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Exact solutions for vibration of stepped circular cylindrical shells

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Abstract

Based on the Flügge thin shell theory, this paper presents exact solutions for the vibration of circular cylindrical shells with step-wise thickness variations in the axial direction. The shell is sub-divided into multiple segments at the locations of thickness variations. The state-space technique is adopted to derive the homogenous differential equations for a shell segment and domain decomposition method is employed to impose the equilibrium and compatibility requirements along the interfaces of the shell segments. To ensure the correctness of the present results, comparisons are made with one paper available in the open literature based on the Donnell–Mushtari theory. Shells with various combinations of end boundary conditions can be analyzed by the proposed method. Furthermore, the influences of the shell thickness ratios, locations of step-wise thickness variations and step thickness ratios on the natural frequencies and mode shapes are examined. The exact vibration results can serve as important benchmark values for researchers to validate their numerical methods for such circular cylindrical shells.

1. Introduction

Circular cylindrical shells are widely used in many fields of engineering, especially in civil, mechanical, aerospace and marine engineering. Vibration of circular cylindrical shells is of interest of a number of different fields and has been extensively studied by many researchers. However, only a handful of references available in the published literature address the effect of the thickness variations on the vibration behavior of shells. Investigation has been made into different forms for analyzing the vibration of the cylindrical shells with variable thickness:

(1) axial thickness variation [1–6];
(2) circumferential thickness variation [7–8];
(3) thickness variation in the direction of generator [9–10]; and
(4) step-wise thickness variation [11–14].

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Most of the studies on the vibration of circular cylindrical shells of non-uniform thickness are for shells with axial direction thickness variations. To the authors' knowledge, there are no exact solutions available in the open literature for vibration of stepped circular cylindrical shells. The main purpose of the present paper is to develop an analytical method for vibration of stepped circular cylindrical shells and to present exact vibration frequencies for such shells under different combinations of boundary conditions. The Flügge shell theory is employed in this study. The state-space technique is adopted to derive the homogenous differential equation system. Comparison studies are carried out to verify the correctness of the proposed method with published results [11,15]. The effect of different step thickness ratios, step locations and the length to radius ratios on frequency parameters and mode shapes of the circular cylindrical shells is highlighted in this study. The results are presented in tabular and graphical forms for easy reference by researchers and engineers.

2. Formulations

Consider an isotropic, circular cylindrical shell with length L. The shell is of three steps with lengths L_1, L_2 and L_3, step thicknesses h_1, h_2 and h_3, midsurface radius R, Young's modulus E, Poisson's ratio υ and mass density ρ as shown in Fig. 1. The displacement fields of the open shell with reference to the coordinate system are denoted by u, v and w in the x, θ and radial directions, respectively.

An analytical method based on the state-space technique has been developed by Xiang et al. [17] to study the vibration of circular cylindrical shells with intermediate ring supports. The Goldenveizer–Novozhilov shell theory was employed in their study [17]. The same approach is employed in this study to determine the vibration frequencies of Flügge shells with step-wise thickness variations. For integrity and convenience, the analytical method is briefly presented in this section.

2.1. Governing differential equations

The shell can be divided into q ring segments. The symbol q denotes the number of total ring segments separated from the whole cylindrical shell at the locations of thickness variations. For the ith ring segment, the governing differential equations based on the Flügge shell theory can be expressed as [16]

$$\frac{\partial^2 u_i}{\partial x^2} + \frac{(1-\upsilon)}{2R^2}\frac{\partial^2 u_i}{\partial \theta^2} + \frac{(1+\upsilon)}{2R}\frac{\partial^2 v_i}{\partial x\,\partial \theta} + \frac{\upsilon}{R}\frac{\partial w_i}{\partial x} + k_i\left[\frac{(1-\upsilon)}{2R^2}\frac{\partial^2 u_i}{\partial \theta^2} - R\frac{\partial^3 w_i}{\partial x^3} + \frac{(1-\upsilon)}{2R}\frac{\partial^3 w_i}{\partial x\,\partial^2 \theta}\right] = \rho\frac{(1-\upsilon^2)}{E}\frac{\partial^2 u_i}{\partial t^2}, \tag{1}$$

$$\frac{(1+\upsilon)}{2R}\frac{\partial^2 u_i}{\partial x\,\partial \theta} + \frac{(1-\upsilon)}{2}\frac{\partial^2 v_i}{\partial x^2} + \frac{1}{R^2}\frac{\partial^2 v_i}{\partial \theta^2} + \frac{1}{R^2}\frac{\partial w_i}{\partial \theta} + k_i\left[\frac{3(1-\upsilon)}{2}\frac{\partial^2 v_i}{\partial x^2} - \frac{(3-\upsilon)}{2}\frac{\partial^3 w_i}{\partial x^2\,\partial \theta}\right] = \rho\frac{(1-\upsilon^2)}{E}\frac{\partial^2 v_i}{\partial t^2}, \tag{2}$$

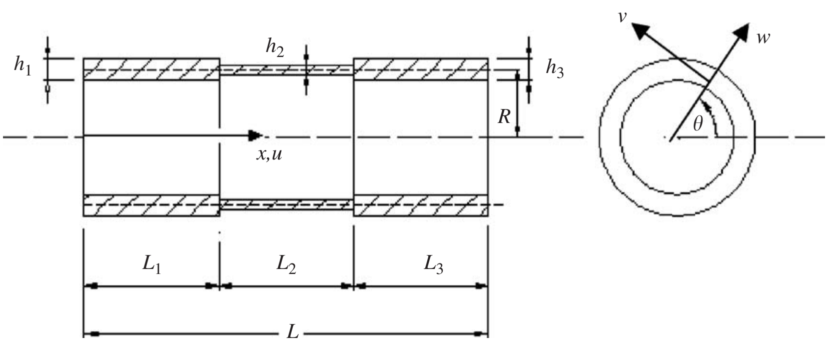

Fig. 1. Geometry and coordinate for a circular cylindrical shell with two-step thickness variations.

$$\frac{\upsilon}{R}\frac{\partial u_i}{\partial x}+\frac{1}{R^2}\frac{\partial v_i}{\partial \theta}+\frac{1}{R^2}w_i+k_i\left[R^2\frac{\partial^4 w_i}{\partial x^4}+2\frac{\partial^4 w_i}{\partial x^2\partial\theta^2}+\frac{1}{R^2}\frac{\partial^4 w_i}{\partial\theta^4}-R\frac{\partial^3 u_i}{\partial x^3}\right.$$
$$\left.+\frac{(1-\upsilon)}{2R}\frac{\partial^3 u_i}{\partial x\partial\theta^2}-\frac{(3-\upsilon)}{2}\frac{\partial^3 v_i}{\partial x^2\partial\theta}+\frac{1}{R^2}w_i+\frac{2}{R^2}\frac{\partial^2 w_i}{\partial\theta^2}\right]=-\rho\frac{(1-\upsilon^2)}{E}\frac{\partial^2 w_i}{\partial t^2} \tag{3}$$

in which $u_i(x,\theta,t)$, $v_i(x,\theta,t)$ and $w_i(x,\theta,t)$ are the displacements of the ith ring segment in the x, θ and radial directions, t is the time and $k_i=h_i^2/(12R^2)$.

2.2. Solutions for the ith ring segment

The displacement fields for the ith ring segment may be expressed as

$$u_i(x,\theta,t)=U_i(x)\cos m\theta\cos\omega t, \tag{4}$$

$$v_i(x,\theta,t)=V_i(x)\sin m\theta\cos\omega t, \tag{5}$$

$$w_i(x,\theta,t)=W_i(x)\cos m\theta\cos\omega t, \tag{6}$$

where the subscript $i\ (=1,2,3,\ldots,q)$ denotes the ith ring segment of the shell, $2m$ ($m=0,1,2,\ldots,\infty$) is the number of half-waves in the circumferential direction of the vibration mode, ω is the angular frequency of vibration, and $U_i(x)$, $V_i(x)$ and $W_i(x)$ are unknown functions to be determined. Note that $m=0$ corresponds to the axisymmetric vibration mode for the shell. We restrict our study in this paper to $m>0$.

Using the state-space technique, a homogenous differential equation system for the ith ring segment can be derived from Eqs. (1) to (3) and (4) to (6) after appropriate algebraic operations:

$$\boldsymbol{\psi}_i'-\mathbf{H}_i\boldsymbol{\psi}_i=\mathbf{0} \tag{7}$$

in which

$$\boldsymbol{\Psi}_i=\begin{bmatrix}U_i & U_i' & V_i & V_i' & W_i & W_i' & W_i'' & W_i'''\end{bmatrix}^{\mathrm{T}}, \tag{8}$$

the prime (′) denotes the derivative with respect to x, and $\mathbf{H}_i$ is an 8×8 matrix with the following non-zero elements:

$$(H_i)_{12}=(H_i)_{34}=(H_i)_{56}=(H_i)_{67}=(H_i)_{78}=1, \tag{9}$$

$$(H_i)_{21}=\frac{m^2(1-\upsilon)(1+k_i)}{2R^2}-\frac{\rho\left(1-\upsilon^2\right)\omega^2}{E}, \tag{10}$$

$$(H_i)_{24}=-\frac{(1+\upsilon)m}{2R}, \tag{11}$$

$$(H_i)_{26}=-\frac{\upsilon}{R}+\frac{(1-\upsilon)}{2R}k_i m^2, \tag{12}$$

$$(H_i)_{28}=k_i R, \tag{13}$$

$$(H_i)_{42}=\frac{(1+\upsilon)m}{R(1-\upsilon)(1+3k_i)}, \tag{14}$$

$$(H_i)_{43}=\frac{2m^2}{R^2(1-\upsilon)(1+3k_i)}-\frac{2\rho\left(1-\upsilon^2\right)\omega^2}{E(1-\upsilon)(1+3k_i)}, \tag{15}$$

$$(H_i)_{45}=\frac{2m}{R^2(1-\upsilon)(1+3k_i)}, \tag{16}$$

$$(H_i)_{47} = \frac{-mk_i(3-\upsilon)}{(1-\upsilon)(1+3k_i)}, \tag{17}$$

$$\begin{aligned}(H_i)_{82} = & -\frac{\upsilon}{k_iR^3(1-k_i)} + \frac{m^2(1-\upsilon)}{R^3(1-k_i)} + \frac{k_im^2(1-\upsilon)}{2R^3(1-k_i)} \\ & -\frac{\rho(1-\upsilon^2)\omega^2}{ER(1-k_i)} + \frac{m^2(1+\upsilon)}{R^3(1-k_i)(1+3k_i)},\end{aligned} \tag{18}$$

$$(H_i)_{83} = -\frac{m}{k_iR^4(1-k_i)} + \frac{2m^3}{R^4(1-k_i)(1+3k_i)} - \frac{2m\rho(1-\upsilon^2)\omega^2}{ER^2(1-k_i)(1+3k_i)}, \tag{19}$$

$$\begin{aligned}(H_i)_{85} = & -\frac{1}{k_iR^4(1-k_i)} - \frac{m^4}{R^4(1-k_i)} + \frac{2m^2}{R^4(1-k_i)} \\ & -\frac{1}{R^4(1-k_i)} + \frac{\rho\omega^2(1-\upsilon^2)}{k_iR^2E(1-k_i)} + \frac{2m^2}{R^4(1-k_i)(1+3k_i)},\end{aligned} \tag{20}$$

$$(H_i)_{87} = \frac{2m^2}{R^2(1-k_i)} - \frac{\upsilon}{R^2(1-k_i)} + \frac{k_im^2(1-\upsilon)}{2R^2(1-k_i)} - \frac{k_im^2(3-\upsilon)}{R^2(1-k_i)(1+3k_i)}. \tag{21}$$

The procedure for solving Eq. (7) has been detailed by Xiang et al. [18] and Liew et al. [19]. The solution for Eq. (7) can be expressed as

$$\boldsymbol{\psi}_i = \mathbf{e}^{\mathbf{H}_i x}\mathbf{c}_i, \tag{22}$$

where $\mathbf{e}^{\mathbf{H}_i x}$ is a general matrix solution of Eq. (7), $\mathbf{c}_i$ is an 8×1 constant column matrix that is to be determined using the boundary conditions and/or interface conditions between the shell ring segments.

2.3. Boundary and interface conditions

It is well known that there are 4 simply supported and 4 clamped boundary conditions associated with a circular cylindrical shell [16]. Although we can obtain exact solutions for circular cylindrical shells with various combinations of end support conditions, in this paper three typical boundary conditions are considered:

(1) Simply supported or shear diaphragms (*S*):

$$w_i = (M_x)_i = (N_x)_i = v_i = 0. \tag{23}$$

(2) Free (*F*):

$$(N_x)_i = (N_{x\theta})_i + \frac{(M_{x\theta})_i}{R} = (Q_x)_i + \frac{1}{R}\frac{\partial(M_{x\theta})_i}{\partial\theta} = (M_x)_i = 0. \tag{24}$$

(3) Clamped (*C*):

$$u_i = v_i = w_i = \frac{\partial w_i}{\partial x} = 0, \tag{25}$$

where i takes the value 1 or q, and the force and moment resultants based on the Flügge shell theory are given by [16]

$$N_x = \frac{Eh}{(1-\upsilon^2)}\left[\varepsilon_x + \upsilon\varepsilon_\theta + \frac{h^2}{12R}\kappa_x\right], \tag{26}$$

$$N_\theta = \frac{Eh}{(1-\upsilon^2)}\left[\varepsilon_\theta + \upsilon\varepsilon_x - \frac{h^2}{12R}\left(\kappa_\theta - \frac{1}{R}\varepsilon_\theta\right)\right], \tag{27}$$

$$N_{x\theta} = \frac{Eh}{2(1+\upsilon)}\left(\varepsilon_{x\theta} + \frac{h^2}{24R}\tau\right), \tag{28}$$

$$M_x = \frac{Eh^3}{12(1-\upsilon^2)}\left(\kappa_x + \upsilon\kappa_\theta + \frac{1}{R}\varepsilon_x\right), \tag{29}$$

$$M_{x\theta} = \frac{Eh^3}{24(1+\upsilon)}\tau, \tag{30}$$

$$M_{\theta x} = \frac{Eh^3}{24(1+\upsilon)}\left(\tau - \frac{1}{R}\varepsilon_{x\theta}\right), \tag{31}$$

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{1}{R}\frac{\partial M_{\theta x}}{\partial \theta} \tag{32}$$

and the strain, curvature and twist of middle surface terms are related to displacement fields by [16]

$$\varepsilon_x = \frac{\partial u}{\partial x}, \tag{33}$$

$$\varepsilon_\theta = \frac{1}{R}\left(\frac{\partial v}{\partial \theta} + w\right), \tag{34}$$

$$\varepsilon_{x\theta} = \frac{1}{R}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x}, \tag{35}$$

$$\kappa_x = -\frac{\partial^2 w}{\partial x^2}, \tag{36}$$

$$\kappa_\theta = \frac{1}{R^2}\left(\frac{\partial v}{\partial \theta} - \frac{\partial^2 w}{\partial \theta^2}\right), \tag{37}$$

$$\tau = -\frac{2}{R}\left(\frac{\partial^2 w}{\partial x\,\partial \theta} - \frac{\partial v}{\partial x}\right). \tag{38}$$

To ensure the continuity along the interface between the ith and the $(i+1)$th ring segments, the following essential and natural continuity conditions must be satisfied:

$$w_i = w_{i+1}, \tag{39}$$

$$u_i = u_{i+1}, \tag{40}$$

Table 1
Comparison of frequency parameters λ for a simply supported cylindrical shell ($\upsilon = 0.3$, $L/R = 20$)

m	$h/R = 0.05$		$h/R = 0.002$	
	Markuš [15]	Present	Markuš [15]	Present
1	0.0161063	0.0161065	0.0161011	0.0161011
2	0.0392332	0.0393038	0.00545243	0.00545325
3	0.109477	0.109853	0.00503724	0.00504188
4	0.209008	0.210345	0.00853409	0.00853408

Table 2
Comparison of frequency parameters $\lambda = \omega R^2 \sqrt{\rho(1-\upsilon)^2/E}$ for cylindrical shells with two thickness variations ($L/R = 1$, $h_1/R = 0.01$, $h_2/h_1 = 3$, $h_3/h_1 = 1$)

	Boundary conditions	Sources	Mode sequence (n)											
			1	2	3	4	5	6	7	8	9	10	11	12
$m = 2$	FF	Zhou and Yang [11]	0.00216	0.00381	0.08557	0.09953	0.10482	0.10737	0.11803	0.15312	0.16913	0.21901	0.24046	0.26512
		Present	0.0021880	0.0033280	0.0855840	0.0995220	0.1048270	0.1073720	0.1180420	0.1532200	0.1691340	0.2190960	0.2404550	0.2651260
$m = 5$	SS	Zhou and Yang [11]	0.02944	0.07115	0.08471	0.11111	0.13660	0.15720	0.21960	0.24255	0.28263	0.33358	0.37058	0.40638
		Present	0.0295000	0.0710958	0.0847365	0.1110633	0.1366046	0.1572884	0.2196003	0.2425447	0.2826692	0.3335553	0.3706131	0.4053841
$m = 5$	CC	Zhou and Yang [11]	0.03343	0.07447	0.09046	0.11869	0.15241	0.17342	0.24489	0.26996	0.32390	0.36426	0.41544	0.43032
		Present	0.0334945	0.0747033	0.0904934	0.1186532	0.1524124	0.1734125	0.2448918	0.2699563	0.3239359	0.3642384	0.4081849	0.4909856

$$v_i = v_{i+1}, \tag{41}$$

$$\frac{\partial w_i}{\partial x} = \frac{\partial w_{i+1}}{\partial x}, \tag{42}$$

$$(M_x)_i = (M_x)_{i+1}, \tag{43}$$

$$(N_x)_i = (N_x)_{i+1}, \tag{44}$$

$$\left(N_{x\theta} + \frac{M_{x\theta}}{R}\right)_i = \left(N_{x\theta} + \frac{M_{x\theta}}{R}\right)_{i+1}, \tag{45}$$

$$\left(Q_x + \frac{1}{R}\frac{\partial M_{x\theta}}{\partial \theta}\right)_i = \left(Q_x + \frac{1}{R}\frac{\partial M_{x\theta}}{\partial \theta}\right)_{i+1}. \tag{46}$$

2.4. Assembling ring segments to shell

In view of Eq. (22), a homogeneous system of equations can be derived by implementing the boundary conditions of the shell [see Eqs. (23)–(25)] and the interface conditions between two ring segments [Eqs. (33)–(46)] when assembling the segments to form the whole shell. We have

$$\mathbf{Kc} = \mathbf{0}, \tag{47}$$

Table 3
Frequency parameters λ for SS shells with one-step thickness variation ($h_1/R = 0.01$, $h_2/h_1 = 0.5$, $L_1/L = 1/2$)

L/R		m							
	n	1	2	3	4	5	6	7	8
1	1	0.549340	0.6224352	0.461341	0.342826	0.264939	0.2204481	0.203730	0.208690
	2	0.852379	0.8663417	0.7771635	0.682271	0.594148	0.5196507	0.462070	0.422981
	3	0.931840	0.9299856	0.8873969	0.835280	0.779992	0.7270372	0.6807859	0.644406
	4	0.958827	0.9826149	0.9593823	0.930105	0.897440	0.8641631	0.8329965	0.806449
	5	0.998036	1.0657842	1.0552058	1.042250	1.028176	1.0143573	1.0022124	0.993095
	6	1.072854	1.1543279	1.1667493	1.161383	1.155950	1.1514385	1.1489364	1.149584
	7	1.174256	1.1720124	1.3678038	1.368910	1.371115	1.3748945	1.3807329	1.389112
	8	1.367307	1.367410	1.5636963	1.570130	1.578938	1.590495	1.6052027	1.623470
5	1	0.176590	0.0729781	0.0411366	0.0415354	0.0537910	0.0643111	0.0779291	0.0964449
	2	0.459537	0.241848	0.137338	0.0914285	0.0823614	0.100462	0.116863	0.126808
	3	0.581714	0.404736	0.253553	0.170766	0.131015	0.122451	0.143158	0.174006
	4	0.670022	0.549270	0.377266	0.266358	0.200257	0.168358	0.166487	0.188167
	5	0.782352	0.650020	0.479877	0.356095	0.274027	0.225921	0.206112	0.212107
	6	0.806931	0.726442	0.570134	0.442843	0.349437	0.287006	0.252786	0.245482
	7	0.845214	0.778058	0.639504	0.516584	0.420041	0.351087	0.308017	0.288069
	8	0.880104	0.816454	0.696293	0.581186	0.484095	0.409330	0.357921	0.330692
10	1	0.056537	0.020741	0.0205316	0.0291277	0.0381839	0.051992	0.069883	0.091162
	2	0.187773	0.077268	0.0425071	0.0443353	0.057613	0.064715	0.077344	0.095833
	3	0.322085	0.151429	0.0822348	0.0600841	0.0701742	0.089500	0.0955837	0.107743
	4	0.459404	0.240994	0.1365716	0.0909894	0.0819812	0.101259	0.122619	0.128460
	5	0.560665	0.324886	0.1936166	0.1284957	0.1030313	0.108994	0.1383343	0.156020
	6	0.599689	0.409942	0.2571894	0.1728271	0.1315572	0.123198	0.1430132	0.180775
	7	0.658756	0.481711	0.3163429	0.2177874	0.1643209	0.143850	0.152024	0.183084
	8	0.663906	0.549150	0.3768955	0.2658855	0.199826	0.168163	0.1658968	0.188585

where **K** is an $8q \times 8q$ matrix and **c** is an $8q \times 1$ vector. The angular frequency ω is evaluated by setting the determinant of **K** in Eq. (47) to zero i.e., solving the eigenvalue problem.

3. Results and discussions

The proposed analytical method is applied to evaluate exact vibration frequencies for circular cylindrical shells having various combinations of edge support conditions and a given number of thickness variations. For convenience, a two-letter symbol is used to describe the shell boundary conditions, i.e. the symbol SF denotes a shell having simply supported and free edge conditions at $x = 0$ and L, respectively. The vibration frequency ω is expressed in terms of a non-dimensionalized frequency parameter $\lambda = \omega R^2\sqrt{\rho(1-\upsilon^2)/E}$. The Poisson ratio υ takes the value 0.3 in this study.

3.1. Verification of solution method

The authors have difficulty to find vibration results for stepped Flügge circular cylindrical shells to compare with in the open literature. The vibration solutions obtained by Zhou and Yang [11] based on the Donnell–Mushtari shell theory, therefore, are used in this paper to confirm the correctness of the solution method. The differential equations and the corresponding **H** matrix based on the Donnell–Mushtari shell theory are given in Appendix.

Table 1 shows the comparison of frequency parameters of SS shells obtained from 3D solutions [15] and the present method without thickness variations. We observe that the present exact solutions based on Flügge shell theory are in close agreement with the exact 3D solutions [15]. The value of $2m$ in Table 1 represents the number of half-waves of a vibration mode in the circumferential direction.

Table 4
Frequency parameters λ for CC shells with one-step thickness variation ($h_1/R = 0.01$, $h_2/h_1 = 0.5$, $L_1/L = 1/2$)

L/R		m							
	n	1	2	3	4	5	6	7	8
1	1	0.851004	0.657065	0.5107107	0.4090616	0.339550	0.294249	0.269134	0.261650
	2	0.940764	0.874865	0.7862245	0.6943636	0.612273	0.545455	0.495197	0.461317
	3	0.973299	0.947067	0.9069743	0.8576169	0.805356	0.755997	0.713800	0.681439
	4	1.025416	1.011561	0.9897311	0.9620549	0.931274	0.900278	0.871804	0.848260
	5	1.128235	1.120170	1.1094318	1.0970959	1.084333	1.072302	1.062182	1.055110
	6	1.247009	1.244317	1.2403934	1.2359467	1.231801	1.228868	1.228131	1.230608
	7	1.472132	1.472034	1.4724456	1.4735819	1.475843	1.479691	1.485605	1.494053
	8	1.689655	1.692319	1.6967675	1.7031928	1.711953	1.723450	1.738083	1.756222
5	1	0.236938	0.124514	0.0735505	0.0572470	0.0624887	0.0724892	0.0837975	0.100156
	2	0.465215	0.270691	0.169777	0.117799	0.0979801	0.106698	0.124965	0.134955
	3	0.654823	0.415764	0.275089	0.194529	0.151438	0.136604	0.148301	0.179224
	4	0.782057	0.552225	0.388211	0.283104	0.218466	0.184255	0.177383	0.193191
	5	0.806531	0.651388	0.485537	0.367180	0.288468	0.240858	0.219084	0.220808
	6	0.850639	0.727157	0.572762	0.449354	0.359618	0.299219	0.264925	0.255656
	7	0.881919	0.779273	0.641570	0.521210	0.427795	0.361233	0.319091	0.298481
	8	0.901664	0.817512	0.697740	0.584146	0.489428	0.417029	0.367247	0.340463
10	1	0.0976584	0.0413561	0.0277287	0.0337158	0.0413595	0.0535962	0.0706732	0.0915757
	2	0.212690	0.102737	0.0581062	0.0494976	0.0624907	0.0692264	0.0801636	0.0974743
	3	0.333290	0.173652	0.100302	0.0712600	0.0730836	0.0943531	0.100375	0.111046
	4	0.462386	0.256602	0.153703	0.104351	0.0894228	0.102621	0.127564	0.133030
	5	0.567189	0.334890	0.208148	0.141944	0.112842	0.113425	0.138934	0.160840
	6	0.657675	0.415509	0.268314	0.185117	0.141950	0.129935	0.145399	0.181488
	7	0.665121	0.484806	0.324627	0.228486	0.174554	0.151733	0.156428	0.184906
	8	0.734091	0.550687	0.382595	0.274615	0.209234	0.176238	0.171445	0.191281

Table 2 presents the frequency parameters λ obtained by Zhou and Yang [11] and the present analytical method for a circular cylindrical shell with two thickness variations and FF, SS and CC boundary conditions. The dimensionless geometric and material parameters used in the calculations are given in Ref. [11] as $E = 100$, $\upsilon = 0.3$, $R = L = 100$ the thicknesses and lengths of the three segments $h_1 = 1$, $h_2 = 3$, $h_3 = 1$, $L_1 = 40$, $L_2 = 20$, $L_3 = 40$, respectively. It is seen that the exact vibration solutions from the proposed analytical method based on the Donnell–Mushtari shell theory are in close agreement with the solutions of the distributed transfer function method [11] except for ($m = 2$, $n = 2$ for FF shells) and ($m = 5$, $n = 12$ for CC shells) where a large discrepancy is observed. These differences could be due to typos in Zhou and Yang [11]. These comparisons confirm the correctness of the present analytical method.

3.2. Vibration and mode shapes of shells with one-step thickness variation

Tables 3–6 present the effect of end boundary conditions on frequency parameters λ of the first 8 modes for circular cylindrical shells with one thickness variation. Four different combinations of shell end support conditions are considered, i.e. SS, CC, CF and FF, respectively. The shell length to radius ratio L/R is set to be 1, 5 and 10, the thickness to radius ratio h_1/R is fixed at 0.01, the step thickness ratio h_2/h_1 is taken to be 0.5, and the location of the thickness variation is at the center of the shell. The value of $2m$ indicates circumferential half-wavenumbers and n denotes the mode sequence number for a given m value. The effect of end boundary conditions on the frequency parameters of the shell is observed. As expected, clamped–clamped end supports will lead to higher frequency parameters, which in turn have higher fundamental frequency parameter due to the larger stiffness resulting from restraining the axial displacement u at the end. An increase

Table 5
Frequency parameters λ for CS shells with one-step thickness variation ($h_1/R = 0.01$, $h_2/h_1 = 0.5$, $L_1/L = 1/2$)

L/R		m							
	n	1	2	3	4	5	6	7	8
1	1	0.839577	0.656477	0.504868	0.396243	0.321475	0.272933	0.246298	0.238393
	2	0.933956	0.869713	0.781849	0.688810	0.603571	0.532784	0.478959	0.442776
	3	0.965861	0.940464	0.900492	0.850694	0.797497	0.746771	0.702844	0.668472
	4	1.007454	0.993190	0.970633	0.942215	0.910706	0.878926	0.849588	0.825164
	5	1.100136	1.093081	1.082152	1.069022	1.055113	1.041658	1.029859	1.020881
	6	1.208588	1.205792	1.201609	1.196915	1.192614	1.189639	1.188980	1.191652
	7	1.410255	1.410090	1.410107	1.410673	1.412241	1.415315	1.420420	1.428088
	8	1.635987	1.638643	1.642884	1.649094	1.657629	1.668857	1.683138	1.700801
5	1	0.221195	0.105694	0.0593263	0.0476538	0.0541674	0.0643185	0.0779305	0.0964449
	2	0.461123	0.254706	0.152747	0.104139	0.0898911	0.102007	0.116863	0.126809
	3	0.639087	0.410565	0.265953	0.184197	0.142141	0.129721	0.145832	0.174035
	4	0.700747	0.550612	0.382209	0.274239	0.209084	0.176363	0.171911	0.190689
	5	0.782707	0.651206	0.483761	0.363153	0.282856	0.234678	0.213203	0.216739
	6	0.844817	0.726854	0.571377	0.445843	0.354255	0.293035	0.259240	0.251152
	7	0.879229	0.779090	0.641121	0.519881	0.425244	0.357559	0.314564	0.293608
	8	0.901525	0.817009	0.697108	0.582722	0.486764	0.413259	0.363031	0.336591
10	1	0.0824356	0.0319945	0.0228416	0.0291299	0.0381847	0.0519924	0.0698830	0.0911617
	2	0.198968	0.0893695	0.0498920	0.0464173	0.0576169	0.0647151	0.0773440	0.0958328
	3	0.327164	0.163758	0.0918323	0.0656422	0.0716962	0.0895010	0.0955837	0.107743
	4	0.460866	0.248201	0.144752	0.0974624	0.0854855	0.102022	0.122619	0.128460
	5	0.566788	0.330983	0.201955	0.135916	0.108174	0.111279	0.138670	0.156020
	6	0.619633	0.412268	0.262247	0.178630	0.136648	0.126468	0.144196	0.180831
	7	0.663673	0.483677	0.321334	0.223989	0.170037	0.147990	0.153993	0.183518
	8	0.708281	0.549796	0.379380	0.269888	0.204327	0.172336	0.168819	0.189701

Table 6
Frequency parameters λ for CF shells with one-step thickness variation ($h_1/R = 0.01$, $h_2/h_1 = 0.5$, $L_1/L = 1/2$)

L/R		m							
	n	1	2	3	4	5	6	7	8
1	1	0.637349	0.409444	0.275022	0.194503	0.147252	0.123524	0.118057	0.125949
	2	0.899615	0.766426	0.619980	0.499541	0.410750	0.349571	0.312104	0.295727
	3	0.948341	0.911952	0.852946	0.779737	0.703082	0.632319	0.573498	0.529699
	4	0.973919	0.954440	0.922660	0.881356	0.835896	0.791600	0.752693	0.722193
	5	1.026355	1.015653	0.999403	0.979354	0.956896	0.933598	0.911320	0.892034
	6	1.125975	1.120843	1.113008	1.103944	1.094922	1.086978	1.081108	1.078300
	7	1.247310	1.245776	1.243603	1.241456	1.240034	1.240078	1.242379	1.247766
	8	1.472043	1.473055	1.474768	1.477654	1.482132	1.488628	1.497559	1.509317
5	1	0.0978355	0.0377954	0.0223843	0.0257202	0.0364883	0.0511895	0.0693894	0.0907467
	2	0.281382	0.141542	0.0800276	0.0594417	0.0635691	0.0742545	0.0859105	0.102387
	3	0.542039	0.308232	0.185817	0.125096	0.101532	0.108724	0.129176	0.140232
	4	0.639836	0.447572	0.295884	0.206714	0.158686	0.141297	0.150311	0.182401
	5	0.747988	0.596853	0.422113	0.304876	0.232025	0.192665	0.182515	0.196523
	6	0.831442	0.674779	0.509023	0.385863	0.302028	0.250553	0.226419	0.225935
	7	0.853341	0.754079	0.603384	0.475842	0.380026	0.314148	0.275317	0.262638
	8	0.894071	0.790750	0.657965	0.538072	0.442711	0.373604	0.329488	0.307703
10	1	0.0294630	0.0108632	0.0129036	0.0217671	0.0342804	0.0499774	0.0686512	0.0902381
	2	0.109438	0.0440413	0.0281391	0.0338680	0.0417656	0.0540995	0.0712107	0.0921111
	3	0.237627	0.108905	0.0598498	0.0501035	0.0636196	0.0706284	0.0813663	0.0985602
	4	0.354481	0.184177	0.104243	0.0729533	0.0735858	0.0960644	0.102538	0.112900
	5	0.496950	0.275639	0.161474	0.107736	0.0909596	0.102990	0.130245	0.135874
	6	0.588074	0.352205	0.217207	0.146490	0.115296	0.114532	0.139113	0.164457
	7	0.633824	0.439996	0.282036	0.192324	0.145814	0.132066	0.146157	0.181716
	8	0.681861	0.501541	0.336617	0.235818	0.178946	0.154575	0.158208	0.185876

in the length to radius ratio L/R from 1 to 10 will lead to a decrease in the frequency parameters. The results reveal that the influence of end boundary conditions on the frequency parameters decreases as the ratio of L/R increases. It is also observed that as the mode sequence number n increases, the frequency parameters become less dependent upon the type of boundary conditions, i.e. for fixed n and m values, the difference among the frequency parameters for shells of various boundary conditions becomes smaller. The frequency parameters λ may increase or decrease as the number of circumferential half-waves $2m$ increases. Therefore, it is not certain that the lowest frequency parameter for each case in Tables 2–5 corresponds to the true fundamental mode of the case. The value of m at which the lowest frequency parameter occurs depends on the length to radius radio L/R and the step thickness ratio h_2/h_1.

The mode shapes corresponding to the first three frequency parameters (for $m = 1$) of CC and CF circular cylindrical shells with one step thickness variation are presented in Fig. 2. The step thickness ratio is set to be $h_2/h_1 = 0.5$ and 2, thickness ratio is taken as $h_1/R = 0.01$ and length to radius ratio is $L/R = 5$ and 10. Although the step thickness variation exists in the cylindrical shells, the modal shapes for the displacement fields u, v and w for all cases are smooth at the location of step thickness variation ($L_1/L = 0.5$). It is obvious that for the lower modes, the amplitude of the displacement in the x direction (u) is quit small compared to ones in the radial (w) and circumferential (v) directions. For higher modes, however the amplitude of the displacement u becomes more pronounced. From this point of view, any assumption that the longitudinal displacement can be neglected may lead to inaccuracy predictions for higher modes. As described in Eqs. (23)–(25), the displacements in x, radial and circumferential directions for free edges are unconstrained while all constrained for clamped edges. It is also observed that the amplitude of mode shapes increases slightly as the step thickness ratio h_2/h_1 increases from 0.5 to 2 for lower modes. However, this increment becomes more significant with higher modes.

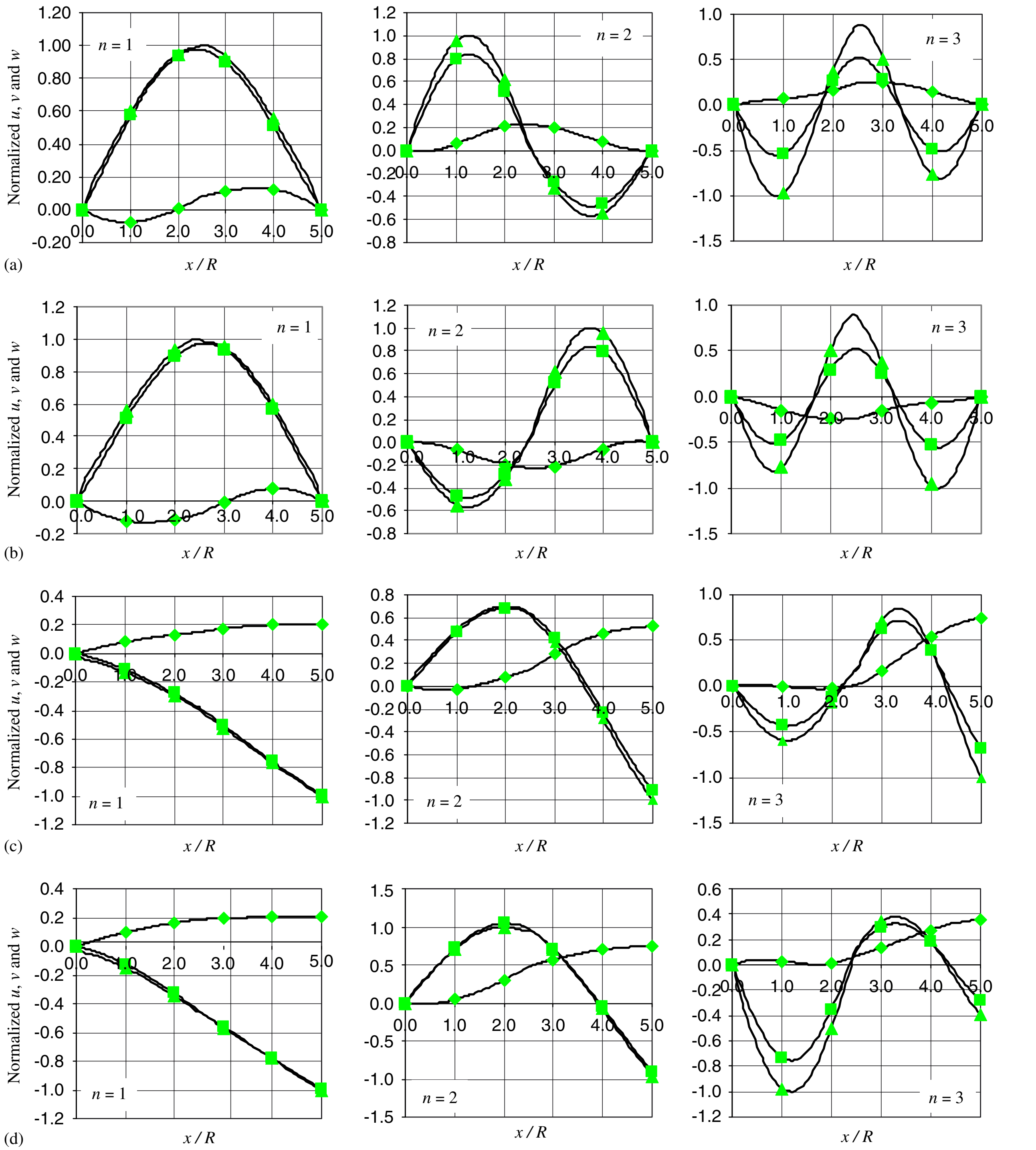

Fig. 2. The first three mode shapes of cylindrical shells with one-step thickness variation with $m = 1$. ▲ w , ■ v, ◆ u ($h_1/R = 0.01$, $L/R = 5$). (a) CC, $h_2/h_1 = 0.5$, (b) CC, $h_2/h_1 = 2$, (c) CF, $h_2/h_1 = 0.5$, (d) CF, $h_2/h_1 = 2$.

3.3. Vibration of shells with one thickness variation at various shell lengths

Figs. 3 and 4 show the variation of the fundamental frequency parameters λ against the shell length to radius ratio L/R for SS and CF shells, respectively. The step thickness to radius ratio $h_1/R = 0.01$, and thickness ratio $h_2/h_1 = 0.5$, 1 and 2 are considered for all shells in Figs. 2–4. For the SS shells, the step thickness variation is located at $L_1/L = 1/4$ and 1/2 and for the CF shell the location of the step thickness

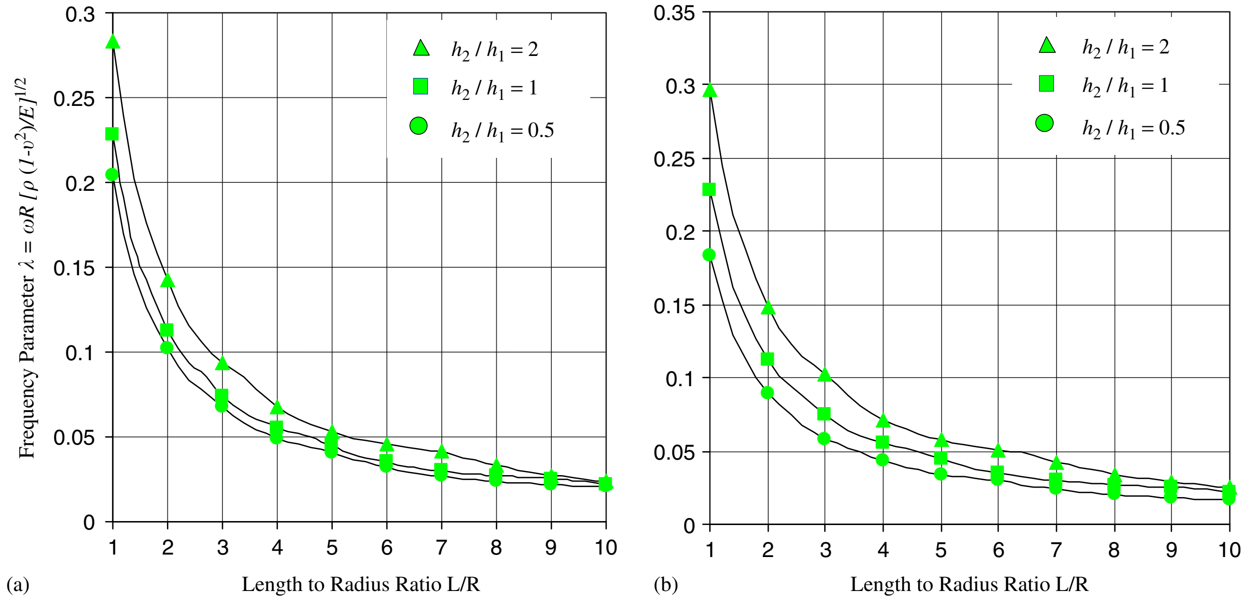

Fig. 3. Frequency parameters $\lambda = \omega R^2\sqrt{\rho(1-\upsilon^2)/E}$ versus the length to radius ratio L/R for SS circular cylindrical shells having one-step thickness variation, thickness ratio $h_1/R = 0.01$, step thickness ratio $h_2/h_1 = 0.5$, 1, 2 and shell thickness variation location $L_1/L = 0.5$ and 0.25. (a) $L_1/L_2 = 1/2$, (b) $L_1/L_2 = 1/4$.

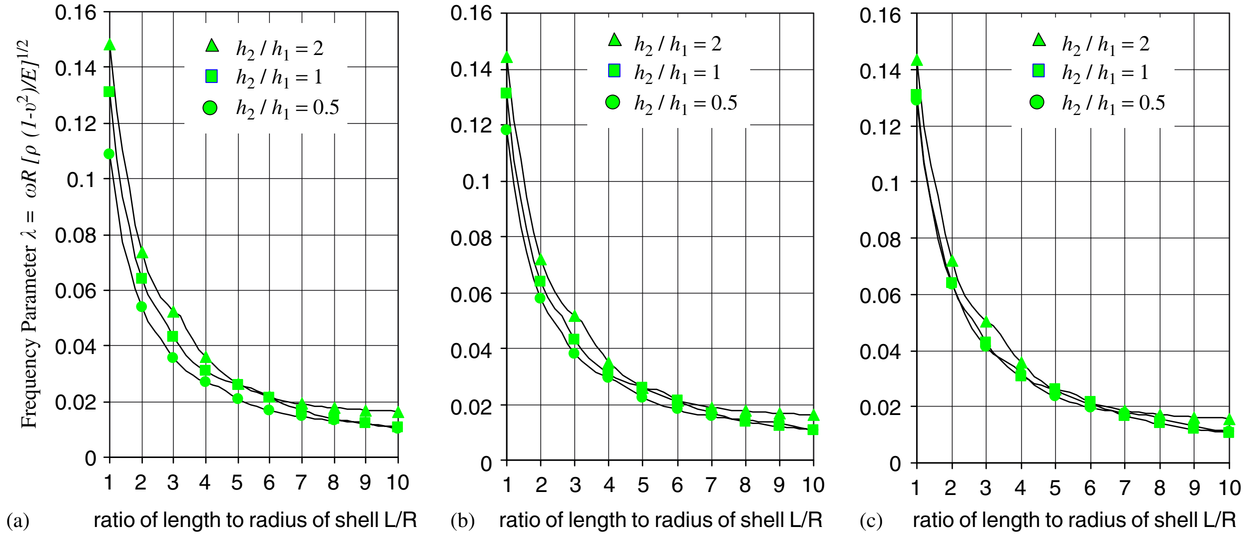

Fig. 4. Frequency parameters $\lambda = \omega R^2\sqrt{\rho(1-\upsilon^2)/E}$ versus the length to radius ratio L/R for CF circular cylindrical shells having one-step thickness variation, thickness ratio $h_1/R = 0.01$, step thickness ratio $h_2/h_1 = 0.5$, 1, 2 and shell thickness variation location $L_1/L = 1/3$, 1/2, 2/3. (a) $L_1/L_2 = 1/3$, (b) $L_1/L_2 = 1/2$, (c) $L_1/L_2 = 2/3$.

variation L_1/L is set to be 1/3, 1/2 and 2/3, respectively. The shell length to radius ratio L/R varies from 1 to 10 with increment of 0.2. It is apparent that the fundamental frequency parameters decrease as the shell length to radius ratio L/R increases. It is found that the fundamental frequency parameters λ decrease significantly as L/R increases from 1 to 5 and then become hardly influenced by the length to radius ratio L/R of the circular cylindrical shell, i.e. the effect of end boundary conditions diminishes for long shells.

3.4. Vibration of shells with one thickness variation at various locations

The effect of the locations of the step thickness variation L_1/L on the fundamental frequency parameters of CF and FF shells is illustrated in Figs. 5 and 6, respectively. The step thickness to radius ratio $h_1/R = 0.01$, thickness ratio $h_2/h_1 = 0.5$ and 2, and the length to radius ratio $L/R = 1$, 5 and 10 are considered. The location of the step thickness variation L_1/L varies from 0.01 to 0.99 with increment of 0.02. It is seen that the

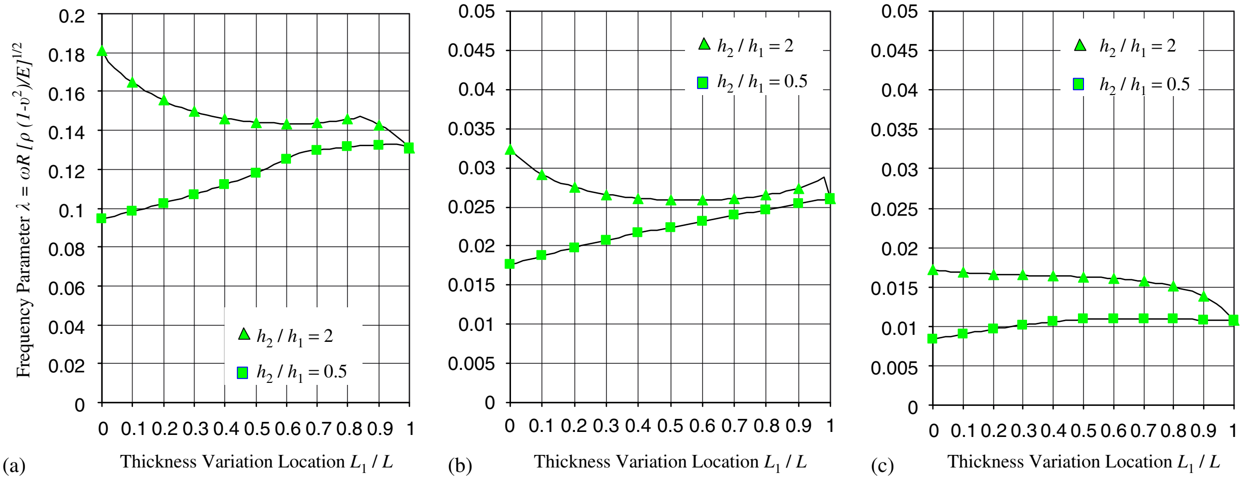

Fig. 5. Frequency parameters $\lambda = \omega R^2\sqrt{\rho(1-\upsilon^2)/E}$ versus the thickness variation ratio L_1/L for CF circular cylindrical shells having one-step thickness variation, thickness ratio $h_1/R = 0.01$, step thickness ratio $h_2/h_1 = 0.5$, 2 and length to radius ratio is set to be 1, 5 and 10. (a) $L/R = 1$, (b) $L/R = 5$, (c) $L/R = 10$.

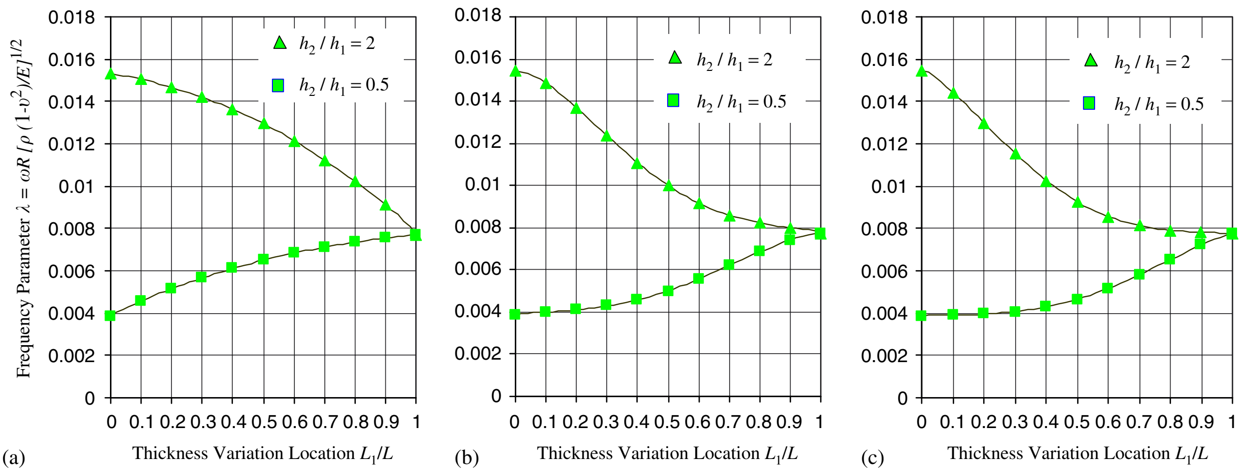

Fig. 6. Frequency parameters $\lambda = \omega R^2\sqrt{\rho(1-\upsilon^2)/E}$ versus the thickness variation ratio L_1/L for FF circular cylindrical shells having one-step thickness variation, thickness ratio $h_1/R = 0.01$, step thickness ratio $h_2/h_1 = 0.5$, 2 and length to radius ratio is set to be 1, 5 and 10. (a) $L/R = 1$, (b) $L/R = 5$, (c) $L/R = 10$.

step thickness ratio h_2/h_1 is of great influence on the fundamental frequency parameters of the shells. The fundamental frequency parameters for shells with a larger step thickness ratio h_2/h_1 are always greater than the ones with a smaller step thickness ratio until the value of L_1/L reaches 1. The position of step thickness variation also has a significant influence on the fundamental frequency and this influence varied with the boundary conditions.

3.5. Vibration of shells with one thickness variation at various step thickness ratios

The influence of step thickness ratio h_2/h_1 on the fundamental frequency parameters of SS and CF circular cylindrical shells with one thickness variation is depicted in Figs. 7 and 8. The step thickness to radius ratio $h_1/R = 0.01$, and the length to radius ratio $L/R = 1$, 5 and 10, respectively. For the SS shells, the step thickness variation is located at $L_1/L = 1/4$ and 1/2 and for the *CF* shell the location of the step thickness

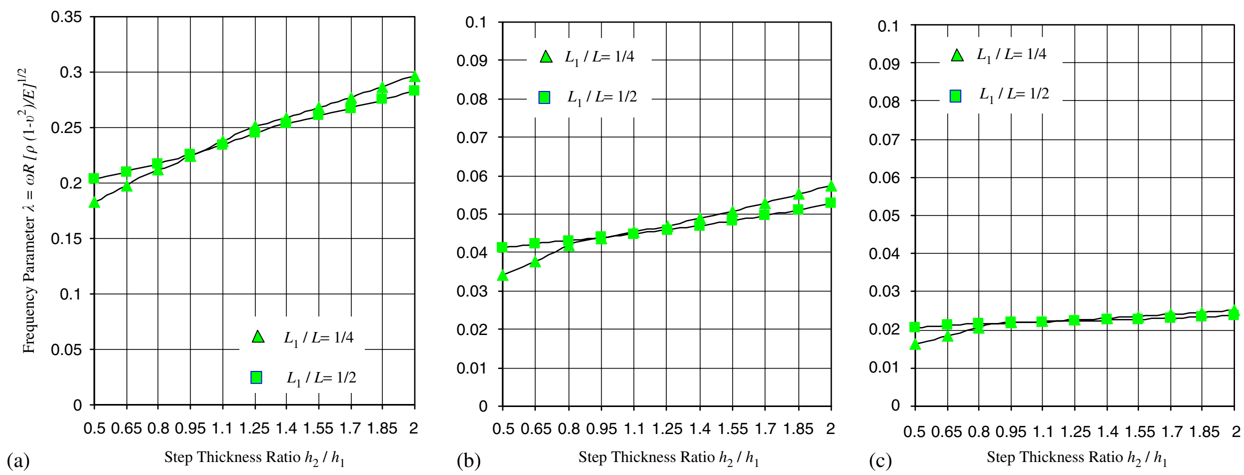

Fig. 7. Frequency parameters $\lambda = \omega R^2\sqrt{\rho(1-\upsilon^2)/E}$ versus the step thickness ratio h_2/h_1 for SS circular cylindrical shells having one-step thickness variation, thickness ratio $h_1/R = 0.01$, thickness variation location is $L_1/L = 0.25$, 0.5 and length to radius ratio is set to be 1, 5 and 10. (a) $L/R = 1$, (b) $L/R = 5$, (c) $L/R = 10$.

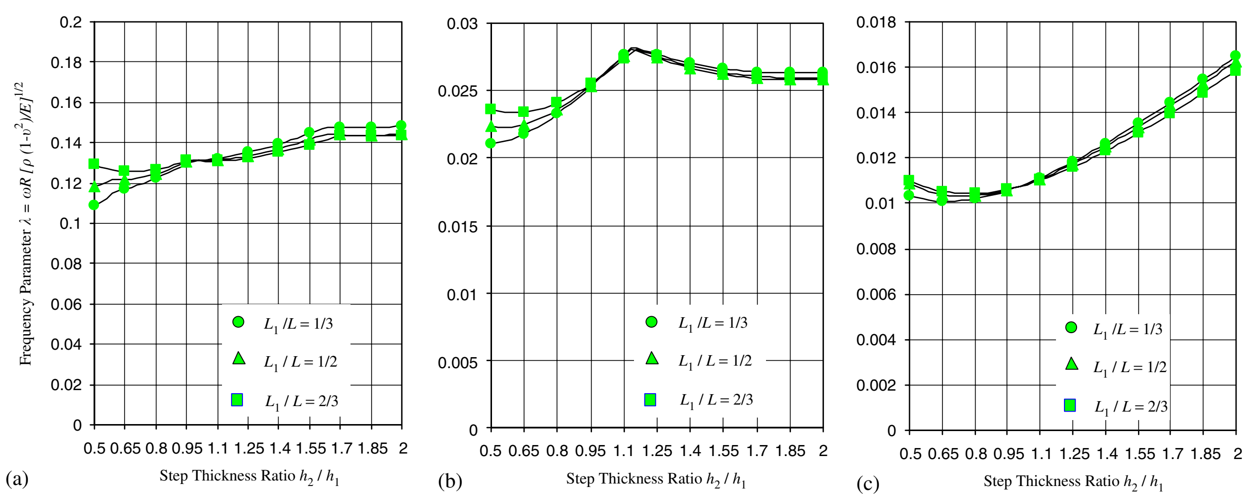

Fig. 8. Frequency parameters $\lambda = \omega R^2\sqrt{\rho(1-\upsilon^2)/E}$ versus the step thickness ratio h_2/h_1 for CF circular cylindrical shells having one-step thickness variation, thickness ratio $h_1/R = 0.01$, thickness variation location is $L_1/L = 1/3$, 0.5, 2/3 and length to radius ratio is set to be 1, 5 and 10. (a) $L/R = 1$, (b) $L/R = 5$, (c) $L/R = 10$.

variation L_1/L is set to be 1/3, 1/2 and 2/3, respectively. The step thickness ratio h_2/h_1 changes from 0.5 to 2 with increment of 0.03. For the SS shells, the fundamental frequency parameters increase monotonically as the step thickness ratio h_2/h_1 increases. For the CF shells, however, the variation of the fundamental frequency parameters against the step thickness ratio h_2/h_1 is quite different from the pattern for the SS shells. The increase of the step thickness ratio h_2/h_1 could lead to an increase or a decrease in the fundamental frequency parameters of the CF shells, depending on the shell length to radius ratio L/R and the location of the step thickness variation L_1/L, as shown in Fig. 8.

Table 7
Frequency parameters λ for SS shells with two-step thickness variation ($h_1/R = 0.01$, $h_2/h_1 = 3$, $h_3/h_1 = 1$, $h_4/h_1 = 4$, $L_1/L = 1/3$, $L_2/L = 1/3$, $L_3/L = 1/3$)

L/R		m							
	n	1	2	3	4	5	6	7	8
1	1	0.517542	0.505186	0.369360	0.296891	0.281642	0.316648	0.386669	0.476253
	2	0.722939	0.862676	0.810485	0.741740	0.684636	0.649386	0.637715	0.645609
	3	0.945073	0.997603	0.961980	0.920945	0.881260	0.849290	0.830802	0.831986
	4	1.021771	1.046677	1.053025	1.040086	1.033496	1.039381	1.062890	1.107311
	5	1.082221	1.101748	1.456421	1.469956	1.487741	1.510148	1.537662	1.570885
	6	1.441044	1.446871	1.594147	1.685246	1.698880	1.717845	1.743747	1.778270
	7	1.661682	1.672300	1.681144	2.112340	2.162205	2.223930	2.297332	2.381787
	8	1.669206	2.047329	2.074175	2.136910	2.684127	2.902007	2.939344	2.982544
5	1	0.144796	0.066942	0.068069	0.110912	0.132059	0.144737	0.272677	0.408392
	2	0.406223	0.235771	0.140607	0.120957	0.136977	0.145298	0.273025	0.519789
	3	0.626946	0.413989	0.266796	0.196154	0.209818	0.265000	0.390353	0.525174
	4	0.663870	0.507014	0.354791	0.269685	0.248519	0.265953	0.395801	0.571296
	5	0.765474	0.673344	0.503700	0.387576	0.326093	0.322347	0.443414	0.603354
	6	0.854751	0.729706	0.577280	0.458844	0.383680	0.364176	0.480228	0.651991
	7	0.881249	0.768055	0.630956	0.521902	0.454301	0.431576	0.535569	0.706801
	8	0.898369	0.834950	0.724973	0.621826	0.534135	0.489303	0.592543	0.738892

Table 8
Frequency parameters λ for SS shells with three-step thickness variation ($h_1/R = 0.01$, $h_2/h_1 = 2$, $h_3/h_1 = 3$, $h_4/h_1 = 4$, $L_1/L = 1/4$, $L_2/L = 1/4$, $L_3/L = 1/4$, $L_4/L = 1/4$)

L/R		m							
	n	1	2	3	4	5	6	7	8
1	1	0.501100	0.574534	0.442315	0.363088	0.342990	0.368892	0.420955	0.485119
	2	0.856108	0.904135	0.832518	0.766221	0.722140	0.710725	0.735860	0.794417
	3	0.959588	1.078523	1.060339	1.045075	1.038738	1.046618	1.072503	1.118770
	4	1.093752	1.140371	1.319586	1.324191	1.338109	1.366104	1.412889	1.482482
	5	1.322024	1.320223	1.699012	1.835803	1.875982	1.928321	1.993600	2.072356
	6	1.776723	1.787504	1.806747	2.281066	2.471077	2.528322	2.596923	2.677201
	7	2.007188	2.364310	2.389465	2.424894	2.870882	3.166407	3.242298	3.333200
	8	2.349746	2.411982	2.850357	3.055443	3.104595	3.463829	4.057334	4.319659
5	1	0.165713	0.074674	0.070909	0.100470	0.129680	0.156930	0.183309	0.216538
	2	0.452733	0.239237	0.151120	0.155030	0.193444	0.230755	0.280540	0.324675
	3	0.597712	0.414403	0.271633	0.222936	0.246002	0.295766	0.337882	0.392473
	4	0.668250	0.552346	0.388570	0.302990	0.300159	0.347869	0.408062	0.454526
	5	0.783467	0.654963	0.494705	0.396661	0.366752	0.403601	0.455143	0.532594
	6	0.838543	0.735681	0.590256	0.488247	0.442550	0.453354	0.510848	0.580966
	7	0.848028	0.797502	0.672848	0.573610	0.513453	0.506372	0.569290	0.626949
	8	0.887245	0.832491	0.723467	0.631647	0.580632	0.578282	0.612149	0.691638

3.6. Vibration of shells with multiple step thickness variations

Tables 7 and 8 present exact frequency parameters λ for SS circular cylindrical shells with two- and three-step thickness variations. The shell length to radius ratio L/R is set to be 1, 5 and the thickness to radius ratios are $h_1/R = 0.01$, $h_2/h_1 = 3$, $h_3/h_1 = 1$ for shells of two-step thickness variations and $h_1/R = 0.01$, $h_2/h_1 = 2$, $h_3/h_1 = 3$, $h_4/h_1 = 4$ for shells of three-step thickness variations, respectively. The segment length ratios are $L_1/L = 1/3$, $L_2/L = 1/3$ and $L_3/L = 1/3$ for the two-step shells and $L_1/L = 1/4$, $L_2/L = 1/4$, $L_3/L = 1/4$ and $L_4/L = 1/4$ for the three-step shells, respectively. The frequency parameters show very similar trends to those for shells with one-step thickness variation. The frequency parameters decrease as the length to radius ratio L/R varies from 1 to 5.

4. Conclusions

This paper presents an investigation on the vibration of circular cylindrical shells with step-wise thickness variations. The Flügge thin shell theory is employed and the state-space technique is used to develop an analytical approach for the shell vibration problem. First-known exact frequency parameters for Flügge circular cylindrical shells with various combinations of boundary conditions and step thickness variations are presented in tables and graphs. It is desirable that the exact vibration frequencies presented in this paper can be used as important benchmark values for researchers to check the validity and accuracy of their numerical methods for such shell vibration problems.

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Appendix A

The governing differential equations for the ith ring segment of a thin circular cylindrical shell based on the Donnell–Mushtari shell theory can be expressed as [1]

$$\frac{\partial^2 u_i}{\partial x^2} + \frac{(1-\upsilon)}{2R^2}\frac{\partial^2 u_i}{\partial \theta^2} + \frac{(1+\upsilon)}{2R}\frac{\partial^2 v_i}{\partial x\,\partial \theta} + \frac{\upsilon}{R}\frac{\partial w_i}{\partial x} = \rho\frac{(1-\upsilon^2)}{E}\frac{\partial^2 u_i}{\partial t^2}, \tag{A.1}$$

$$\frac{(1+\upsilon)}{2R}\frac{\partial^2 u_i}{\partial x\,\partial \theta} + \frac{(1-\upsilon)}{2}\frac{\partial^2 v_i}{\partial x^2} + \frac{1}{R^2}\frac{\partial^2 v_i}{\partial \theta^2} + \frac{1}{R^2}\frac{\partial w_i}{\partial \theta} = \rho\frac{(1-\upsilon^2)}{E}\frac{\partial^2 v_i}{\partial t^2}, \tag{A.2}$$

$$\frac{\upsilon}{R}\frac{\partial u_i}{\partial x} + \frac{1}{R^2}\frac{\partial v_i}{\partial \theta} + \frac{1}{R^2}w_i + k_i\left[R^2\frac{\partial^4 w_i}{\partial x^4} + 2\frac{\partial^4 w_i}{\partial x^2\,\partial \theta^2} + \frac{1}{R^2}\frac{\partial^4 w_i}{\partial \theta^4}\right] = -\rho\frac{(1-\upsilon^2)}{E}\frac{\partial^2 w_i}{\partial t^2}. \tag{A.3}$$

The **H** matrix used in the proposed analytical method based on the Donnell–Mushtari shell theory can be derived as follows:

$$(H_i)_{12} = (H_i)_{34} = (H_i)_{56} = (H_i)_{67} = (H_i)_{78} = 1, \tag{A.4}$$

$$(H_i)_{21} = \frac{m^2(1-\upsilon)}{2R^2} - \frac{\rho(1-\upsilon^2)\omega^2}{E}, \tag{A.5}$$

$$(H_i)_{24} = -\frac{(1+\upsilon)m}{2R}, \tag{A.6}$$

$$(H_i)_{26} = -\frac{\upsilon}{R}, \tag{A.7}$$

$$(H_i)_{42} = \frac{(1+\upsilon)m}{R(1-\upsilon)}, \tag{A.8}$$

$$(H_i)_{43} = \frac{2m^2}{R^2(1-\upsilon)} - \frac{2\rho\left(1-\upsilon^2\right)\omega^2}{E(1-\upsilon)}, \tag{A.9}$$

$$(H_i)_{45} = \frac{2m}{R^2(1-\upsilon)(1+3k_i)}, \tag{A.10}$$

$$(H_i)_{82} = -\frac{\upsilon}{k_i R^3}, \tag{A.11}$$

$$(H_i)_{83} = -\frac{m}{k_i R^4}, \tag{A.12}$$

$$(H_i)_{85} = -\frac{1}{k_i R^4} - \frac{m^4}{R^4} + \frac{\rho\omega^2\left(1-\upsilon^2\right)}{k_i R^2 E}. \tag{A.13}$$

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