceedingly cumbersome.

This paper presents a method for reducing the computational effort due to finite element (FE) analysis of vibrating linear systems with a fractional derivative viscoelastic model, namely the Fractional Kelvin–Voigt.

The proposed method may be applied also to problems involving other fractional derivative linear models, and it takes under control the computational effort by reducing the main eigenproblem of large dimension to the solution of two standard related eigenproblems of lower size.

Numerical examples are provided in order to validate both the accuracy and the efficiency of the proposed methodology.

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1. Introduction

Linear viscoelasticity is usually studied by considering rheological models, consisting of integer-order differential constitutive equations, which have deficiencies when applied to large time or frequency intervals, and an improvement of adaptivity with respect to measured data can be obtained by introducing non-integer-order differential operators [1–3]. The application of fractional calculus to viscoelasticity results to be physically consistent and the fractional-order differential stress–strain relations provide good curve-fitting properties, require only few parameters and lead to causal behaviour [4–6].

As a consequence, the possibility of implementing fractional constitutive equations in FE formulations has been studied by several authors [7–9], being a topic of great interest in physics and engineering. Some authors suggested a solution through direct numerical integration of the equations of motion, using the discrete-time Grünwald definition of fractional derivative in order to reduce the computational effort [10]. Other authors proposed iterative solutions in terms of non-standard eigenproblems [11].

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In this paper the problem is addressed in terms of modal analysis [12], assuming generalized damping distributions, leading to complex mode shapes [13]. In the formulation the Fractional Kelvin–Voigt model is considered, according to the Riemann–Liouville and Caputo definitions of fractional derivatives [14,15]. The proposed methodology may be applied also to problems involving different fractional derivative linear models, although not detailed here.

Some fundamentals of modal analysis for fractionally damped systems are presented, including a statement of the expansion theorem, and closed form expressions for the response functions in both the time and the frequency domain [16].

Since this form of expressing the responses requires a fractional state-space expansion, which in general carries too cumbersome calculations for FE applications, a methodology for reducing the computational effort is then presented. Essentially, it consists in replacing the main eigenproblem of large dimension with a couple of related standard eigenproblems of lower size.

A numerical example shows an application of the proposed technique to the study of a viscoelastic cantilever Euler–Bernoulli beam. Numerical FE results are compared to exact analytical solutions.

Finally, an appendix provides the essential information about the definitions of fractional differential operators, their Fourier and Laplace transform properties and the higher transcendent functions adopted within the paper [15].

2. Modal analysis

In this section some fundamentals of modal analysis for fractionally damped systems are presented, including a statement of the expansion theorem, and closed form expressions for the response functions in both the time and the frequency domain.

2.1. Statement of the expansion theorem

The equation describing the dynamic behaviour of a generic continuous linear system with the Fractional Kelvin–Voigt viscoelastic model can be written in the following general form [13]:

$$M \left[\frac{\partial^2}{\partial t^2} w(x, t) \right] + C_f \left[\frac{\partial^\alpha}{\partial t^\alpha} w(x, t) \right] + K[w(x, t)] = f(x, t), \quad x \in D, \quad (1)$$

where M , C_f , K are linear homogeneous differential operators and are referred to as mass operator, generalized fractional damping operator [17,18] and stiffness operator, respectively, f is the external force density, t is time, w and x are the displacement and the spatial coordinate in a domain of extension D , and $\alpha \in (0,1)$ is the non-integer derivative order, in general varying with respect to x . It is important noting that the domain D is not necessarily one dimensional, the spatial coordinate x as well as the displacement w and the external force density f can be vectors, and the differential operators M , C_f , K can take a matrix form. For example, x is a vector in the case of a vibrating plate, and w is a vector in the case of a Timoshenko beam (displacement and rotation). Nevertheless in Eq. (1) the vectorial notation is not adopted, since in the following developments this does not represent a loss of generality and it simplifies the notation when defining the fractional state and the external force density vectors in Eq. (4).

To solve Eq. (1), appropriate boundary and initial conditions must be satisfied by w .

Recalling that self-adjointness of differential operators corresponds to symmetry of matrices [13], in the following the operators M , C_f and K will be supposed to be self-adjoint. This assumption is unnecessary and could easily be removed, as explained in Ref. [13], nevertheless it will be adopted since such properties hold in most of the existing models, carrying less cumbersome analytical developments.

With the notation:

$$(w_1, w_2) = \int_D w_1 w_2 \, dD, \quad (2)$$

denoting the inner product between two scalar functions w_1 and w_2 over a domain of extension D (if w_1 and w_2 are vector functions, then the integrand in Eq. (2) represents their scalar product), a linear differential operator

L is said to be self-adjoint if

$$(w_1, L[w_2]) = (w_2, L[w_1]), \tag{3}$$

which is a property of symmetry with respect to the inner product [19].

Note that the following developments are valid for discrete systems as well: in that case the operators M , C_f and K are square matrices and the functions w and f are vectors.

In order to rewrite Eq. (1) by analogy with a state-space representation, rational values for the fractional derivative order α are considered, that is $\alpha = h/q$, which in itself is not a restriction for applications. Since α can vary on the spatial domain, it is assumed to be piecewise constant on D , $\mu = 1/q$ being the minimum common fractional derivative order.

The fractional state vector and the external force density vector have both dimension $2q$ and can be defined according to [12]:

$$\begin{aligned} \mathbf{w} &= [w \quad w^{(\mu)} \quad w^{(2\mu)} \quad \dots \quad w^{(2-\mu)}]^T, \\ \mathbf{f} &= [f \quad 0 \quad 0 \quad \dots \quad 0]^T, \end{aligned} \tag{4}$$

where the superscripts denote the time derivative orders. So the fractional state vector contains the displacement w and all its time derivatives of order multiple of μ up to $2-\mu$. Consequently, Eq. (1) can be rewritten in a fractional state-space form as follows:

$$A_f[\mathbf{w}^{(\mu)}] + B_f[\mathbf{w}] = \mathbf{f}, \tag{5}$$

where the homogeneous linear differential operators A_f and B_f can be expressed in the matrix-form:

$$\begin{aligned} A_f &= \underbrace{\begin{bmatrix} 0 & \dots & 0 & C_f & 0 & \dots & 0 & M \\ \dots & 0 & C_f & 0 & \dots & 0 & M & 0 \\ 0 & C_f & 0 & \dots & 0 & M & 0 & \dots \\ C_f & 0 & \dots & 0 & M & 0 & \dots & 0 \\ 0 & \dots & 0 & M & 0 & \dots & 0 & 0 \\ \dots & 0 & M & 0 & \dots & 0 & 0 & 0 \\ 0 & M & 0 & \dots & 0 & 0 & 0 & 0 \\ M & 0 & \dots & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{2q}, \\ B_f &= \underbrace{\begin{bmatrix} K & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & -C_f & 0 & \dots & 0 & -M \\ \dots & 0 & -C_f & 0 & \dots & 0 & -M & 0 \\ 0 & -C_f & 0 & \dots & 0 & -M & 0 & \dots \\ 0 & 0 & \dots & 0 & -M & 0 & \dots & 0 \\ 0 & \dots & 0 & -M & 0 & \dots & 0 & 0 \\ \dots & 0 & -M & 0 & \dots & 0 & 0 & 0 \\ 0 & -M & 0 & \dots & 0 & 0 & 0 & 0 \end{bmatrix}}_{2q}. \end{aligned} \tag{6}$$

The position of the antidiagonals containing the operator C_f depends on the fractional derivative order (or more precisely, on its numerator h). Note that if M , C_f and K are self-adjoint, A_f and B_f result to be self-adjoint too.

As usual in modal analysis, the differential problem is reduced to a differential eigenvalue problem by separating the variables. With the solution in form of a linear combination of terms $w(x, t) = \phi(x)q(t)$, Eq. (5) yields

$$s^h A_f[\mathbf{z}] + B_f[\mathbf{z}] = \mathbf{0}, \tag{7}$$

leading to the differential eigenproblem:

$$r = s^h \Rightarrow r A_f[\mathbf{z}] + B_f[\mathbf{z}] = \mathbf{0}, \tag{8}$$

where the eigenvectors \mathbf{z} take the form:

$$\mathbf{z} = \left[\phi(x) \quad r\phi(x) \quad r^2\phi(x) \quad \dots \quad r^{2q-1}\phi(x) \right]^T. \tag{9}$$

They are known unless a scaling factor, generally complex, depending on the initial conditions. The solution of this eigenproblem forms an infinite set of pairs of discrete values, each pair being related to a pair of eigenvectors (i.e. to a pair of eigenfunctions).

The eigenvector orthogonality properties can be derived rewriting Eq. (8) for the n th and m th eigenvectors, respectively, pre-multiplying the first by \mathbf{z}_m^T and the second by \mathbf{z}_n^T , then integrating them both over the spatial domain D , i.e.:

$$\begin{cases} r_n A_f[\mathbf{z}_m] + B_f[\mathbf{z}_n] = 0 \\ r_n A_f[\mathbf{z}_n] + B_f[\mathbf{z}_m] = 0 \end{cases} \Rightarrow \begin{cases} r_n(\mathbf{z}_m, A_f[\mathbf{z}_n]) + (\mathbf{z}_m, B_f[\mathbf{z}_n]) = 0, \\ r_n(\mathbf{z}_n, A_f[\mathbf{z}_m]) + (\mathbf{z}_n, B_f[\mathbf{z}_m]) = 0, \end{cases} \tag{10}$$

which, taking into account the self-adjointness of A_f and B_f , yield

$$\begin{cases} (r_n - r_m)(\mathbf{z}_m, A_f[\mathbf{z}_n]) = 0, \\ (r_n^{-1} - r_m^{-1})(\mathbf{z}_m, B_f[\mathbf{z}_n]) = 0, \end{cases} \tag{11}$$

thus

$$\text{if } n = m \quad \text{then} \quad \begin{cases} (\mathbf{z}_n, A[\mathbf{z}_n]) = a_n, & \begin{cases} (\mathbf{z}_n^*, A[\mathbf{z}_n]) = 0, \\ (\mathbf{z}_n^*, B[\mathbf{z}_n]) = 0, \end{cases} \end{cases} \tag{12}$$

$$\text{if } n \neq m \quad \text{then} \quad \begin{cases} (\mathbf{z}_m, A[\mathbf{z}_n]) = 0, \\ (\mathbf{z}_m, B[\mathbf{z}_n]) = 0, \end{cases} \tag{13}$$

where the modal parameters a_n and b_n can be expressed as functions of the eigenvalues r and of the inner products of the operators M , C_f , K and the eigenfunctions ϕ :

$$\begin{aligned} a_n &= 2q(\phi_n, M[\phi_n])r_n^{2q-1} + h(\phi_n, C[\phi_n])r_n^{h-1}, \\ b_n &= (\phi_n, K[\phi_n]) - (2q - 1)(\phi_n, M[\phi_n])r_n^{2q} - (h - 1)(\phi_n, C[\phi_n])r_n^h. \end{aligned} \tag{14}$$

Note that b_n can be found directly knowing a_n and r_n , recalling that $b_n/a_n = -r_n$.

Due to the orthogonality properties of the eigenvectors \mathbf{z}_n , any other vector in the same space can be expressed as their linear combination, which constitutes an extension to fractional systems of the expansion theorem.

2.2. Time domain analysis

According to the expansion theorem, the general solution of Eq. (1) in the time domain can be written as a linear combination of modes:

$$w(x, t) = \sum_{n=1}^{\infty} \gamma_n \phi_n(x) q_n(t), \tag{15}$$

where γ_n is a scaling factor, generally complex, depending on the initial conditions.

Introducing the expression (15) into Eq. (5), the decoupling follows by applying the orthogonality relations (12), (13) and the factors $\gamma_n q_n(t)$ can then be evaluated according to the Laplace transform technique. The unit impulse response can then be expressed in the following form:

$$h(x, x_f; t) = \sum_{n=1}^{\infty} \frac{\phi_n(x_f)\phi_n(x)}{a_n} e_n(t), \tag{16}$$

x_f being the acting point of the impulse. The functions $e_n(t)$ can be defined according to

$$e_n(t) = \sum_{j=0}^{q-1} r_n^{2q-j-1} E_t(1 - j\mu, r_n^q), \tag{17}$$

where E_t is a fractional integral of the exponential function, which can be computed by means of a series, that is

$$E_t(v, a) = I^{(v)}[e^{at}] = t^v \sum_{i=0}^{\infty} \frac{(at)^i}{\Gamma(v + i + 1)}, \tag{18}$$

and $\Gamma(\cdot)$ is the Gamma function. For further details, the reader is referred to the appendix.

On the basis of the unit impulse response (16), the steady-state response to an arbitrary load $g(x)f(t)$ could be computed by means of a convolution integral:

$$w_{reg}(x, t) = \sum_{n=1}^{\infty} \frac{(\phi_n, g)\phi_n(x)}{a_n} \int_0^t f(\xi)e_n(t - \xi) d\xi, \tag{19}$$

where (ϕ_n, g) is the modal force defined through an inner product involving the external density force $g(x)$, either distributed or concentrated.

2.3. Frequency response functions

A harmonic excitation force of amplitude f_0 acting with angular frequency ω at a coordinate x_f is considered. Since the system is linear-time-invariant, the steady-state response w will still be a harmonic oscillation at the same angular frequency ω . So, taking into account the expansion theorem and dropping the time dependent terms, the state-space equation of motion (5) can be rewritten as

$$\sum_{n=1}^{\infty} \Psi_n \{ (i\omega)^\mu A[\mathbf{z}_n] + B[\mathbf{z}_n] \} = \mathbf{f}_0, \tag{20}$$

where Ψ_n is a scaling factor and $\mathbf{f}_0 = [f_0\delta(x - x_f)0]^\top$, δ being the Dirac distribution.

Pre-multiplying by \mathbf{z}_n^\top , integrating over the spatial domain D and recalling the orthogonality properties (12) and (13), Eq. (20) gives

$$\Psi_n = \frac{(\mathbf{z}_n, \mathbf{f}_0)}{a_n[(i\omega)^\mu - r_n]} = \frac{\phi_n(x_f)f_0}{a_n[(i\omega)^\mu - r_n]}, \tag{21}$$

where the expression of the modal force $f_n = (\mathbf{z}_n, \mathbf{f}_0)$ in terms of ϕ_n and f_0 is due to the Dirac distribution properties. By means of the definition of modal force through an inner product, the analysis can be extended to study distributed harmonic loads.

Taking into account again the expansion theorem and Eqs. (20) and (21), it is possible to express the system receptance as follows:

$${}_x H_{x_f}(\omega) = \sum_{n=1}^{\infty} \left[\frac{\phi_n(x_f)\phi_n(x)}{a_n[(i\omega)^\mu - r_n]} \right]. \tag{22}$$

The expressions of other frequency response functions follow immediately from Eq. (22).

3. Computational approach

The growth of matrix dimensions carried by the previously described fractional state-space expansions is in general so fast to make the calculations too cumbersome for FE applications. So it is necessary to find alternative solutions.

First of all, it is convenient to distinguish between systems with ‘proportional’ damping distributions and systems with generalized or ‘non-proportional’ damping distributions.

Clearly, these definitions still hold when fractional derivative rheological models are considered: in such cases the model is said to be proportional if the undamped system eigenfunctions φ (obtained taking into account the operators M and K only) result to be real, i.e. orthogonal with respect to the fractional damping operator C_f . In this case the computational effort can be cut down simply by using the eigenfunctions φ to reduce the problem to N equivalent uncoupled single degree of freedom (dof) systems, one for each mode to be taken into account in the modal expansion.

It is worth pointing out that for discrete fractionally damped systems with many dofs, this procedure carries even more advantages than in the case of viscous damping, avoiding to solve a large eigenproblem of dimension $2qn$.

In presence of generalized fractional damping distributions, the problem can be solved by considering that also in this case it is possible to expand the solution on the basis of the undamped system eigenfunctions φ (vectors, for discrete systems) and then to reduce an eigenproblem of dimension $2qn$ to two distinct smaller eigenproblems of dimensions n and $2qN$, respectively, being N the number of modes taken into account to approximate the solution. This means that the computational effort can be cut down by taking $N \ll n$.

Clearly this technique can be applied to distributed parameter models provided that the related undamped system eigenfunctions are known.

According to this condensation method, as clearly explained in [20], the solution can be approximated by a finite expansion in terms of the (known) undamped system eigenfunctions φ , not to be confused with the eigenfunctions ϕ of Eq. (9)

$$w(x, t) \cong \sum_{j=1}^N \varphi_j(x)r_j(t) = \boldsymbol{\Phi}^T \mathbf{r}. \tag{23}$$

Substituting this expanded form of the solution in Eq. (1) and taking into account the orthogonality relations of the eigenfunctions φ with respect to M and K , it is possible to rewrite the state-space equation (5) as follows:

$$\mathbf{A}_f \mathbf{v}^{(\mu)} + \mathbf{B}_f \mathbf{v} = \mathbf{g}, \tag{24}$$

where the $2qN \times 2qN$ matrices \mathbf{A}_f and \mathbf{B}_f can be defined by analogy with the differential operators (6):

$$\mathbf{A}_f = \underbrace{\begin{bmatrix} 0 & \dots & 0 & \mathbf{C}_f & 0 & \dots & 0 & \mathbf{M} \\ \dots & 0 & \mathbf{C}_f & 0 & \dots & 0 & \mathbf{M} & 0 \\ 0 & \mathbf{C}_f & 0 & \dots & 0 & \mathbf{M} & 0 & \dots \\ \mathbf{C}_f & 0 & \dots & 0 & \mathbf{M} & 0 & \dots & 0 \\ 0 & \dots & 0 & \mathbf{M} & 0 & \dots & 0 & 0 \\ \dots & 0 & \mathbf{M} & 0 & \dots & 0 & 0 & 0 \\ 0 & \mathbf{M} & 0 & \dots & 0 & 0 & 0 & 0 \\ \mathbf{M} & 0 & \dots & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{2q}, \tag{25}$$

$$\mathbf{B}_f = \underbrace{\begin{bmatrix} \mathbf{K} & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & -\mathbf{C}_f & 0 & \dots & 0 & -\mathbf{M} \\ \dots & 0 & -\mathbf{C}_f & 0 & \dots & 0 & -\mathbf{M} & 0 \\ 0 & -\mathbf{C}_f & 0 & \dots & 0 & -\mathbf{M} & 0 & \dots \\ 0 & 0 & \dots & 0 & -\mathbf{M} & 0 & \dots & 0 \\ 0 & \dots & 0 & -\mathbf{M} & 0 & \dots & 0 & 0 \\ \dots & 0 & -\mathbf{M} & 0 & \dots & 0 & 0 & 0 \\ 0 & -\mathbf{M} & 0 & \dots & 0 & 0 & 0 & 0 \end{bmatrix}}_{2q},$$

and the fractional state-space and density force vectors can be written in the form:

$$\mathbf{v} = \begin{bmatrix} \mathbf{r} \\ \mathbf{r}^{(1)} \\ \dots \\ \mathbf{r}^{(2q-1)} \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} (\varphi, f(x, t)) \\ 0 \\ \dots \\ 0 \end{bmatrix}. \tag{26}$$

The $N \times N$ matrices \mathbf{M} , \mathbf{C}_f and \mathbf{K} are built up by means of the following inner products involving the differential operators M , C_f , K and the eigenfunctions φ :

$$\mathbf{M} = [(\varphi_i, M[\varphi_j])], \quad \mathbf{C}_f = [(\varphi_i, C_f[\varphi_j])] = \mathbf{C}_f^T, \quad \mathbf{K} = [(\varphi_i, K[\varphi_j])], \tag{27}$$

with $i, j = 1, \dots, N$. It is worth noting that both \mathbf{M} and \mathbf{K} are diagonal.

The solution of the related algebraic eigenvalue problem, consisting of a set of $2qN$ eigenvalues $r_n^{(r)}$ and $2qN$ eigenvectors \mathbf{u}_n , allows to decouple the equations of motion (24) by introducing the usual coordinate transformation $\mathbf{v} = \mathbf{U}\boldsymbol{\eta}$ (\mathbf{U} denoting the eigenvector matrix and $\boldsymbol{\eta}$ the modal coordinate vector), which in the Laplace domain yields:

$$\eta_n = \frac{f_n}{a_n^{(r)} [s^\mu - r_n^{(r)}]}, \tag{28}$$

where $a_n^{(r)}$ is the n th element of the diagonalization of \mathbf{A}_f and f_n is the n th component of the modal force vector $\mathbf{U}^T \mathbf{g}$.

If a single external force of amplitude f_0 acting at a coordinate x_f is considered, according to Eq. (26) f_n can be expressed by means of the eigenfunctions φ of the undamped system as follows:

$$f_n = \sum_{i=1}^N u_{in} \varphi_i(x_f) f_0. \tag{29}$$

Introducing Eq. (29) in the expression of the modal coordinates, and taking into account backwards the links among $\boldsymbol{\eta}$, \mathbf{v} and \mathbf{r} , the expansion (23) yields the unit impulse response in the form:

$${}_x h_{x_f}(t) \cong \sum_{n=1}^{2qN} \left[\sum_{i=1}^N \hat{u}_{in} \varphi_i(x_f) \right] \left[\sum_{j=1}^N \hat{u}_{jn} \varphi_j(x) \right] e_n(t), \tag{30}$$

and the receptance:

$${}_x H_{x_f}(\omega) \cong \sum_{n=1}^{2qN} \frac{\left[\sum_{i=1}^N \hat{u}_{in} \varphi_i(x_f) \right] \left[\sum_{j=1}^N \hat{u}_{jn} \varphi_j(x) \right]}{(i\omega)^\mu - r_n^{(r)}}, \tag{31}$$

the superscript $\hat{}$ denoting normalization with respect to the square root of $a_n^{(r)}$.

In case of FE applications, the expressions (30) and (31) can be rewritten in terms of the eigenvectors of the undamped discrete system (namely \mathbf{p}_j , not to be confused with the eigenfunctions φ). To cut down the computational effort, the order of the system can be reduced by taking into account a subset of N eigenvectors \mathbf{p}_j with $N \leq M$, M being the number of dofs of the FE model. It should be stressed that the selected sequence of eigenvectors does not necessarily include the first N or even a set of N eigenvectors close to \mathbf{p}_j , albeit this has been the choice for the numerical examples herein presented. The response at a dof h due to a unit impulse applied in the dof m takes the form:

$${}_h h_m(t) \cong \sum_{n=1}^{2qN} \left[\sum_{i=1}^N \hat{u}_{in} p_{mi} \right] \left[\sum_{j=1}^N \hat{u}_{jn} p_{hj} \right] e_n(t), \tag{32}$$

whilst the receptance is

$${}_h H_m(\omega) \cong \sum_{n=1}^{2qN} \frac{\left[\sum_{i=1}^N \hat{u}_{in} p_{mi} \right] \left[\sum_{j=1}^N \hat{u}_{jn} p_{hj} \right]}{(i\omega)^\mu - r_n^{(r)}}. \tag{33}$$

The responses can therefore be expressed as functions of a subset of real eigenvectors \mathbf{p}_j of the undamped system, and of the complex eigenvalues and eigenvectors $s_n^{(r)}$, \mathbf{u}_n of the (low order) damped system.

Finally, it should be noticed that the eigenproblem (8) in the case of discrete systems (and so also in the case of the operator Eq. (25)) is generally ill-conditioned as the matrix \mathbf{A}_f structure suggests. This difficulty can be overcome by considering the inverse of \mathbf{A}_f :

$$\mathbf{A}_f^{-1} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & \mathbf{M}^{-1} \\ 0 & 0 & 0 & 0 & \dots & 0 & \mathbf{M}^{-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & \mathbf{M}^{-1} & 0 & \dots \\ 0 & 0 & \dots & 0 & \mathbf{M}^{-1} & 0 & \dots & 0 \\ 0 & \dots & 0 & \mathbf{M}^{-1} & 0 & \dots & 0 & -\mathbf{M}^{-1} \mathbf{C}_f \mathbf{M}^{-1} \\ \dots & 0 & \mathbf{M}^{-1} & 0 & \dots & 0 & -\mathbf{M}^{-1} \mathbf{C}_f \mathbf{M}^{-1} & 0 \\ 0 & \mathbf{M}^{-1} & 0 & \dots & 0 & -\mathbf{M}^{-1} \mathbf{C}_f \mathbf{M}^{-1} & 0 & 0 \\ \mathbf{M}^{-1} & 0 & \dots & 0 & -\mathbf{M}^{-1} \mathbf{C}_f \mathbf{M}^{-1} & 0 & 0 & 0 \end{bmatrix}}_{2q-h}, \tag{34}$$

that is, by solving the related eigenproblem $(r\mathbf{I} + \mathbf{A}_f^{-1} \mathbf{B}_f) \mathbf{v} = \mathbf{0}$.

4. Numerical examples

In this section some numerical examples are presented regarding Euler–Bernoulli beams and the description of an analytical method to solve the differential eigenproblem for non-homogeneous Euler–Bernoulli beams in bending vibrations is included.

4.1. Solution of the eigenproblem

For an Euler–Bernoulli beam in bending vibration, the mass, damping and stiffness operators consist of

$$M = m(x), \quad C = c_f(x) \quad \text{or} \quad C = \frac{\partial^2}{\partial x^2} \left[c_{fin}(x) \frac{\partial^2}{\partial x^2} \right], \quad K = \frac{\partial^2}{\partial x^2} \left[k(x) \frac{\partial^2}{\partial x^2} \right], \quad (35)$$

where $m(x)$ is the mass per unit length of beam, $c_f(x)$ is the external fractional damping distribution, $c_{fin}(x)$ is the internal fractional damping distribution (according to the Fractional Kelvin–Voigt model, used in conjunction with the assumption that cross-sectional areas remain plane during deformation) and $k(x) = EI(x)$ is the bending stiffness, or flexural rigidity, in which E is the Young’s modulus of the material and I is the area moment of inertia [13].

The distributions $m(x)$, $c_f(x)$ (or $c_{fin}(x)$) and $k(x)$ are considered piecewise constant on D . So, dividing the beam into P segments of length $\Delta x_p = x_p - x_{p-1}$ (where $x_0 = 0$, $x_P = l$, length of the beam), the differential eigenvalue problem can be reduced to a set of P fourth-order ordinary differential equations with constant coefficients:

$$\phi_p^{IV}(x) = a_p^4 \phi_p(x), \quad (36)$$

with appropriate boundary conditions, where

$$a_p = \sqrt[4]{-\frac{m_p s^2 + c_{fp} s^\alpha}{c_{fin,p} s^\alpha + k_p}}. \quad (37)$$

Eq. (36) can be conveniently converted into a set of four first-order equations.

According to the state vector definition:

$$\mathbf{y}(x) = \left[\phi^{III}(x) \quad \phi^{II}(x) \quad \phi^I(x) \quad \phi(x) \right]^T, \quad (38)$$

the solution for each segment can then be expressed as

$$\mathbf{y}_p(x) = \mathbf{\Phi}_p e^{\Lambda_p x} \mathbf{c}_p, \quad (39)$$

where, $\mathbf{\Phi}_p$ is the p th segment eigenvector matrix, Λ_p is the p th segment eigenvalue matrix (with eigenvalues $\lambda_{1p} = a_p$, $\lambda_2 = -a_p$, $\lambda_3 = ia_p$, $\lambda_4 = -ia_p$) and \mathbf{c}_p is the p th segment constant vector.

Moreover, it is possible to show [21] that the solution at any point x_p can be written as

$$\mathbf{y}_p(x_p) = \prod_p \mathbf{y}_1(0) \quad \text{with} \quad \prod_p = \prod_{i=p}^1 \left[\mathbf{\Phi}_i e^{\Lambda_i(x_i - x_{i-1})} \mathbf{\Phi}_i^{-1} \mathbf{B}_{i-1} \right], \quad (40)$$

where the i th segment eigenvector matrix and its inverse can be written as functions of a_i , and the \mathbf{B}_{i-1} are 4×4 matrices obtained by imposing the continuity of displacement, rotation, moment and shear in $x = x_{i-1}$. Clearly, these constraints represent the inner boundary conditions between the adjacent beam segments. For analytical expressions of the above-mentioned matrices $\mathbf{\Phi}$ and \mathbf{B} , the reader is referred to Ref. [21].

It is now possible to relate the solution $\mathbf{y}(l)$ at one end of the beam to the solution $\mathbf{y}(0)$ at the other end, which enables to express the boundary conditions at the ends of the beam in the following form:

$$\begin{cases} \mathbf{B}_{e0} \mathbf{y}_1(0) = \mathbf{0}, \\ \mathbf{B}_{el} \prod_p \mathbf{y}_1(0) = \mathbf{0}, \end{cases} \quad (41)$$

where \mathbf{B}_e are 2×4 matrices depending on the kind of constraints and $\mathbf{y}_1(0) = \mathbf{\Phi}_1 \mathbf{c}_1$. In the case of a clamped-free beam they take the form:

$$\mathbf{B}_e = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{B}_e = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \quad (42)$$

clamped free

Eq. (41) form a linear homogeneous system of four algebraic equations in four unknowns (i.e. the constants \mathbf{c}_1). Thus the solution of the eigenproblem follows directly by setting to zero the determinant of the coefficient matrix, recalling that its elements depend on the (unknown) eigenvalues s .

4.2. Frequency response functions through direct integration

Besides the modal approach, the analytical tools just developed allow to express the FRFs through direct integration of the equation of motion, avoiding the solution of the eigenproblem.

The Euler–Bernoulli beam model with piecewise constant distributions is now considered under the effect of a harmonic force of amplitude f_0 acting with angular frequency ω at a coordinate x_f . Since the steady-state response results to be still a harmonic oscillation at the same angular frequency ω , dropping the time dependent terms, the equation of motion for each segment of the beam reduces to

$$W^{IV}(x) + a_{\omega p}^4 W(x) = \kappa_{\omega p}^{-1} f_0 \delta(x - x_f), \tag{43}$$

where, the coefficients $a_{\omega p}$ and $\kappa_{\omega p}$, which are constant within each segment, can be defined according to Eq. (37) by substituting s with $i\omega$ (κ being the denominator of the radicand in Eq. (37)).

Eq. (43) is an ordinary differential equation with constant coefficients, since the angular frequency is considered as a given parameter. As previously stated, in order to find the global solution, the four coefficients \mathbf{c}_1 have to be determined by imposing four boundary conditions.

In the case of a homogeneous beam with two different external damping levels forced in x_f (with $x_1 \leq x_f \leq l$), by assuming, without loss of generality, that the external force acts at a separation point between two segments (say: $x_f = x_p$), and defining the external force vector in the state-space as follows:

$$\mathbf{f} = \begin{bmatrix} \kappa_{\omega f}^{-1} f_0 & 0 & 0 & 0 \end{bmatrix}^T, \tag{44}$$

($\kappa_{\omega f}$ being κ_{ω} evaluated in x_f), the system yielding the unknown coefficients \mathbf{c}_1 is [18]

$$\begin{cases} \mathbf{B}_{el} [\Phi_{2\omega} e^{\Lambda_{2\omega}(l-x_1)} \Phi_{2\omega}^{-1} \Phi_{1\omega} e^{\Lambda_{1\omega} x_1}] \mathbf{c}_1 = -\mathbf{B}_{el} [\Phi_{2\omega} e^{\Lambda_{2\omega}(l-x_f)} \Phi_{2\omega}^{-1}] \mathbf{f}, \\ \mathbf{B}_{e0} \Phi_{1\omega} \mathbf{c}_1 = 0. \end{cases} \tag{45}$$

It is important pointing out that:

- the matrices \mathbf{B}_e are the same as in Eq. (41); and
- the matrices retain their own definitions as in Eq. (41), but the subscript ω means that a_{np} has been substituted by $a_{\omega p}$ (i.e. in every definition the eigenvalues s_n has been changed in $i\omega$).

So the receptance at a coordinate x (with $0 \leq x \leq x_1$) can be written in function of the four coefficients \mathbf{c}_1 :

$${}_x H_{x_f}(\omega) = \frac{1}{f_0} [c_{11} e^{a_{1\omega} x} + c_{12} e^{-a_{1\omega} x} + c_{13} e^{i a_{1\omega} x} + c_{14} e^{-i a_{1\omega} x}]. \tag{46}$$

4.3. FE model and numerical results

A standard beam element is the basis to the entire FE model. Each node i exhibits two dofs, i.e. the transverse displacement w_i and the rotation ϑ_i , and the shape functions of the element are third degree polynomials, so that mass and stiffness matrices can simply be computed or even found in any textbook [22,23].

As regards the damping matrix, it can be written with the same structure of the mass (external damping) or the stiffness (internal damping) matrix, or even with a combination of the two. The damping distribution along the beam can be different from the mass and stiffness distributions.

In the numerical examples herein presented the FE model is composed of 30 elements with constant distribution of mass and stiffness ($m = 0.243 \text{ kg/m}$, $k = 4.725 \text{ N m}^2$). The number of elements is largely sufficient to give an optimum accuracy on the first five modes and allows placing the input and output

observation points exactly in the same places as in the analytical model. In the following examples the input force has always been applied at the free end of the beam whilst the output displacement is detected at 1/4 of the beam length.

To match the analytical model, the damping distribution is constant on two segments of the beam as shown in Fig. 1.

Fig. 2 shows the comparison of the FRFs computed according to Eq. (46) (exact solution) and to Eq. (33) (approximated FE solution) when the damping is internal, which in the FE formulation means that the damping and the stiffness matrices have the same structure in both segments of the beam ($c_{fm1} = 31.62 \times 10^{-5} \text{ N s m}^2$, $c_{fm2} = 10.00 \times 10^{-5} \text{ N s m}^2$, where 1 is referred to the segment of the beam near the clamped end, 2 to the segment near the free end). Some minor differences can be noticed near the anti-resonance at about 1700 Hz and very similar results can be obtained also by implementing Eq. (22), applied to a discrete system. It is worth noting that the exact fractional solution (Eq. (22)) is approximately 50 times slower to run than its approximated equivalent (Eq. (33)) albeit not achieving really different results.

Fig. 3 shows the effects of the fractional derivative order. The line marked with $h_1 = h_2 = q$ represents the behaviour of the system with derivative order equal to one and both internal and external damping ($c_{fm1} = 31.62 \times 10^{-5} \text{ N s m}^2$, $c_{fm2} = 10.00 \times 10^{-5} \text{ N s m}^2$, $c_{f1} = 200 \text{ N s/m}^2$, $c_{f2} = 100 \text{ N s/m}^2$). The other lines are obtained by varying h_1 and h_2 in order to demonstrate the effect of reducing the derivative order. No particular distortion of these latter FRFs can be observed and in fact they can accurately be fitted by any traditional frequency domain method involving a modal parameter extraction. Clearly the parameters thus obtained loose any link to the fractional order, being based on an assumed viscous or hysteretical model. These observations make clear that the problem of identifying even the presence, not to speak of the type, of a fractional derivative system has to be faced by more sophisticated techniques and will possibly be addressed in the future.

Fig. 4 shows the contribution of the first three modes to the impulse response. The system has derivative order 0.5 and proportional internal damping ($c_{fm1} = c_{fm2} = 3 \times 31.62 \times 10^{-5} \text{ N s}^{0.5} \text{ m}^2$). A close look to the time history of mode 3 reveals that its oscillation, just like the oscillation of the other modes, is superimposed to a decaying behaviour approaching zero towards infinity.

Fig. 5 shows the influence of the derivative order on the part of the impulse response due to the second mode only. With α passing from 1/16 to 1/2 the effect of the damping becomes obviously more evident.

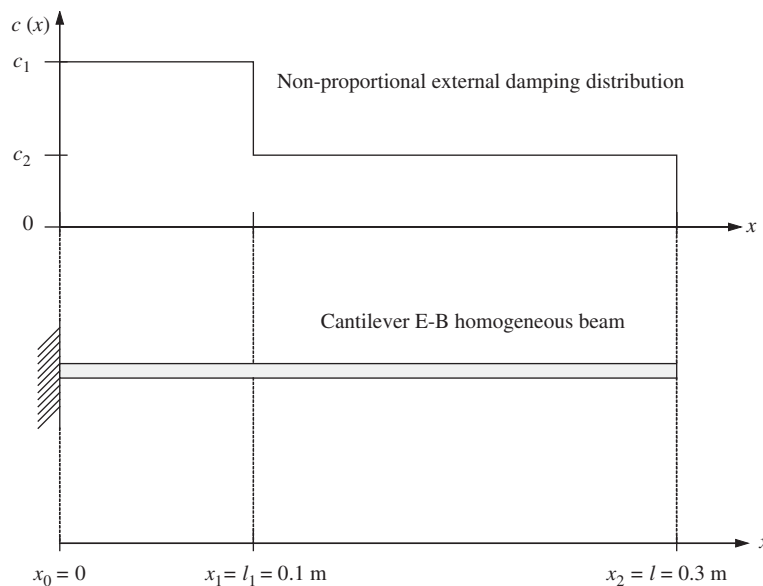


Fig. 1. Cantilever homogeneous beam with non-proportional external damping.

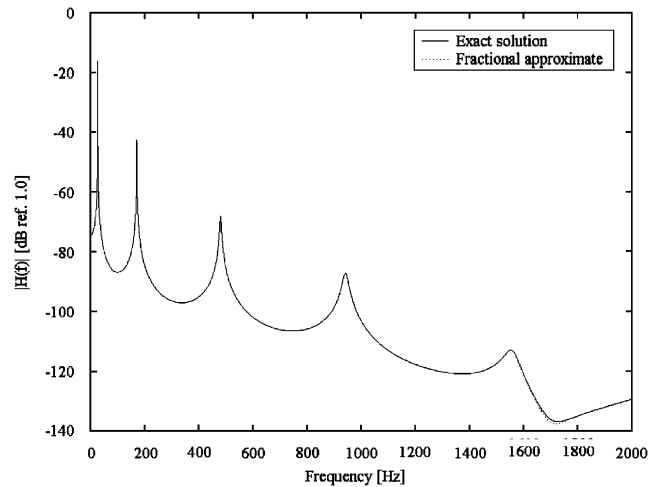


Fig. 2. Receptance modulus for proportional internal damping (approximate vs. exact solution).

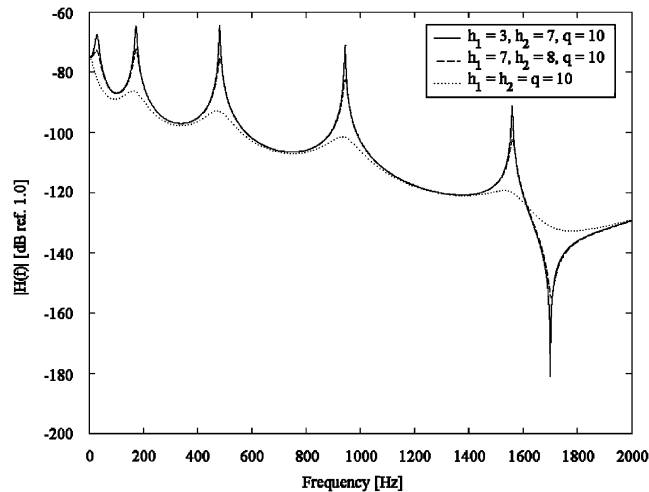


Fig. 3. Receptance modulus for different fractional derivative orders (FE approximate solution).

5. Conclusions

In Section 2 a technique has been presented for the modal analysis of linear vibrating systems with fractional derivative damping distributions. According to an expansion of the equations of motion analogous to a state-space representation, the orthogonality properties of the eigenfunctions have been defined and discussed, leading to the statement of the expansion theorem for fractionally damped systems, valid for both discrete and continuous systems.

The proposed method is of general validity, due to the assumption of generalized fractional damping distributions, the adoption of an operator notation and the possibility to take into account fractional viscoelastic models other than the Fractional Kelvin–Voigt one.

In Section 3, a methodology for reducing the computational effort due to finite element analysis has been presented. It consists in replacing the main eigenproblem of large dimension with two related standard eigenproblems of lower size. The choice of the most suitable procedure for the solution depends on the damping distribution properties:

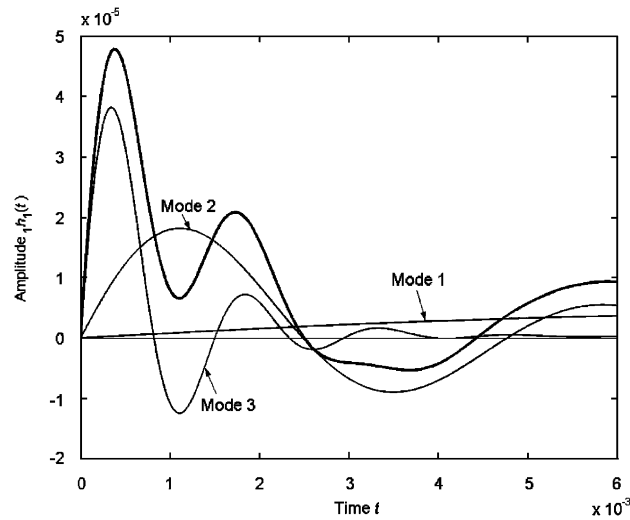


Fig. 4. Impulse response function for $\alpha = 0.5$ (contribution of first 3 modes).

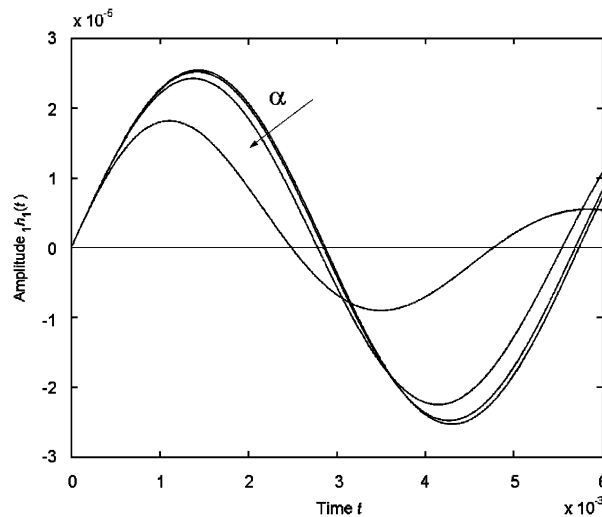


Fig. 5. Impulse response function for $\alpha = 1/16-1/8-1/4-1/2$ (contribution of the second mode only).

Case 1: ‘Proportional’ fractional damping. The solution can be expanded on the basis of the undamped system eigenfunctions, reducing the problem to the solution of N single dof systems, N being the number of modes taken into account in the expansion. The computational effort for the time domain responses, however, is in any case a little higher with respect to the case of viscous damping, due to the presence of Γ functions in the series defining the response.

Case 2: Generalized fractional damping. If the dimensions of the eigenproblem following the fractional state-space expansion are not too high, the standard procedure can be adopted. Otherwise, the technique presented and described in Section 3 can reduce significantly the computational effort without carrying relevant losses of accuracy.

Both the accuracy and efficiency of the proposed methodology have been verified in Section 4 by means of numerical examples regarding Euler–Bernoulli beams.

Appendix

According to the Riemann–Liouville theory [15], an integral of non-integer order $\beta > 0$ of $x(t)$ can be defined by means of the following integral transform:

$$I_{t_0}^{(\beta)}[x(t)] = \frac{1}{\Gamma(\beta)} \int_{t_0}^t x(\tau)(t - \tau)^{\beta-1} d\tau, \quad (\text{A1})$$

valid for $t > t_0$, where Γ is the gamma function and t_0 is conventionally settled to 0 if $x(t)$ represents the response due to a causal system.

Since the integral in Eq. (A1) generally does not converge for $\beta < 0$, a fractional derivative of order $\alpha \in (0, 1)$ of $x(t)$ can be obtained taking the first derivative of the integral of order $1 - \alpha$, that is

$$\frac{d^\alpha}{dt^\alpha} x(t) = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_0^t (t - \tau)^{-\alpha} x(\tau) d\tau, \quad (\text{A2})$$

which is known as the Riemann–Liouville definition of fractional derivative of order $\alpha \in (0, 1)$, for $t > 0$. Taking the n th derivative of the integral in Eq. (A2), with n positive integer, yields the fractional derivative of order $n + \alpha$.

Following the same approach, another possibility exists of defining a fractional derivative of order $\alpha \in (0, 1)$. It consists in taking the integral of order $1 - \alpha$ of the first derivative of $x(t)$, that is

$$\frac{d^\alpha}{dt^\alpha} x(t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - \tau)^{-\alpha} \left[\frac{d}{d\tau} x(\tau) \right] d\tau, \quad (\text{A3})$$

which represents the dual of the definition (A2), and which is known as the Caputo definition of fractional derivative of order $\alpha \in (0, 1)$, for $t > 0$. The two definitions (A2) and (A3) are not equivalent since the index law does not apply for fractional derivatives [15].

A fractional derivative being an integer order derivative of an integral function, or vice-versa, it results to be linear and time-invariant [5].

The Fourier and Laplace transforms of non-integer order integrals and derivatives can be obtained by means of rules similar to those that hold for integer order operators [15]:

$$F \left[\frac{d^\alpha}{dt^\alpha} x(t) \right] = (i\omega)^\alpha F[x(t)] = (i\omega)^\alpha X(\omega) \quad (\text{A4})$$

as regards the Fourier transform and:

$$L \left[\frac{d^\alpha}{dt^\alpha} x(t) \right] = s^\alpha L[x(t)] = s^\alpha X(s) \quad (\text{A5})$$

as regards the bilateral Laplace transform, both valid for every real value of α and for both the Riemann–Liouville and Caputo definitions.

The study of fractional differential systems carries the replacement of the traditional trigonometric and exponential functions with higher transcendent functions. In this paper the function E_t has been adopted, which is the integral of order ν of the exponential function [15]:

$$E_t(\nu, a) = \frac{1}{\Gamma(\nu)} \int_0^t \xi^{\nu-1} e^{a(t-\xi)} d\xi \quad \text{Re}[\nu] > 0 \quad \text{or} \quad E_t(\nu, a) = t^\nu \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(\nu + k + 1)}. \quad (\text{A6})$$

In place of E_t , other higher transcendent function may be adopted, in particular the Mittag–Leffler function [15].

The Laplace transform of the function E_t can be determined by applying the convolution theorem to its integral representation in Eq. (A6), yielding:

$$L[E_t(\nu, a)] = \frac{1}{\Gamma(\nu)} L[t^{\nu-1}] L[e^{at}] = \frac{1}{s^\nu (s - a)} \quad \text{with } \text{Re}[\nu] > -1, \quad (\text{A7})$$

where the existence condition $\text{Re}[\nu] > 0$ can be extended to $\text{Re}[\nu] > -1$ [15].

It is now possible finding the inverse Laplace transform of the basic component of the transfer function of the problems under study, i.e.:

$$X(s) = \frac{1}{s^\mu - a}, \quad (\text{A8})$$

where, $\mu = 1/q$, with q positive integer. By means of the partial factor decomposition of the right-hand side term in Eq. (A8):

$$\frac{1}{s^\mu - a} = \sum_{k=1}^q \frac{a^{k-1}}{s^{k\mu-1}(s - a^q)}, \quad (\text{A9})$$

the inverse transform of $X(s)$ follows immediately from Eq. (A7):

$$L^{-1} \left[\frac{1}{s^\mu - a} \right] = \sum_{k=1}^q a^{k-1} E_t(k\mu - 1, a^q). \quad (\text{A10})$$

The case of coincident eigenvectors carries the problem of the calculation of the inverse Laplace transform of powers of the right-hand side term in Eq. (A8). But it can be handled according to the same technique, using different partial factor decompositions. The final result will again be a combination of E_t functions [15].

References

- [1] P.G. Nutting, A new general law of deformation, *Journal of the Franklin Institute* 191 (1921) 679–685.
- [2] A. Gemant, A method of analyzing experimental results obtained from elasto-viscous bodies, *Physics* 7 (1936) 311–317.
- [3] M. Caputo, F. Mainardi, Linear models of dissipation in anelastic solids, *Rivista del Nuovo Cimento* 1 (1971) 161–198.
- [4] H. Beyer, S. Kempfle, Definition of physically consistent damping laws with fractional derivatives, *Zeitschrift für Angewandte Mathematic und Mechanic* 75 (1995) 623–635.
- [5] D. Frammartino, Modelli analitici evoluti per lo studio di sistemi smorzati. Degree Thesis, Politecnico di Torino, 2000 (in Italian).
- [6] L. Debnath, Recent applications of fractional calculus to science and engineering, *International Journal of Mathematics and Mathematical Sciences* 54 (2003) 3413–3442.
- [7] J. Padovan, Computational algorithms for FE formulations involving fractional operators, *Computational Mechanics* 2 (1987) 271–287.
- [8] A. Schmidt, L. Gaul, FE implementation of viscoelastic constitutive stress–strain relations involving fractional time derivatives, in: D. Bresdo, R.H. Schuster, J. Ihlemann (Eds.), *Constitutive Models for Rubbers II*, Balkema, Rotterdam, 2001.
- [9] M. Enelund, L. Josefson, Time-domain finite element analysis of viscoelastic structures with fractional derivative constitutive relations, *American Institute of Aeronautics and Astronautics Journal* 35 (1999) 1630–1637.
- [10] A. Schmidt, L. Gaul, Application of fractional calculus to viscoelastically damped structures in the finite element method. *Proceedings of the International Conference on Structural Dynamics Modelling*, Madeira, Portugal, 2002, pp. 297–306.
- [11] A. Fenander, Modal synthesis when modeling damping by use of fractional derivatives, *American Institute of Aeronautics and Astronautics Journal* 34 (1996) 1051–1054.
- [12] R.L. Bagley, P.J. Torvik, Fractional calculus—a different approach to the analysis of viscoelastically damped structures, *American Institute of Aeronautics and Astronautics Journal* 21 (1983) 741–748.
- [13] L. Meirovitch, *Principles and Techniques of Vibrations*, Prentice Hall, New Jersey, 1997.
- [14] F. Mainardi, Fractional calculus: some basic problems in continuum and statistical mechanics, in: A. Carpinteri, F. Mainardi (Eds.), *Fractals and Fractional Calculus in Continuum Mechanics*, Springer, New York, 1997.
- [15] K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.
- [16] S. Sorrentino, L. Garibaldi, Modal analysis of continuous systems with fractional derivative damping distributions. *Proceedings of the International Seminar on Modal Analysis*, Leuven, Belgium, September 2004, pp. 1133–1147.
- [17] K. Foss, Co-ordinates which uncouple the equations of motions of damped linear dynamic systems, *Journal of Applied Mechanics* 25 (1958) 361–364.
- [18] S. Sorrentino, A. Fasana, S. Marchesiello, Frequency domain analysis of continuous systems with viscous generalized damping, *Shock and Vibration* 11 (3–4) (2004) 243–259.
- [19] J. Mason, *Methods of Functional Analysis for Application in Solid Mechanics*, Elsevier, New York, 1985.
- [20] L. Meirovitch, *Dynamics and Control of structures*, Wiley, New York, 1990.
- [21] S. Sorrentino, S. Marchesiello, B.A.D. Piombo, A new analytical technique for vibration analysis of non-proportionally damped beams, *Journal of Sound and Vibration* 265 (2003) 765–782.
- [22] R.D. Cook, *Concepts and Applications of Finite Element Analysis*, Wiley, New York, 1981.
- [23] K.J. Bathe, *Finite Element Procedures in Engineering Analysis*, Prentice Hall, New Jersey, 1982.