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The frequency response of a rectangular cantilever plate vibrating in a viscous fluid

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Abstract

The fluid reaction on a wide rectangular cantilever plate vibrating in a viscous fluid is constructed by superposition of a potential flow solution and an asymptotic correction built from the Navier–Stokes equations for an incompressible fluid with nonlinear terms neglected. Using the Wiener–Hopf method, a simple analytic expression for the fluid reaction is obtained for high values of the dimensionless parameter β, which is the frequency times the length of the plate squared divided by the kinematic viscosity. Fluid and structure motion are then coupled when solving the equation of motion of the plate with a fluid reaction expressed in terms of weighted averages of the velocity of the plate. The frequency response is then calculated for plates of different sizes and materials and the results are shown for a gas and a liquid.

1. Introduction

The objective of this article is to provide a method to calculate the frequency response around the first mode of vibration of a thin rectangular plate vibrating in a viscous fluid. The plate is clamped on one edge while the other three edges are free, and performs harmonic vibrations.

The first mode of vibration corresponds to pure bending, where the amplitude of the movement is greater the farther from the clamped edge and all the points in the plate move in the same direction (upwards or downwards) at every instant.

The motivation of this work is to provide a theoretical model that can be used in the design and interpretation of density and viscosity sensors. The particular sensor considered here consists of a plate where by design only the pure bending modes can be excited, and for practical reasons the first mode is preferred for measuring purposes.

Since the plate considered in this study is wider than it is long (see Fig. 1), it is assumed that all longitudinal cross sections behave equally, with negligible effect of the two parallel free edges and a predominant importance of the third free edge. Therefore, the problem is reduced to planar flow, treating only one longitudinal cross section. However, it is important to single out that this planar flow is not representative of

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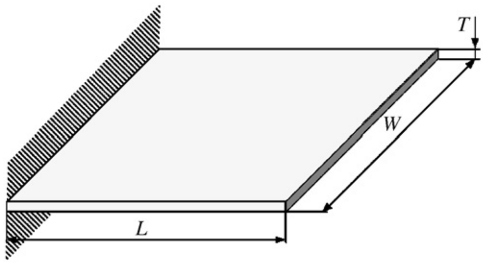

Fig. 1. Schematic drawing of the cantilever plate, clamped on one edge and with thickness T, length L and width W, where $T \ll L, W$ and $L \approx W$ or $L < W$.

geometries where the width of the plate is smaller than the length, because then the effect on the parallel free edges is not negligible.

In the literature there exists work on vibrating plates of different geometries interacting with fluids in the inviscid limit. In this study, the viscosity cannot be neglected because the sensor measures it together with the fluid density. Geometries similar to the cantilever plate in viscous fluids have been reported, notably cantilever beams.

The complexity of the models varies widely, owing to different approximations for the structure, the fluid behaviour and the coupling of the two motions. The reaction of the fluid on the beam is modelled as the reaction on a sphere in Refs. [1–3]. Other works are based on the vibration of a cylinder modified empirically for a rectangular cross section [4]. Another method is to approximate the fluid reaction for every cross section as the reaction on an infinite beam with the same cross section and amplitude of vibration [5].

In Refs. [2,6,7], the structure of the cantilever is modelled as a spring–mass system leading to a simple harmonic oscillator motion, where in general the action of the fluid is reflected in the equations by an added mass and a damping coefficient. In other cases the equations of a vibrating beam as in Weaver et al. [8] are used, for example in Ref. [9] or Ref. [10]. Nevertheless the effect of the fluid is again represented as an added mass and damping coefficients that are independent of position along the beam. The most complete treatment to date is proposed by Sader [5], where the dependence of the fluid on position is kept. However, the fluid reaction is modelled as a complex added mass coefficient and not as an external force.

In this paper, the equation of motion of the plate is obtained from the balance of forces acting on it. The driving force together with the fluid reaction force and the elastic restoring force are balanced by the plate acceleration force. This driving force is usually known, also the elastic restoring force takes a known expression for thin plates but the fluid reaction is to be determined.

The first part of this work deals with obtaining an expression that relates the pressure of the surrounding fluid on the plate to the vertical velocity of the plate u_z, for a given frequency of vibration. This expression constitutes the fluid reaction needed to solve the balance of forces on the plate.

The second part of this work explains a method for solving the equation of motion and obtaining the amplitude of the vibration as a function of position and the frequency ω.

2. Asymptotic solution for an infinitely wide cantilever

This section deals with the asymptotic study of an infinitely wide cantilever of length L for large values of the dimensionless parameter β characterising the flow, defined as

$$\beta = \frac{\omega L^2}{\nu}, \tag{1}$$

where ω is the frequency of vibration, L is the length of the plate and ν is the kinematic viscosity of the surrounding fluid.

First of all, a potential flow solution for the geometry of an infinitely wide plate is obtained in Section 2.1. This solution represents the first term in the asymptotic expansion, because it is the solution in the limit when β

tends to infinity. Section 2.2 shows the correction to the potential solution, by solving a semi-infinite problem with the Wiener–Hopf method.

The plate is located in the plane $z = 0$ along the x-axis between $x = 0$ (clamped end) and $x = 1$ (free end) when x and z are made dimensionless by dividing by the length of the plate, L (see Fig. 2). All variables are dimensionless. The velocity components u_x, u_z are scaled with respect to a reference velocity U and the pressure p is dimensionless after being divided by the product $\omega\rho LU$.

An important magnitude that leads to the fluid reaction on the plate is called the pressure differential across the plate and is defined as

$$[p] = p|_{z=0^+} - p|_{z=0^-}. \tag{2}$$

2.1. Potential flow approximation

The limit when β tends to infinity corresponds to inviscid flow. We make the following assumptions:

- the amplitude of the vibrations is very small,
- the thickness of the plate is negligible,
- the medium is considered to be infinite, with no wall effects,
- the fluid is incompressible with density ρ.

These assumptions together with the fact that we impose the condition that the viscosity is zero, lead to the potential flow approximation. The Navier–Stokes equations and the continuity equation are then reduced to [11]

$$u_x = \frac{\partial\phi}{\partial x}, \tag{3}$$

$$u_z = \frac{\partial\phi}{\partial z}, \tag{4}$$

$$p = \mathrm{i}\phi. \tag{5}$$

The potential ϕ has to satisfy Laplace's equation

$$\nabla^2\phi = 0. \tag{6}$$

Laplace's equation has solution

$$\phi = \theta = \arctan\left(\frac{z}{x}\right). \tag{7}$$

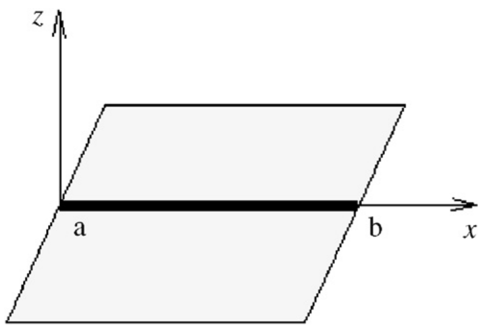

Fig. 2. The plate is represented by the line going from **a** ($x = 0,\ z = 0$), which is the clamped end, to **b** ($x = 1,\ z = 0$), which is the free end, in the scaled coordinate system.

Moving the origin to $x = s$ and superposing over s one obtains

$$\phi = \int_{-\infty}^{+\infty} f(s) \arctan\left(\frac{z}{x-s}\right) \mathrm{d}s, \tag{8}$$

where $f(s)$ is a function to be determined from the boundary conditions. The appropriate branch of the arctangent is chosen and at $z = 0^+$ for $x < s$ it takes the value π while for $x > s$ the arctangent is zero. At $z = 0^-$ the arctangent takes the opposite sign.

The boundary conditions are imposed on the plane $z = 0$ where the plate is located

- The tangential velocity, u_x, is continuous and equal to zero

$$u_x|_{z=0^+} = u_x|_{z=0^-} = 0. \tag{9}$$

  This condition is not satisfied by the potential flow solution, and, to take account of it, the solution of the potential flow will need to be corrected for.
- The vertical velocity, u_z, is continuous everywhere and known for the plate, but unknown outside the plate

$$u_z|_{z=0^+} = u_z|_{z=0^-} = \begin{cases} \text{unknown function}, & x < 0, \\ u_z(x), & 0 < x < 1, \\ \text{unknown function}, & x > 1. \end{cases} \tag{10}$$

- The pressure differential is unknown for the plate, but known and equal to zero outside the plate for $x > 1$. At the other side of the plate (beyond the clamped end) we do not expect the pressure to be zero, but to be finite. It will be shown later that for $x < 0$ the pressure differential takes a constant value. This value represents the effect of the clamping on $x = 0$, thus

$$[p] = \begin{cases} \text{finite} & x < 0, \\ \text{unknown function to be determined}, & 0 < x < 1, \\ 0, & x > 1. \end{cases} \tag{11}$$

From Eq. (8) and condition (11) the potential at $z = 0^+$ can be written as

$$\phi = \pi \int_x^1 f(s) \, \mathrm{d}s. \tag{12}$$

The vertical velocity can be obtained as the derivative of the potential from Eq. (8) with respect to z. Applying the boundary condition on the vertical velocity from Eq. (10) one obtains the following integral equation for the plate

$$u_z(x) = \int_0^1 \frac{f(s)}{x-s} \, \mathrm{d}s, \quad 0 < x < 1, \tag{13}$$

where the singular integral on the right is defined as a Cauchy principal value as are subsequent integrals. The general solution of such an integral equation can be found in Ref. [12] and gives the following expression for $f(s)$

$$f(s) = \frac{1}{\pi^2 \sqrt{s(1-s)}} \int_0^1 u_z(x') \frac{\sqrt{x'(1-x')}}{x'-s} \mathrm{d}x' + \frac{A}{\sqrt{s(1-s)}}, \tag{14}$$

where A is a constant that is determined here by imposing a finite pressure differential (finite potential) at $x = 0$. Thus,

$$A = \frac{-1}{\pi^2} \int_0^1 u_z(x') \sqrt{\frac{1-x'}{x'}} \mathrm{d}x'. \tag{15}$$

The distribution $f(s)$ can thus be re-written as

$$f(s) = \frac{\sqrt{s}}{\pi^2\sqrt{1-s}}\int_0^1 \frac{u_z(x')}{x'-s}\sqrt{\frac{1-x'}{x'}}\,\mathrm{d}x'. \tag{16}$$

Once the function $f(s)$ is known, the potential can be determined using Eq. (12) and the tangential velocity can be obtained by taking the derivative of the potential above with respect to x, from Eq. (12) one gets,

$$u_x = -\pi f(x), \quad x<1. \tag{17}$$

From Eqs. (17) and (16) one can see that the no-slip condition is violated, and moreover the tangential velocity has a singular behaviour at the edge of the plate with

$$u_x|_{x=1_-} = \frac{B}{\sqrt{1-x}}, \tag{18}$$

where B is a weighted average of the vertical velocity over the plate,

$$B = \frac{1}{\pi}\int_0^1 \frac{u_z(x')}{\sqrt{x'(1-x')}}\,\mathrm{d}x'. \tag{19}$$

This tangential velocity is not acceptable and needs to be corrected for.

Though the interest of the fluid behaviour is centred on the pressure differential across the plate, the vertical velocity outside the plate on the plane $z=0$ can also be calculated with the same equations as before. The vertical velocity u_z can be obtained from Eq. (13). The first term of the expansion for the vertical velocity at the free edge just outside the edge is singular and takes the expression

$$u_z|_{x=1^+} \sim \frac{-B}{\sqrt{x-1}}. \tag{20}$$

2.2. Correction to the potential flow solution

The equations that we need to solve here are the Navier–Stokes equations and the equation of continuity with the following assumptions:

- the amplitude of the vibrations is very small,
- the thickness of the plate is negligible,
- the medium is considered to be infinite, with no wall effects,
- the fluid is incompressible with density ρ,
- the fluid is assumed to be Newtonian with kinematic viscosity v.

Neglecting the nonlinear terms and with no influence of the variable y, we write [11]

$$-\mathrm{i}u_x = -\frac{\partial p}{\partial x} + \frac{1}{\beta}\nabla^2 u_x, \tag{21}$$

$$-\mathrm{i}u_z = -\frac{\partial p}{\partial z} + \frac{1}{\beta}\nabla^2 u_z, \tag{22}$$

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} = 0, \tag{23}$$

with

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}. \tag{24}$$

A stream function ψ is introduced such that

$$u_x = -\frac{\partial\psi}{\partial z}, \tag{25}$$

$$u_z = \frac{\partial\psi}{\partial x}. \tag{26}$$

After eliminating the pressure term in the Navier–Stokes equations one finds that

$$\nabla^2(\nabla^2 + \mathrm{i}\beta)\psi = 0. \tag{27}$$

The interest of this study lies in the cases where β is large. Indeed the potential flow solution corresponds to a first-order approximation where β tends to infinity. In this correction, the behaviour at the edges needs special attention, where the previous solution is least satisfactory. Consequently the change of coordinates illustrated in Fig. 3 is proposed, we write

$$x = 1 + \varepsilon X, \tag{28}$$

$$z = \varepsilon Z, \tag{29}$$

where ε is a small dimensionless number. In this new coordinate system the plate is located on the negative X-axis and

$$\nabla^2 = \frac{1}{\varepsilon^2}\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Z^2}\right). \tag{30}$$

In order to make both terms in Eq. (27) comparable we take

$$\varepsilon = \frac{1}{\sqrt{\beta}}. \tag{31}$$

Using the Fourier transform $\overline{F} = \int_{-\infty}^{+\infty} F\mathrm{e}^{\mathrm{i}\zeta X}\mathrm{d}X$ and knowing that we seek solutions that tend to zero at infinity, from Eq. (27) one finds

$$\overline{\psi} = \begin{cases} B_1\mathrm{e}^{-\sqrt{\zeta^2-\mathrm{i}}\,Z} - \mathrm{i}A_1\mathrm{e}^{-|\zeta|Z}, & Z>0, \\ B_2\mathrm{e}^{+\sqrt{\zeta^2-\mathrm{i}}\,Z} - \mathrm{i}A_2\mathrm{e}^{+|\zeta|Z}, & Z<0, \end{cases} \tag{32}$$

with $\sqrt{\zeta^2 - \mathrm{i}}$ and $|\zeta|$ defined to have positive real part; and where A_1, A_2, B_1 and B_2 are functions of ζ to be determined from the boundary conditions. From $\overline{\psi}$, the expressions for $\overline{u_x}$ and $\overline{u_z}$ are easily obtained, also $\overline{p}$

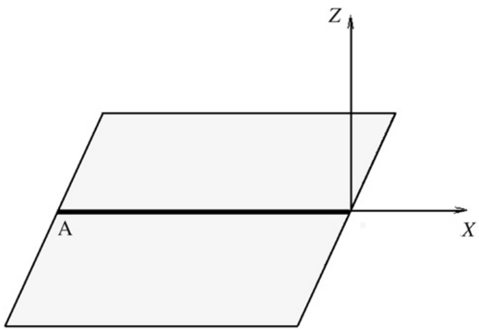

Fig. 3. In the coordinate system proposed in Eqs. (28) and (29) the plate is represented by the line from **A** ($X = -1/\varepsilon,\ Z = 0$) to **B** ($X = 0,\ Z = 0$). These points correspond respectively to **a** and **b** from Fig. 2. Note that when β tends to infinity, ε tends to zero and the plate is located in the negative semi-infinite plane $Z = 0,\ X < 0$.

using Eq. (22).

$$\overline{u_x} = \begin{cases} +B_1\sqrt{\zeta^2 - \mathrm{i}}\,\mathrm{e}^{-\sqrt{\zeta^2-\mathrm{i}}\,Z} - \mathrm{i}|\zeta|A_1\mathrm{e}^{-|\zeta|Z}, & Z>0, \\ -B_2\sqrt{\zeta^2 - \mathrm{i}}\,\mathrm{e}^{+\sqrt{\zeta^2-\mathrm{i}}\,Z} + \mathrm{i}|\zeta|A_2\mathrm{e}^{+|\zeta|Z}, & Z<0, \end{cases} \tag{33}$$

$$\overline{u_z} = \begin{cases} -\mathrm{i}\zeta B_1\mathrm{e}^{-\sqrt{\zeta^2-\mathrm{i}}\,Z} - \zeta A_1\mathrm{e}^{-|\zeta|Z}, & Z>0, \\ -\mathrm{i}\zeta B_2\mathrm{e}^{+\sqrt{\zeta^2-\mathrm{i}}\,Z} - \zeta A_2\mathrm{e}^{+|\zeta|Z}, & Z<0, \end{cases} \tag{34}$$

$$\overline{p} = \begin{cases} +\mathrm{i}\varepsilon\frac{|\zeta|}{\zeta}A_1\mathrm{e}^{-|\zeta|Z}, & Z>0, \\ -\mathrm{i}\varepsilon\frac{|\zeta|}{\zeta}A_2\mathrm{e}^{+|\zeta|Z}, & Z<0. \end{cases} \tag{35}$$

We write the boundary conditions at $Z=0$ for this new solution as

- The velocity components are continuous across the plane $Z=0$,

$$u_x|_{Z=0^+} = u_x|_{Z=0^-}, \tag{36}$$

$$u_z|_{Z=0^+} = u_z|_{Z=0^-}. \tag{37}$$

- The tangential velocity outside the plate is zero and on the plate cancels the velocity u_x from Eq. (18), hence

$$u_x|_{X<0} = -\frac{B}{\sqrt{-\varepsilon X}}, \tag{38}$$

$$u_x|_{X>0} = 0. \tag{39}$$

- The vertical velocity is zero on the plate and unknown outside (the potential flow solution accounted exactly for the vertical component of the velocity, so no additional contribution is needed),

$$u_z|_{X<0} = 0, \tag{40}$$

$$u_z|_{X>0} = \text{unknown function}. \tag{41}$$

- The pressure differential on the plate is unknown and zero outside,

$$[p]|_{X<0} = \text{unknown function}, \tag{42}$$

$$[p]|_{X>0} = 0. \tag{43}$$

The first consequence of these boundary conditions is that $A_1 = A_2$ and $B_1 = B_2$. Thus the pressure differential takes the following expression

$$\overline{[p]} = 2\mathrm{i}\frac{|\zeta|}{\zeta}A_1. \tag{44}$$

Now the expression of the pressure differential correction is obtained using the Wiener–Hopf method [13, p. 376]. In this method half Fourier transforms are introduced as

$$\overline{F}_- = \int_{-\infty}^{0} F\mathrm{e}^{\mathrm{i}\zeta X}\mathrm{d}X, \tag{45}$$

where the integral converges for $\mathrm{Im}(\zeta)<0$, and $\overline{F}_-$ is a 'minus function' because it is analytic in the semi-infinite plane $\mathrm{Im}(\zeta)<0$ and it is denoted with a subindex '-'. In the same way

$$\overline{F}_+ = \int_{0}^{+\infty} F\mathrm{e}^{\mathrm{i}\zeta X}\mathrm{d}X, \tag{46}$$

where the integral converges for $\text{Im}(\zeta)>0$, so $\overline{F}_+$ is a 'plus function' to signify that it is analytic in the semi-infinite plane $\text{Im}(\zeta)>0$.

From the boundary conditions, the transform of the velocity components and the pressure differential can be classified as plus or minus functions depending on the region where they are not zero. Thus there are two minus functions $\overline{u_x} = \overline{u_{x-}}$ and $\overline{[p]} = \overline{[p]}_-$ and one plus function $\overline{u_z} = \overline{u_{z+}}$. In addition the expression of the tangential velocity is known from Eq. (38). The Fourier transform of u_x takes the form

$$\overline{u_{x-}} = \mathrm{e}^{\mathrm{i}3\pi/4}\sqrt{\pi}\frac{B}{\sqrt{\varepsilon}}\frac{1}{\sqrt{\zeta_-}}. \tag{47}$$

Then at $Z = 0$ we have three equations (33), (34) and (44) with 4 unknowns A_1, B_1, $\overline{u_{z+}}$ and $\overline{[p]}_-$. After elimination of A_1 and B_1 and using Eq. (47) for the expression of $\overline{u_{x-}}$ the following functional equation is obtained:

$$\overline{u_{z+}} = \frac{|\zeta|}{2(\zeta^2-\mathrm{i})}\frac{\sqrt{\zeta^2-\mathrm{i}}}{|\zeta|+\sqrt{\zeta^2-\mathrm{i}}}\overline{[p]}_- + \mathrm{e}^{\mathrm{i}\pi/4}\sqrt{\pi}\frac{B}{\sqrt{\varepsilon}}\frac{\sqrt{\zeta_-}}{\sqrt{\zeta^2-\mathrm{i}}}. \tag{48}$$

This equation is rearranged to have all the plus functions on the left-hand side and all the minus functions on the right-hand side as follows

$$\frac{\zeta+\mathrm{e}^{\mathrm{i}\pi/4}}{\sqrt{\zeta_+}}H_+\overline{u_{z+}} - \mathrm{e}^{\mathrm{i}\pi/4}\sqrt{\pi}\frac{B}{\sqrt{\varepsilon}}(S_+ + H_+) = \frac{\sqrt{\zeta_-}}{4(\zeta-\mathrm{e}^{\mathrm{i}\pi/4})}\frac{1}{H_-}\overline{[p]}_- + \mathrm{e}^{\mathrm{i}\pi/4}\sqrt{\pi}\frac{B}{\sqrt{\varepsilon}}S_-, \tag{49}$$

where the functions H_+ and H_- come from writing the function $H(\zeta)$ as the product of plus and minus functions

$$H(\zeta) = \frac{|\zeta|+\sqrt{\zeta^2-\mathrm{i}}}{2\sqrt{\zeta^2-\mathrm{i}}} = H_+(\zeta)H_-(\zeta), \tag{50}$$

and S_+ and S_- come from writing the function $S(\zeta)$ as the sum of plus and minus functions

$$S(\zeta) = \left[\frac{\sqrt{\zeta_-}}{\sqrt{\zeta_+}}\frac{\sqrt{\zeta+\sqrt{\mathrm{i}}}}{\sqrt{\zeta-\sqrt{\mathrm{i}}}} - 1\right]H_+ = S_+(\zeta) + S_-(\zeta). \tag{51}$$

The decomposition of H and S in plus and minus functions is explained in the Appendices A and B, respectively.

It is shown in Appendix A that as ζ tends to infinity, H_+ and H_- tend to unity in respective half-planes of regularity and in Appendix B that S_+ and S_- tend to zero. The total velocity u_z must be finite at the edge of the plate so $\overline{u_{z+}}$ must correct for that of the potential solution, i.e. $\overline{u_{z+}} \sim \text{constant}/\sqrt{\zeta_+}$, from Eq. (20) as ζ tends to infinity in the plus region.

At the same time, the pressure differential at $X = 0$ can be singular but has to be integrable (giving a finite force on the plate). Hence the left-hand side of Eq. (49) is bounded by a constant as $|\zeta| \to \infty$ in the plus half-plane, while the right-hand side tends to zero as $|\zeta| \to \infty$ in the minus half-plane, both expressions coexisting on the real line. Thus by Liouville's theorem both sides of Eq. (49) are equal to an entire function that must be identically zero.

From Eq. (49) we obtain the two equations with two unknowns, and the pressure differential and the vertical velocity are calculated. Using the expansions of the product and sum decomposition of functions $H(\zeta)$ and $S(\zeta)$ from Appendices A and B, taking the inverse transform and changing the coordinates back to x and z one obtains

$$[p]|_{x\to 1} \sim 2\sqrt{2}\mathrm{e}^{-\mathrm{i}\pi/4}\varepsilon B\frac{1}{\sqrt{1-x}}. \tag{52}$$

This correction to the pressure differential now has a damping component in phase with the velocity of the plate, and an added mass contribution out of phase with the velocity of the plate. Singular behaviour of this pressure differential is observed at the free edge of the plate, as expected.

The correction to the vertical velocity of the plate can also be calculated from the functional equation using the expansions for the functions H_+, H_-, S_+ and S_- from Appendices A and B. Once the inverse transform is taken, it gives in x and z coordinates

$$u_z|_{x=1^+} \sim \frac{B}{\sqrt{x-1}}. \tag{53}$$

One can see that this result cancels the singular behaviour of the vertical velocity on the free edge, just outside the plate, presented in Eq. (20).

3. Equation of motion of the vibrating cantilever plate

The balance of forces per unit surface area on the plate gives the equation of motion of the plate. Now the fluid reaction on the plate, f_{fluid}, can be calculated by superposing the solutions of the pressure differential in the potential flow problem and the correction to the potential flow solution from Eq. (52). Since the full expression from the potential problem is quite complex, a simplification is made to facilitate the coupling of the motion of the plate with the fluid. Since the correction is important mainly at the edge $x = 1$, the pressure differential coming from the potential solution is approximated by the value of the potential at the other end of the plate, $x = 0$. Using Eqs. (12) and (16) to get the value of the pressure differential at $x = 0$ and superposing it to the correction from Eq. (52), one obtains the following fluid force per unit surface area, in units of pressure,

$$f_{\text{fluid}} = -\omega\rho LU[p] = -\omega\rho LU\left(2\pi^2 \mathrm{i}A + 2\sqrt{2}\mathrm{e}^{-\mathrm{i}\pi/4}\varepsilon B\frac{1}{\sqrt{1-x}}\right), \tag{54}$$

where A and B are weighted averages of u_z and take the values from Eqs. (15) and (19).

The driving force f_{drive} is modelled as a sinusoid of frequency ω on a line parallel to the clamping at a distance x_f from the clamping. For simplicity, the magnitude of the force is taken to be 1, so

$$f_{\text{drive}} = \delta(x - x_f)\mathrm{e}^{-\mathrm{i}\omega t}, \tag{55}$$

and is expressed in units of pressure.

Using the expression of the elastic restoring force for a thin bent plate with a small deflection in the z direction from Ref. [14] while keeping the assumption that all longitudinal cross sections behave in the same way, the equation of motion of the plate can be reduced to

$$\rho_s h L^2 \frac{\partial^2 w}{\partial t^2} = f_{\text{fluid}} + f_{\text{drive}} - \frac{1}{12}Eh^3\frac{\partial^4 w}{\partial x^4}, \tag{56}$$

where ρ_s is the density of the plate, w is the dimensionless deflection of the plate, and h is the dimensionless thickness, both scaled with respect to the plate length L, and E is the Young's modulus of the plate.

We are interested in the steady-state vibration and thus the time dependence can be removed by taking, in an abuse of notation, $w = w\mathrm{e}^{-\mathrm{i}\omega t}$ and dropping the factor $\mathrm{e}^{-\mathrm{i}\omega t}$. Eq. (56) can then be rewritten as

$$\frac{\partial^4 w(x,\omega)}{\partial x^4} - c(\omega)^4 w(x,\omega) = g(x,\omega), \tag{57}$$

where c takes the following expression:

$$c = \left(\frac{3\rho_s}{E}\right)^{1/4}\left(\frac{2\omega L}{h}\right)^{1/2} \tag{58}$$

and

$$g(x,\omega) = \frac{12}{Eh^3}\left(\delta(x - x_f) - \omega\rho LU\left(2\pi^2 \mathrm{i}A + 2\sqrt{2}\mathrm{e}^{-\mathrm{i}\pi/4}\varepsilon\omega\rho B\frac{1}{\sqrt{1-x}}\right)\right), \tag{59}$$

where A and B are weighted averages of u_z.

The solution of Eq. (57) can be constructed from the general solution and a particular solution w_p,

$$w(x) = w_p(x) + C_1 \sinh cx + C_2 \cosh cx + C_3 \sin cx + C_4 \cos cx, \tag{60}$$

where C_1 to C_4 are constants to be determined from the boundary conditions. The particular solution w_p can be obtained from the Green's function $G(x, x')$ for Eq. (57) before applying the boundary conditions. This Green's function is given in Appendix C, Eq. (C.2), and using it one can write

$$w_p = \int_0^1 G(x, x')g(x')\,\mathrm{d}x'. \tag{61}$$

This expression of w_p depends on A and B which are unknown because u_z is unknown.

The boundary conditions for the plate are zero displacement and the first derivative of the displacement equal to zero at the clamped end,

$$w|_{x=0} = 0, \tag{62}$$

$$\left.\frac{\partial w}{\partial x}\right|_{x=0} = 0. \tag{63}$$

The two boundary conditions at the free end are no bending moments and no shearing forces, which can be written as

$$\left.\frac{\partial^2 w}{\partial x^2}\right|_{x=1} = 0, \tag{64}$$

$$\left.\frac{\partial^3 w}{\partial x^3}\right|_{x=1} = 0. \tag{65}$$

For a given frequency, one can calculate the displacement by solving a system of six equations, four corresponding to the boundary conditions of the plate, Eqs. (62)–(65), and the two expressions for A and B, Eqs. (15) and (19). In the expressions for A and B, the vertical velocity can be written as the first derivative of w over time, and taking into account the scaling of the variables,

$$u_z = -\mathrm{i}\omega \frac{L}{U} w, \tag{66}$$

where the displacement w takes the expression shown in Eq. (60). The system of six equations can then be solved to find the values of C_1 to C_4, A and B for the given frequency. Then back in Eq. (60) the displacement for that frequency at any x can be calculated. The displacement is a complex number with a certain modulus giving the amplitude of the vibration and an argument giving the phase of the displacement with respect to the driving force. To obtain the frequency response, the complex displacements corresponding to an array of frequencies have to be calculated.

4. Another proposal for calculation of the fluid reaction force

In this section we propose another way of obtaining the fluid force. The method is asymptotic and similar to the previous one, and differs only in the boundary conditions in the potential flow problem. Previously the clamping was represented by a finite constant pressure on the negative side of the x-axis. In this section the clamping is approximated by a rigid semi-infinite plate and no vertical velocity is allowed at $x<0$, $z=0$.

The boundary conditions are described as follows:

- The tangential velocity, u_x, is continuous and equal to zero

$$u_x|_{z=0^+} = u_x|_{z=0^-} = 0. \tag{67}$$

  This condition is not satisfied by the potential flow solution, and to account for it the solution of the potential flow will need to be corrected for.

- The vertical velocity, u_z, is continuous everywhere and known for the plate, unknown outside the plate for $x>1$ and zero for $x<0$

$$u_z|_{z=0^+} = u_z|_{z=0^-} = \begin{cases} 0, & x<0, \\ u_z(x), & 0<x<1, \\ \text{unknown function}, & x>1. \end{cases} \tag{68}$$

- The pressure differential is unknown for the plate ($0<x<1$) and the semi-infinite region ($x<0$), but known and equal to zero outside the plate for $x>1$, hence

$$[p] = \begin{cases} \text{unknown function}, & x<1, \\ 0, & x>1. \end{cases} \tag{69}$$

We use the solution of Laplace's equation from Eq. (8), and together with condition (69) the potential at $z=0^+$ can be expressed again as in Eq. (12). The expression for the vertical velocity is then obtained by taking the first derivative of Eq. (8) and taking into account condition (68) as follows

$$u_z = \int_{-\infty}^{1} \frac{f(s)}{x-s}\,\mathrm{d}s, \tag{70}$$

where a Cauchy principal value is taken as before. The distribution $f(s)$ is obtained from the general solution of the integral equation from Ref. [12] and can be written

$$f(s) = \frac{1}{\pi^2\sqrt{1-s}} \int_0^1 u_z(x') \frac{\sqrt{1-x'}}{x'-s}\,\mathrm{d}x', \tag{71}$$

since $u_z = 0$ for $x<0$. From Eqs. (12) and (71) the pressure differential for the plate is expressed as a function of the vertical velocity distribution on the plate,

$$[p] = \frac{2}{\pi}\mathrm{i} \int_0^1 u_z(x') \ln\left|\frac{\sqrt{1-x}-\sqrt{1-x'}}{\sqrt{1-x}+\sqrt{1-x'}}\right| \mathrm{d}x'. \tag{72}$$

From Eqs. (17) and (71) the no-slip condition is violated, and moreover the tangential velocity has a singular behaviour at the edge of the plate with

$$u_x|_{x=1_-} = \frac{B'}{\sqrt{1-x}}, \tag{73}$$

where B' is a weighted average of the velocity u_z over the plate,

$$B' = \frac{1}{\pi} \int_0^1 \frac{u_z(x')}{\sqrt{1-x'}}\,\mathrm{d}x'. \tag{74}$$

This unwanted tangential velocity takes the same form as in Section 2, only the value of the constant changes. To obtain the correction from the semi-infinite problem, it is enough to substitute B by B' in Eq. (52).

The expression for the vertical velocity outside the plate at $z=0$ at the edge $x=1$ has a singular behaviour giving

$$u_z|_{x=1^+} \sim \frac{-B'}{\sqrt{x-1}} \tag{75}$$

and it is cancelled by the vertical velocity introduced by the semi-infinite problem, Eq. (53).

Finally we construct the expression for the fluid force to be introduced in the solution for the motion of the plate in a fluid. Eq. (72) is simplified to facilitate the solution of the vibrational problem and replaced by a constant value, which is the value of the pressure differential at $x=0$,

$$[p] \cong 2\pi^2 \mathrm{i} A', \tag{76}$$

where A' is again a weighted average of u_z for the plate,

$$A' = \frac{1}{\pi^3}\int_0^1 u_z(x')\ln\left|\frac{1-\sqrt{1-x'}}{1+\sqrt{1-x'}}\right|\mathrm{d}x'. \tag{77}$$

If we superpose this pressure differential with the correction from Eq. (52) we obtain the same equation as Eq. (54) with A and B substituted by A' and B',

$$f_{\text{fluid}} = -\omega\rho LU[p] = -\omega\rho LU\left(2\pi^2\mathrm{i}A' + 2\sqrt{2}\mathrm{e}^{-\mathrm{i}\pi/4}\varepsilon\omega\rho B'\frac{1}{\sqrt{1-x}}\right). \tag{78}$$

The method proposed in Section 3 to solve the motion of the plate in a fluid can again be used, with Eqs. (77) and (74) instead of Eqs. (15) and (19).

5. Results and discussion

The method described above was programmed in MATLAB. Some of the results are shown here in Tables 1 and 2, for plates with different dimensions and made of two different materials vibrating in two different fluids. The results are presented in terms of resonance frequency of the first eigenmode, ω_0, and quality factor, Q, defined as

$$Q = \frac{\omega_0}{\omega_1 - \omega_2}, \tag{79}$$

where ω_1 and ω_2 are the frequencies at each side of the resonance that present an amplitude equal to the amplitude at the resonance divided by $\sqrt{2}$.

The resonance frequency and the quality factor are calculated following [15] using the Levenberg–Marquardt algorithm [16]. In the tables the resonance frequency of the first eigenmode in a vacuum, ω_{vac}, is also shown (Fig. 4). This value provides a quick check of the results, the resonance frequency in any fluid being always lower than ω_{vac},

$$\omega_{\text{vac}} = \frac{1.8751^2}{2\sqrt{3}}\frac{h}{L}\sqrt{\frac{E}{\rho_s}}. \tag{80}$$

This resonance frequency in a vacuum is easily obtained from the homogeneous equation of the plate with no external forces, as shown in Ref. [17]. Note that to be consistent with our notation, the thickness h in Eq. (80)

Table 1
Results for various plate dimensions with the clamping represented as a finite pressure, for two different materials (‘m1’ and ‘m2’), and four different fluids (‘f1’, ‘f2’, ‘f3’and ‘f4’)

Material	L (mm)	h (mm)	Fluid	ω_{vac} (rad/s)	ω_0 (rad/s)	Q	β_0
m1	1	0.05	f1	407189	405520	4230	28699
m1	1	0.05	f2	407189	144820	91	144820
m1	10	0.01	f1	814	680	49	4814
m1	10	0.01	f2	814	42	14	4240
m1	10	0.05	f1	4072	3913	446	27694
(*)m1	10	0.05	f2	4072	484	47	48375
m1	10	0.05	f3	4072	3916	1676	391632
m1	10	0.05	f4	4072	469	13	3322
m1	50	0.05	f1	163	137	109	24210
m1	10	0.5	f1	40719	40555	13375	287010
m2	10	0.05	f1	5325	5261	1641	37234
m2	10	0.05	f2	5325	1144	74	114410

The parameter β has been calculated at the resonance frequency (ρ_s(m1) = 2330 kg/m^3, E(m1) = 150 GPa, ρ_s(m2) = 7810 kg/m^3, E(m2) = 860 GPa, ρ(f1) = ρ(f3) = 1.2 kg/m^3, v(f1) = v (f4) = 1.413 × 10^{-5} kg^2/s, ρ(f2) = ρ(f4) = 1000 kg/m^3, v(f2) = v(f3) = 1 × 10^{-6} kg^2/s).

Table 2
Results for a plate with the clamping represented as a semi-infinite rigid plate, two different materials (‘m1’ and ‘m2’), and two different fluids (‘f1’ and ‘f2’)

Material	L (mm)	h (mm)	Fluid	ω_{vac} (rad/s)	ω_0 (rad/s)	Q	β_0
m1	10	0.05	f1	4072	3991	498	28251
m1	10	0.05	f2	4072	3065	4	306523
m2	10	0.05	f1	5325	5290	1841	37438
m2	10	0.05	f2	5325	4581	12	458104

The parameter β has been calculated at the resonance frequency. ($\rho_s(\text{m1}) = 2330\,\text{kg/m}^3$, $E(\text{m1}) = 150\,\text{GPa}$, $\rho_s(\text{m2}) = 7810\,\text{kg/m}^3$, $E(\text{m2}) = 860\,\text{GPa}$, $\rho(\text{f1}) = 1.2\,\text{kg/m}^3$, $\nu(\text{f1}) = 1.413 \times 10^{-5}\,\text{kg}^2/\text{s}$, $\rho(\text{f2}) = 1000\,\text{kg/m}^3$, $\nu(\text{f2}) = 1 \times 10^{-6}\,\text{kg}^2/\text{s}$).

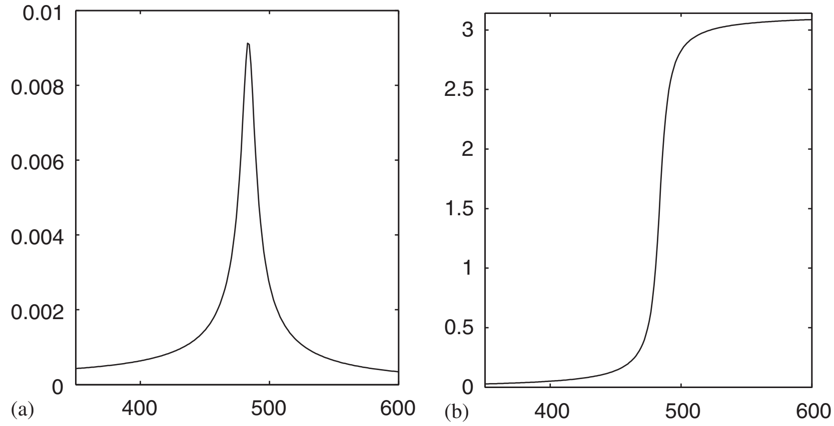

Fig. 4. Example of resonance peak for the first eigenmode of a plate vibrating in a fluid, corresponding to the 6th row in Table 1 marked with (*). The plots correspond to the displacement at $x = 1$ when the plate is driven by a sinusoid of amplitude 1 which is also the reference for the phase angles. Part (a) shows the amplitude of this displacement and but (b) shows the phase, as a function of the frequency ω.

is dimensionless, and is obtained by dividing the dimensional thickness by L, as defined in Section 3. Consequently h/L in Eq. (80) is actually the dimensional thickness divided by L^2.

These results show the expected behaviour for all combinations of parameters. The resonance frequency in a fluid is always lower than in vacuum, getting lower with increasing fluid density. The effect of the viscosity on the resonance frequency is very small, while the combined effect of density and viscosity on the quality factor is quite important.

The location of the driving force, x_f, has an effect only on the amplitude of the displacement, giving a much wider amplitude when closer to the free end, $x = 1$. Also the two cases for modelling the clamping here described, provide quite a different answer, especially in terms of the resonance frequency. For the semi-infinite rigid plate (Section 4), frequencies are much closer together for fluids ‘f1’ and ‘f2’, while for the finite pressure clamping approximation (Section 2), the resonance frequencies for these two fluids are much further from each other. The semi-infinite rigid plate model described in Section 4 constrains much more the behaviour of the plate and the influence of the fluid loses importance with respect to the other method.

When coding this method, it is very important to give care and attention to the complex numbers that build the system of equations to be solved. One can check the numerical results by looking at the phases of the displacement w, and of A and B ($\arg(w) = \arg(A) - \pi/2 = \arg(B) + \pi/2$).

6. Conclusion

We have presented analytical expressions for the fluid reaction on a wide cantilever plate vibrating in a viscous fluid. These equations allow one to couple the structural motion of the cantilever and the fluid motion in order to obtain the frequency response of the plate. The method explained above can be extended to apply to situations where the frequency is not so high at the expense of deriving more terms for the fluid reaction as a function of β.

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Appendix A. Product decomposition of function $H(\zeta)$

The functions H_+ and H_- from Eq. (50) are constructed by taking first the logarithm,

$$\ln H = \ln(H_+H_-) = \ln H_+ + \ln H_- \tag{A.1}$$

and using the Cauchy integral representation for $\ln H_+$ and $\ln H_-$ giving

$$\ln H_\pm = \pm\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{+\infty}\frac{\ln H(z)}{z-\zeta}\,\mathrm{d}z. \tag{A.2}$$

The principal value of $\ln H(z)$ is taken and $\operatorname{Im}\zeta>0$ for H_+ and $\operatorname{Im}\zeta<0$ for H_-. Thus H_+ is analytic in the positive imaginary half-plane and H_- is analytic in the negative imaginary half-plane. The integral expressions have indented contours of integration when $\operatorname{Im}\zeta\to 0$.

Finally, functions $H_+(\zeta)$ and $H_-(\zeta)$ are obtained after identifying the branch points and choosing convenient branch cuts for the integration

$$H_+(\zeta) = \exp\left(-\frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\pi}\int_0^1\frac{\arcsin(r)}{r\mathrm{e}^{\mathrm{i}\pi/4}+\zeta}\,\mathrm{d}r\right), \tag{A.3}$$

$$H_-(\zeta) = \exp\left(-\frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\pi}\int_0^1\frac{\arcsin(r)}{r\mathrm{e}^{\mathrm{i}\pi/4}-\zeta}\,\mathrm{d}r\right), \tag{A.4}$$

where the function $\arcsin(r)$ returns a value in the interval $[0,\pi/2]$.

Since the main interest lies in the free edge of the plate, corresponding to ζ tending to infinity, the expansion of functions H_+ and H_- for ζ tending to infinity are calculated, i.e.,

$$H_+(\zeta)|_{\zeta\to\infty}\sim 1-\frac{(\pi-2)}{2\pi}\mathrm{e}^{\mathrm{i}\pi/4}\frac{1}{\zeta}+\cdots, \tag{A.5}$$

$$H_-(\zeta)|_{\zeta\to\infty}\sim 1+\frac{(\pi-2)}{2\pi}\mathrm{e}^{\mathrm{i}\pi/4}\frac{1}{\zeta}+\cdots \tag{A.6}$$

Appendix B. Sum decomposition of function $S(\zeta)$

The functions S_+ and S_- from Eq. (51) are constructed by using the Cauchy integral representation giving

$$S_\pm = \pm\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{+\infty}\frac{S(z)}{z-\zeta}\,\mathrm{d}z. \tag{B.1}$$

$\operatorname{Im}\zeta>0$ for S_+ and $\operatorname{Im}\zeta<0$ for S_-. Thus S_+ is analytic in the positive imaginary half-plane and S_- is analytic in the negative imaginary half-plane. The integral expressions have indented contours of integration when $\operatorname{Im}\zeta\to 0$.

Finally, the function $S_-(\zeta)$ is obtained after identifying the branch points and choosing convenient branch cuts for the integration

$$S_-(\zeta) = -\frac{e^{i\pi/4}}{\pi}\int_0^1 \sqrt{\frac{1+r}{1-r}}\frac{H_+(re^{i\pi/4})}{re^{i\pi/4}-\zeta}dr \tag{B.2}$$

and

$$H_+(re^{i\pi/4}) = \exp\left(-\frac{1}{\pi}\int_0^1 \frac{\arcsin(r')}{r'+r}dr'\right), \tag{B.3}$$

where the function $\arcsin(r')$ returns a value in the interval $[0, \pi/2]$.

Since the main interest lies in the free edge of the plate, corresponding to ζ tending to infinity, the expansion of function S_- at ζ tending to infinity is calculated as

$$S_-(\zeta)|_{\zeta\to\infty} \sim \frac{e^{i\pi/4}}{\sqrt{2}}\frac{1}{\zeta} + \frac{i}{2}\frac{1}{\zeta^2} + \cdots \tag{B.4}$$

The expansion for S_+ for ζ tending to infinity is obtained by subtracting the expansion for S_- (from Eq. (B.4) from the expansion of $S(\zeta)$),

$$S_+(\zeta)|_{\zeta\to\infty} \sim e^{i\pi/4}\left(1-\frac{1}{\sqrt{2}}\right)\frac{1}{\zeta} + i\left(\frac{1}{\pi}-\frac{1}{2}\right)\frac{1}{\zeta^2} + \cdots \tag{B.5}$$

Appendix C. Green's function for the equation of motion of the plate

The Green's function for Eq. (57), $G(x, x')$ satisfies the following continuity conditions at $x = x'$ with G, $\partial G/\partial x$ and $\partial^2 G/\partial x^2$ continuous, and the third derivative presents a unit jump

$$\lim_{\varepsilon\to 0^+}\left[\frac{\partial^3 G}{\partial x^3}\right]_{x=x'-\varepsilon}^{x=x'+\varepsilon} = 1. \tag{C.1}$$

Using the symmetry property, we obtain the following Green's function:

$$G(x, x') = \begin{cases} \frac{1}{4c^3}\{\sinh(c(x-x')) - \sin(c(x-x'))\}, & x > x', \\ \frac{-1}{4c^3}\{\sinh(c(x-x')) - \sin(c(x-x'))\}, & x < x'. \end{cases} \tag{C.2}$$

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