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Optimal sinusoidal disturbances damping for singularly perturbed systems with time-delay

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Abstract

This paper considers an optimal damping vibration control problem for linear time-delay singularly perturbed systems affected by external sinusoidal disturbances. By using the slow–fast decomposition theory of singularly perturbed systems, the system is first decomposed into a fast subsystem without time-delay and a slow subsystem with time-delay affected by external sinusoidal disturbances. Then, the successive approximation approach (SAA) and the feedforward compensation techniques are proposed to solve the slow-time scale time-delay optimal control problem and damp the external sinusoidal disturbances, respectively. The conditions of existence and uniqueness of the finite- and infinite-horizon feedforward and feedback composite control (FFCC) laws are presented, and the design approaches are given. The FFCC laws consist of nondelay feedback terms, disturbances compensation terms and a time-delay compensation term which is the limit of the solution sequence of the adjoint vector equations. Simulation examples indicate that the SAA is valid, and the FFCC laws are easy to implement, and more effective with respect to damping the external sinusoidal disturbances than that of the classical feedback composite control (FCC) laws.

1. Introduction

It is well known that almost all practical systems are affected by external disturbances. One example is that the offshore structures [1,2] are exposed to various loads: wind, wave and current in hostile environment. Others include sinusoidal forcing in flight control through wind shear [3] and periodic disturbances in optical and magnetic disk drives [4]. Various reliable approaches to the disturbance rejection and cancellation have been well documented in many literatures. For example, the controller based on internal model in Ref. [5] and the adaptive compensator in Ref. [6] are designed to regulate the output to zero while an internal model structure with adaptive frequency in Ref. [7] is proposed to cancel the periodic disturbances. By using the disturbance decoupling algorithm, a disturbance decoupling feedback controller is given in Ref. [8]. Tang et al.

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[9–11] and Lindquist and Yakubovich [12] proposed feedforward and feedback optimal damping controllers based on quadratic average performance index for linear continuous and discrete systems with sinusoidal disturbances.

On the other hand, time-delay is quite common in practical control systems. The optimal control problems of time-delay systems always attract attention of scientific researches. In resent years, some research results have been obtained in both theory and application fields. For example, Kolmanovsky and Maizenberg [13] investigated a finite-horizon optimal control problem for randomly varying time-delay systems. Cai et al. [14] provided an optimal control method for linear time-delay systems in vibration control. We all know that the optimal control problems for time-delay systems about quadratic cost functional (QCF) generally lead to a two-point boundary value (TPBV) problem involving both delay and advance terms whose exact solution is almost impossible. So looking for an approximate solution to the TPBV problem is one of the important aims of researchers. Recently, many better results in the approximate approach of optimal control for nonlinear and/or time-delay systems have been obtained. Among them, Tang proposed the successive approximation approach (SAA) to obtain the suboptimal control laws for nonlinear systems [15,16] as well as time-delay systems [17].

In recent decades, the theory of optimal control for singularly perturbed systems has been of considerable concern, and well-developed, very efficient optimal control methods for singularly perturbed systems are obtained. For example, Fridman [18,19] studied the nonlinear singularly perturbed optimal control via invariant manifolds and a descriptor system approach, respectively. In [20], Kim et al. presented a composite control law for singularly perturbed bilinear systems via successive Galerkin approximation. Based on the bilinear transformation, Bidani et al. [21] presented an algorithm for solving the optimal control of discrete-time singularly perturbed systems. Unfortunately, we have not seen the research results concerning optimal disturbance damping control for singularly perturbed systems with time-delay.

In this note, the finite- and infinite-horizon quadratic composite control of linear time-delay singularly perturbed systems affected by external sinusoidal disturbances are investigated. On the basis of the singular perturbation theory, the full-order linear time-delay singularly perturbed system is decomposed into reduced slow and fast subsystems, and optimal control laws are designed for each subsystem. By using the SAA [15–17] and the feedforward compensation techniques, we obtain the condition of the existence and uniqueness of the optimal control law of the slow linear time-delay subsystem. Then the feedforward and feedback composite control (FFCC) laws of the original system are designed.

The note is organized as follows. In next section, the problem to be considered will be given. Section 3 will propose in detail the SAA and the design processes of the finite- and infinite-horizon FFCC laws. The validity of the FFCC laws will be illustrated by a numerical example in Sections 4. Section 5 concludes this note.

Notation: Throughout this note, the superscript T stands for matrix transposition, R^n denotes the n-dimensional *Euclidean* space, $R^{n\times m}$ is the set of all $n\times m$ real matrices, and $R_T=(t_0, t_f]$.

2. Problem formulation

Consider the linear time-delay singularly perturbed system with external disturbances described by

$$\begin{aligned}\dot{x}(t) &= A_{11}x(t)+A_{12}z(t)+A_{13}x(t-\tau)+B_1u(t)+Dv(t),\\ \varepsilon\dot{z}(t) &= A_{21}x(t)+A_{22}z(t)+A_{23}x(t-\tau)+B_2u(t), \quad t>0,\\ x(t) &= \varphi(t), \quad -\tau\leqslant t\leqslant 0, \quad z(0)=z_0,\end{aligned} \tag{1}$$

where $x(t)\in R^n$ and $z(t)\in R^m$ are the state vectors, $u(t)\in R^r$ the control input, τ the positive time-delay, $\varepsilon>0$ the small positive parameter, and $\varphi(t)$ the continuously differentiable initial function, A_{ij}, B_i $(i=1,2; j=1,2,3)$ and D the constant matrices of appropriate dimensions. We assume that the matrix A_{22} is nonsingular. $v\in R^p$ the exogenous known sinusoidal disturbance with the form

$$v(t)=\begin{bmatrix}\alpha_1\sin(\omega_1 t+\varphi_1) & \alpha_2\sin(\omega_2 t+\varphi_2) & \cdots & \alpha_p\sin(\omega_p t+\varphi_p)\end{bmatrix}^T. \tag{2}$$

The finite-horizon QCF is given by

$$J = \frac{1}{2}\begin{bmatrix} x(t_f) \\ z(t_f) \end{bmatrix}^T F \begin{bmatrix} x(t_f) \\ z(t_f) \end{bmatrix} + \frac{1}{2}\int_0^{t_f} \left(\begin{bmatrix} x(t) \\ z(t) \end{bmatrix}^T Q \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + u^T(t)Ru(t) \right) \mathrm{d}t. \tag{3}$$

For the case of infinite-horizon, we can choose the quadratic average cost functional as

$$J = \lim_{t_f \to \infty} \frac{1}{t_f} \int_0^{t_f} \left\{ \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}^T Q \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + u(t)^T Ru(t) \right\} \mathrm{d}t, \tag{4}$$

where $R \in R^{r \times r}$ is the positive definite matrix, F, $Q \in R^{(n+m)(n+m)}$ the positive semi-definite matrices with the block diagonal structure as

$$F = \begin{bmatrix} F_1 & F_2 \\ F_2^T & F_3 \end{bmatrix}, \quad Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix}.$$

Without loss of generality, we assume that $Q_2 = 0$ and $F_2 = 0$ in the following discussion. Our objective is to find the optimal control u^* such that the finite-horizon QCF in Eq. (3) or the infinite-horizon quadratic average cost functional in Eq. (4) is minimized.

3. SAA and damping FFCC laws design

In this section, we will present the SAA and discuss the design approaches of the finite- and infinite-horizon FFCC laws, respectively. In order to obtain the main results, the following technical lemmas will be very useful in the sequel.

Lemma 1. [15]: *Consider autonomous nonlinear systems described by*

$$\dot{y}(t) = G(t)y(t) + f(y(t), t), \quad t \in R_T \quad y(t_0) = y_0, \tag{5}$$

where $y \in R^n$ *is the state vector,* y_0 *is the initial state vector,* $f: C(R^n \times R_T) \to U \in R^n$, $f(0,t) \equiv 0$, $G: C(R_T) \to R^{n \times n}$. *Assume that* f *satisfies the Lipschitz conditions on* $R^n \times R_T$. *Define the vector function sequence* $\{y^{(k)}(t)\}$ *as:*

$$\begin{aligned} y^{(0)}(t) &= \Phi(t, t_0)y_0, \quad t \geqslant t_0, \\ y^{(k)}(t) &= \Phi(t, t_0)y_0 + \int_{t_0}^{t} \left[\Phi(t, r)f(y^{(k-1)}, r)\right] \mathrm{d}r, \quad t \in R_T \\ y^{(k)}(t_0) &= y_0, \quad k = 1, \ 2, \ldots, \end{aligned} \tag{6}$$

where $\Phi(t, t_0)$ *is state-transition matrix corresponding to* $G(t)$. *Then the sequence* $\{y^{(k)}(t)\}$ *uniformly converges to the solution of Eq.* (5).

Lemma 2. [22]: *Assume that* $H \in R^{m \times m}$, $E \in R^{n \times n}$, $L \in R^{n \times m}$, *then the Sylvester matrix equation*

$$EX + XH + L = 0, \tag{7}$$

has unique solution X *if and only if* $\lambda + \mu \neq 0$ *for any* $\lambda \in \sigma(E)$ *and* $\mu \in \sigma(H)$ *with* $\sigma(\cdot)$ *denoting the spectra of matrix.*

3.1. Finite-time damping FFCC law

In view of the slow–fast decomposition theory of singular perturbation, the fast optimal subproblem of order m is given by

$$\begin{aligned} \varepsilon \dot{z}_f(t) &= A_{22} z_f(t) + B_2 u_f(t), \quad t > 0, \\ z_f(0) &= z_0 - z_s(0), \end{aligned} \tag{8}$$

and the QCF as

$$J_f = \frac{1}{2} z_f^T(t_f) F_3 z_f(t_f) + \frac{1}{2} \int_0^{t_f} \left[z_f^T(t) Q_3 z_f(t) + u_f^T(t) R u_f(t) \right] \mathrm{d}t. \tag{9}$$

We all know that, the fast optimal control law is given by

$$u_f^*(t) = -R^{-1} B_2^T P_f(t) z_f(t), \tag{10}$$

where $P_f(t)$ satisfies the differential Riccati equation

$$\begin{aligned} -\dot{P}_f(t) &= A_{22}^T P_f(t) + P_f(t) A_{22} - P_f(t) S_2 P_f(t) + Q_3, \\ P_f(t_f) &= F_3, \end{aligned} \tag{11}$$

with $S_2 = B_2 R^{-1} B_2^T$.

The slow optimal subproblem of order n is given by

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + A_3 x(t-\tau) + B_0 u_s(t) + D v(t), \quad 0 < t \leqslant t_f \\ x(t) &= \varphi(t), \quad -\tau \leqslant t \leqslant 0, \end{aligned} \tag{12}$$

with

$$z_s(t) = -A_{22}^{-1}[A_{21} x(t) + A_{23} x(t-\tau) + B_2 u_s(t)], \tag{13}$$

and the QCF

$$J_s = \frac{1}{2} x^T(t_f) F_0 x(t_f) + \frac{1}{2} \int_0^{t_f} \left[x^T(t) Q_0 x(t) + 2 u_s^T(t) D_s x(t) + u_s^T(t) R_s u_s(t) \right] \mathrm{d}t, \tag{14}$$

where

$$\begin{aligned} &A_0 = A_{11} - A_{12} A_{22}^{-1} A_{21}, \quad A_3 = A_{13} - A_{12} A_{22}^{-1} A_{23}, \\ &B_0 = B_1 - A_{12} A_{22}^{-1} B_2, \quad F_0 = F_1 + A_{21}^T A_{22}^{-T} F_3 A_{22}^{-1} A_{21}, \\ &Q_0 = Q_1 + A_{21}^T A_{22}^{-T} Q_3 A_{22}^{-1} A_{21}, \\ &D_s = B_2^T A_{22}^{-T} Q_3 A_{22}^{-1} A_{21}, \quad R_s = R + B_2^T A_{22}^{-T} Q_3 A_{22}^{-1} B_2. \end{aligned} \tag{15}$$

The slow optimal control law has the form

$$u_s^*(t) = -R_s^{-1}[D_s x(t) + B_0^T \lambda(t)], \tag{16}$$

where $\lambda(t) \in R^n$ satisfies the TPBV problem:

$$\begin{aligned} \dot{x}(t) &= A_s x(t) + A_3 x(t-\tau) - S_0 \lambda(t) + D v(t), \\ -\dot{\lambda}(t) &= Q_s x(t) + A_s^T \lambda(t) + \mathrm{sgn}(t) A_3^T \lambda(t+\tau), \\ \lambda(t_f) &= F_0 x(t_f), \\ x(t) &= \varphi(t), \quad -\tau \leqslant t \leqslant 0, \end{aligned} \tag{17}$$

with

$$\begin{aligned} &A_s = A_0 - B_0 R_s^{-1} D_s, \quad S_0 = B_0 R_s^{-1} B_0^T, \\ &Q_s = Q_0 - D_s^{\mathrm{T}} R_s^{-1} D_s, \\ &\mathrm{sgn}(t) = \begin{cases} 1, & 0 < t \leqslant t_f - \tau, \\ 0, & t_f - \tau < t \leqslant t_f. \end{cases} \end{aligned} \tag{18}$$

In order to solve the TPBV problem in Eq. (17), let

$$\lambda(t) = P_s(t) x(t) + P_v(t) v(t) + P_\omega(t) v_\omega(t) + g(t), \tag{19}$$

where $g{:}C[0,t_f]\to R^n$ is an adjoint vector to be solved, and

$$v_\omega(t) = -\Omega\begin{bmatrix} v_1\left(t-\dfrac{\pi}{2\omega_1}\right) & v_2\left(t-\dfrac{\pi}{2\omega_2}\right) & \cdots & v_p\left(t-\dfrac{\pi}{2\omega_p}\right)\end{bmatrix}^{\mathrm{T}}, \tag{20}$$

with $\Omega = \mathrm{diag}\{\omega_1, \omega_2, \ldots, \omega_p\}$. $P_s(t)$ is the unique positive semi-definite solution matrix of the following Riccati matrix differential equation:

$$\begin{aligned} -\dot{P}_s(t) &= A_s^T P_s(t) + P_s(t)A_s - P_s(t)S_0 P_s(t) + Q_s, \\ P_s(t_f) &= F_0. \end{aligned} \tag{21}$$

From Eq. (20), we obtain

$$\dot{v}(t) = v_\omega(t), \quad \dot{v}_\omega(t) = -\Omega^2 v(t). \tag{22}$$

Thus, the slow optimal $u_s^*(t)$ in Eq. (16) can be rewritten as

$$u_s^*(t) = -R_s^{-1}\{[D_s + B_0^T P_s(t)]x(t) + B_0^T[P_v(t)v(t) + P_\omega(t)v_\omega(t) + g(t)]\}. \tag{23}$$

Substitution of Eq. (19) into Eq. (17), together with Eqs. (21) and (22), we obtain the matrix differential equations with subject to $P_v(t)$ and $P_\omega(t)$:

$$\begin{aligned} -\dot{P}_v(t) &= [A_s^T - P_s(t)S_0]P_v(t) + P_s(t)D - P_\omega(t)\Omega^2, \\ P_v(t_f) &= 0, \end{aligned} \tag{24}$$

$$\begin{aligned} -\dot{P}_\omega(t) &= [A_s^T - P_s(t)S_0]P_\omega(t) + P_v(t), \\ P_\omega(t_f) &= 0, \end{aligned} \tag{25}$$

and a new TPBV problem described by the adjoint vector differential equation

$$\begin{aligned} \dot{g}(t) = {} & [S_0P_s(t) - A_s]^T g(t) - P_s(t)A_3x(t-\tau) \\ & - \mathrm{sgn}(t)A_3^T[P_s(t+\tau)x(t+\tau) + P_v(t+\tau)v(t+\tau) \\ & + P_\omega(t+\tau)v_\omega(t+\tau) + g(t+\tau)], \\ g(t_f) = {} & 0, \end{aligned} \tag{26}$$

and the state differential equation

$$\begin{aligned} \dot{x}(t) = {} & [A_s - S_0P_s(t)]x(t) + A_3x(t-\tau) \\ & + [D - S_0P_v(t)]v(t) - S_0P_\omega(t)v_\omega(t) - S_0g(t), \quad 0 < t \leqslant t_f, \\ x(t) = {} & \varphi(t), \quad -\tau \leqslant t \leqslant 0. \end{aligned} \tag{27}$$

Obviously, in order to obtain the optimal control law in Eq. (23) of slow subsystem in Eq. (12), we need solve the matrices $P_s(t)$, $P_v(t)$, $P_\omega(t)$, and the adjoint vector $g(t)$. By using the approximation approaches to matrix ordinary differential equations, we can get the numerical solution $P_s(t)$ of Eq. (21). Further, we get the approximation solutions $P_v(t)$ and $P_\omega(t)$ from Eqs. (24) and (25), respectively. Unfortunately, note that both time-delay and time-advance terms are involved in the TPBV problem described by Eqs. (26) and (27), the exact solution of this TPBV problem is almost impossible. In the following, we propose the SAA to solve it.

Construct the following adjoint vector differential equation sequence as follows:

$$\begin{aligned} \dot{g}^{(k)}(t) = {} & [S_0P_s(t) - A_s]^T g^{(k)}(t) - P_s(t)A_3x^{(k-1)}(t-\tau) \\ & - \mathrm{sgn}(t)A_3^T[P_s(t+\tau)x^{(k-1)}(t+\tau) + P_v(t+\tau)v(t+\tau) \\ & + P_\omega(t+\tau)v_\omega(t+\tau) + g^{(k-1)}(t+\tau)], \\ g^{(k)}(t_f) = {} & 0, \quad k = 1, 2, \ldots, \\ g^{(0)}(t) = {} & 0, \quad 0 \leqslant t \leqslant t_f, \end{aligned} \tag{28}$$

where $x^{(k)}$ satisfy the state vector differential equation as follows:

$$\begin{aligned}\dot{x}^{(k)}(t) &= [A_s - S_0P_s(t)]x^{(k)}(t) + A_3x^{(k)}(t-\tau)\\ &\quad + [D - S_0P_v(t)]v(t) - S_0P_\omega(t)v_\omega(t) - S_0g^{(k)}(t), \quad 0<t\leqslant t_f,\\ x^{(k)}(t) &= \varphi(t), \quad -\tau\leqslant t\leqslant 0, \quad k=0,1,2,\ldots\ . \end{aligned} \tag{29}$$

Correspondingly, the slow optimal control sequence constructed as

$$u_s^{(k)}(t) = -R_s^{-1}\left\{\left[D_s + B_0^TP_s(t)\right]x^{(k)}(t) + B_0^T\left[P_v(t)v(t) + P_\omega(t)v_\omega(t) + g^{(k)}(t)\right]\right\}. \tag{30}$$

For the kth optimal problem, optimal state trajectory and optimal control law of the slow subsystem in Eq. (12) are $x^{(k)}(t)$ and $u_s^{(k)}(t)$, respectively. We present the following theorem.

Theorem 1. *Consider the slow time scale optimal control problem described by Eqs.* (12) *and* (14). *Assume that* $\{x^{(k)}(t)\}$ *and* $\{u_s^{(k)}(t)\}$ *are the solution sequences of Eqs.* (29) *and* (30), *respectively. Then* $\{u_s^{(k)}(t)\}$ *uniformly converge to the optimal control law* $u_s^*(t)$ *formulated as*

$$u_s^*(t) = -R_s^{-1}\left\{\left[D_s + B_0^TP_s(t)\right]x(t) + B_0^T\left[P_v(t)v(t) + P_\omega(t)v_\omega(t) + g^{(\infty)}(t)\right]\right\}, \tag{31}$$

where $g^{(\infty)}(t)\triangleq \lim_{k\to\infty} g^{(k)}(t)$, and $g^{(k)}(t)$ is given by Eq. (28).

Proof. By the Lemma 1, together with the fact that the sinusoidal disturbances vector $v(t)$ is independent of iteration variable k in the sequences $\{x^{(k)}(t)\}$ and $\{u_s^{(k)}(t)\}$, we have

$$g(t) = \lim_{k\to\infty} g^{(k)}(t), \quad x(t) = \lim_{k\to\infty} x^{(k)}(t). \tag{32}$$

Note from Eq. (30) that the control sequence $\{u_s^{(k)}(t)\}$ is only related to $\{x^{(k)}(t)\}$ and $\{g^{(k)}(t)\}$, thus, the sequence $\{u_s^{(k)}(t)\}$ uniformly converges to the slow optimal control law $u_s^*(t)$, namely

$$u_s^*(t) = \lim_{k\to\infty} u_s^{(k)}(t). \tag{33}$$

In view of Eqs. (30) and (32), we can obtain optimal control law in Eq. (31) directly. The proof is complete.

On the other hand, from Eq. (10) we obtain the fast optimal control law $u_f^*(t)$ of the fast subsystem. Further, we get the damping FFCC law $u_c(t)$ of the original time-delay singularly perturbed system in Eq. (1) as follows:

$$\begin{aligned}u_c(t) &= K_x(t)x(t) + K_z(t)z(t) + K_\tau(t)x(t-\tau)\\ &\quad + K_c(t)\left[P_v(t)v(t) + P_\omega(t)v_\omega(t) + g^{(\infty)}(t)\right],\end{aligned} \tag{34}$$

where

$$\begin{aligned}K_z(t) &= -R^{-1}B_2^TP_f(t),\\ K_\tau(t) &= K_z(t)A_{22}^{-1}A_{23},\\ K_x(t) &= K_z(t)A_{22}^{-1}A_{21} - \left[I + K_z(t)A_{22}^{-1}B_2\right]R_s^{-1}\left[D_s + B_0^TP_s(t)\right],\\ K_c(t) &= -\left[I + K_z(t)A_{22}^{-1}B_2\right]R_s^{-1}B_0^T.\end{aligned} \tag{35}$$

Based on the above detailed analyses, we have

Theorem 2. *For the finite-horizon quadratic optimal sinusoidal disturbances damping control problem of the standard linear time-delay singularly perturbed systems described by Eqs.* (1) *and* (3), *there exists uniquely FFCC law formulated by Eqs.* (34) *and* (35), *where* $P_s(t)$ *and* $P_f(t)$ *are the unique positive semi-definite solution matrices of Eqs.* (21) *and* (11), *respectively.* $P_v(t)$ *and* $P_\omega(t)$ *satisfy the matrix differential Eqs.* (24) *and* (25), *and* $g^{(k)}(t)$ *is determined by Eq.* (28).

3.2. Infinite-horizon damping FFCC law

Similarly, for the case of the infinite-horizon, we have the following results.

Theorem 3. *Consider the infinite-horizon feedforward and feedback optimal sinusoidal disturbances damping control problem of the linear time-delay singularly perturbed system in Eq.* (1) *with subject to the QCF in Eq.* (4). *Assume that the triples* $(A_{22},\ B_2,\ Q_3^{1/2})$ *and* $(A_0,\ B_0,\ Q_s^{1/2})$ *are controllable-observable completely, then there exists uniquely FFCC law* $u_c(t)$ *formulated as*

$$u_c(t) = K_x x(t) + K_z z(t) + K_\tau x(t-\tau) + K_c[P_v v(t) + P_\omega v_\omega(t) + g^{(\infty)}(t)], \tag{36}$$

with

$$\begin{aligned} K_z &= -R^{-1}B_2^T P_f, \quad K_\tau = K_z A_{22}^{-1} A_{23}, \\ K_c &= -[I + K_z A_{22}^{-1} B_2] R_s^{-1} B_0^T, \\ K_x &= K_z A_{22}^{-1} A_{21} - [I + K_z A_{22}^{-1} B_2] R_s^{-1} [D_s + B_0^T P_s], \end{aligned} \tag{37}$$

where P_s *and* P_f *are the unique positive definite solution matrices of the slow algebraic Riccati equation*

$$A_s^T P_s + P_s A_s - P_s S_0 P_s + Q_s = 0, \tag{38}$$

and the fast algebraic Riccati matrix equation

$$A_{22}^T P_f + P_f A_{22} - P_f S_2 P_f + Q_3 = 0, \tag{39}$$

respectively. P_ω *is the unique solution of the Sylvester matrix equation*

$$(P_s S_0 - A_s^T)^2 P_\omega + P_\omega \Omega^2 + P_s D \Omega = 0, \tag{40}$$

and P_v *determined by*

$$P_v = (P_s S_0 - A_s^T) P_\omega. \tag{41}$$

$g^{(k)}(t)$ *satisfies the adjoint vector differential equation*

$$\begin{aligned} \dot{g}^{(k)}(t) &= (S_0 P_s - A_s)^T g^{(k)}(t) - P_s A_3 x^{(k-1)}(t-\tau) \\ &\quad - \operatorname{sgn}(t) A_3^T [P_s x^{(k-1)}(t+\tau) + P_v v(t+\tau) + P_\omega v_\omega(t+\tau) + g^{(k-1)}(t+\tau)] \\ \lim_{t_f \to \infty} & g^{(k)}(t_f) = 0, \quad k = 1, 2, \ldots \\ g^{(0)}(t) &= 0, \quad t \geqslant 0, \end{aligned} \tag{42}$$

where $x^{(k)}(t)$ *satisfies the state vector differential equation as follows:*

$$\begin{aligned} \dot{x}^{(k)}(t) &= (A_s - S_0 P_s) x^{(k)}(t) + A_3 x^{(k)}(t-\tau) \\ &\quad + (D - S_0 P_v) v(t) - S_0 P_\omega v_\omega(t) - S_0 g^{(k)}(t), \quad t > 0, \\ x^{(k)}(t) &= \varphi(t), \quad -\tau \leqslant t \leqslant 0, \quad k = 0, 1, 2, \ldots\ . \end{aligned} \tag{43}$$

Proof. Let

$$\lambda(t) = P_s x(t) + P_v v(t) + P_\omega v_\omega(t) + g(t). \tag{44}$$

Proceeding in a manner similar to that of the case of finite-horizon in section A, we obtain Eqs. (36), (38)–(43), which are analogs of Eqs. (34), (21), (11), (25), (24), (28) and (29), respectively.

Note that the triples $(A_{22},\ B_2,\ Q_3^{1/2})$ and $(A_0,\ B_0,\ Q_s^{1/2})$ are controllable-observable completely, then the algebraic *Riccati* matrix Eqs. (38) and (39) have unique positive definite matrix solutions P_s and P_f, respectively.

On the other hand, from the regulator theory of linear system, it follows that

$$\mathrm{Re}\,\gamma < 0, \quad \gamma \in \sigma(A_s - S_0 P_s). \tag{45}$$

Note the fact that $\mathrm{Re}\,\mu \geqslant 0$ for any $\mu \in \sigma(\Omega^2)$, together with Eq. (45), we can obtain easily that $\lambda + \mu \neq 0$ for any $\lambda \in \sigma[(A_s - S_0 P_s)^2]$ and $\mu \in \sigma(\Omega^2)$. Thus, in view of Lemma 2, the *Sylvester* Eq. (40) has unique solution P_ω. Further, we obtain the matrix P_v directly from Eq. (41).

Consequently, by the Lemma 1, $g^{(k)}(t)$ are uniquely determined by Eqs. (42) and (43). And the infinite-horizon damping FFCC law is obtained uniquely by Eqs. (36) and (37). This completes the proof.

Remark 1. In practical applications, finding the FFCC laws in Eqs. (34) and (36) are almost impossible in the case of $k \to \infty$. By replacing ∞ with integer M in Eqs. (34) and (36), we may obtain the finite-horizon approximate FFCC law

$$u_M(t) = K_x(t)x(t) + K_z(t)z(t) + K_\tau(t)x(t-\tau) + K_c(t)\left[P_v(t)v(t) + P_\omega(t)v_\omega(t) + g^{(M)}(t)\right], \tag{46}$$

and the infinite-horizon FFCC law

$$u_M(t) = K_x x(t) + K_z z(t) + K_\tau x(t-\tau) + K_c\left[P_v v(t) + P_\omega v_\omega(t) + g^{(M)}(t)\right], \tag{47}$$

respectively, where the iteration times M is determined by the given tolerance error bound α, $0 < \alpha < 1$.

Remark 2. In Eqs. (46) and (47), the action of the approximate terms $K_c g^{(M)}$ is to compensate the effect of the state delay terms to the system, while the terms $K_c(P_v v + P_\omega v_\omega)$ compensate the external sinusoidal disturbances. Especially, if $P_v(t) = P_\omega(t) \equiv 0$ in Eq. (46) and $P_v = P_\omega = 0$ in Eq. (47), then we, respectively, obtain the finite- and infinite-horizon feedback composite control (FCC) laws with the forms:

$$u_M(t) = K_x(t)x(t) + K_z(t)z(t) + K_\tau(t)x(t-\tau) + K_c(t)g^{(M)}(t), \tag{48}$$

$$u_M(t) = K_x x(t) + K_z z(t) + K_\tau x(t-\tau) + K_c g^{(M)}(t). \tag{49}$$

4. A numerical example

To demonstrate the feasibility and effectiveness of the proposed FFCC, a numerical example is carried out in this section. We consider the optimal damping control problems described by Eqs. (1) and (4) with the specific matrices:

$$\begin{gathered}
A_{11} = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} -3 & 0.5 \\ -1 & -2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \\
A_{21} = \begin{bmatrix} -0.5 & 0.8 \\ 0 & -0.4 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -1 & 0 \\ 0 & -0.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\
A_{13} = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad A_{23} = \begin{bmatrix} 0.2 & 0.1 \\ -0.5 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\end{gathered} \tag{50}$$

The initial conditions are $\varphi(t) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$, $-\tau \leqslant t \leqslant 0$, $z(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$, the singularly perturbed parameter ε is to take 0.05, and the external sinusoidal disturbances is given by

$$v(t) = \left[\sin\left(\frac{2}{3}\pi t\right) \quad 0.6\sin\left(\frac{5}{6}\pi t\right)\right]^T, \quad t \geqslant 0. \tag{51}$$

The QCF is chosen as

$$J = \int_0^{50} \left\{ \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} Q_1 & 0 \\ 0 & Q_3 \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + 5u^2(t) \right\} \mathrm{d}t, \tag{52}$$

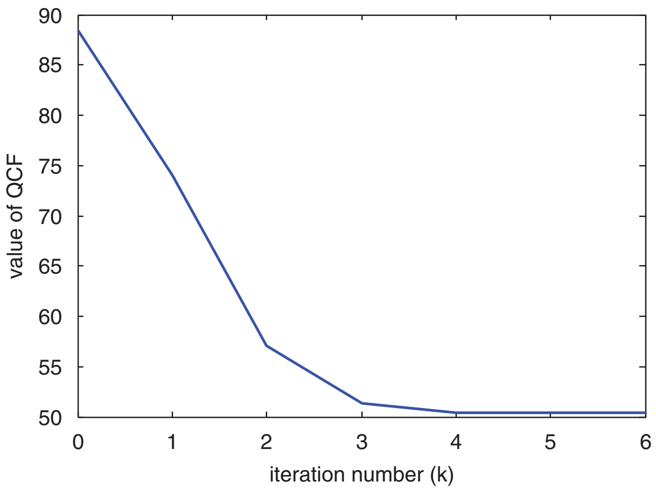

Fig. 1. Value of QCF J_k versus iteration number k.

where

$$Q_1 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is easy to obtain that

$$P_s = \begin{bmatrix} 7.3954 & -1.0168 \\ -1.0168 & 1.6490 \end{bmatrix}, \quad P_f = \begin{bmatrix} 1.0000 & 0 \\ 0 & 0.8541 \end{bmatrix},$$
$$P_v = \begin{bmatrix} 1.3752 & -0.4887 \\ -0.1409 & 0.2844 \end{bmatrix}, \quad P_\omega = \begin{bmatrix} 1.4309 & -0.0016 \\ -0.2279 & 0.1879 \end{bmatrix}.$$

For the time-delay $\tau = 0.2$, the curve of QCF J_k versus the iteration times k is shown in Fig. 1. It is shown that when the iteration times increase, the values of QCF will converge to 50.4217, which is the optimal value of QCF.

Letting the tolerance error bound to be $\alpha = 0.001$, then we have $|(J_6 - J_5)/J_6| < \alpha$. Thus $u_6(t)$ may be considered as a suboptimal control law. When the sixth FFCC law in Eq. (47) and the corresponding FCC law in Eq. (49) are adopted to compare the effectiveness of external sinusoidal disturbances damping, the simulation results of the composite control variable u_c, the corresponding slow state components x_1, x_2, and the fast state components z_1, z_2 are presented in Fig. 2, where the solid lines for the suboptimal trajectories of the FFCC law, while the dash-dotted lines for the FCC law.

From Figs. 1 and 2, we can see clearly that the SAA proposed in this note is valid for the optimal control problem for the singularly perturbed linear time-delay systems, and preserves very good convergence in this example. Moreover, the control effect of the FFCC is much better than that of the FCC, and the former is more effective with respect to rejecting the external sinusoidal disturbances than that of the latter.

5. Conclusion

In this paper, the approximation method for the finite- and infinite-horizon composite damping control scheme of the linear time-delay singularly perturbed systems affected by external sinusoidal disturbances is investigated. The optimization problem of the linear time-delay singularly perturbed systems is replaced by a nondelay sequence of the singularly perturbed optimization problems via the SAA, and the feedforward and

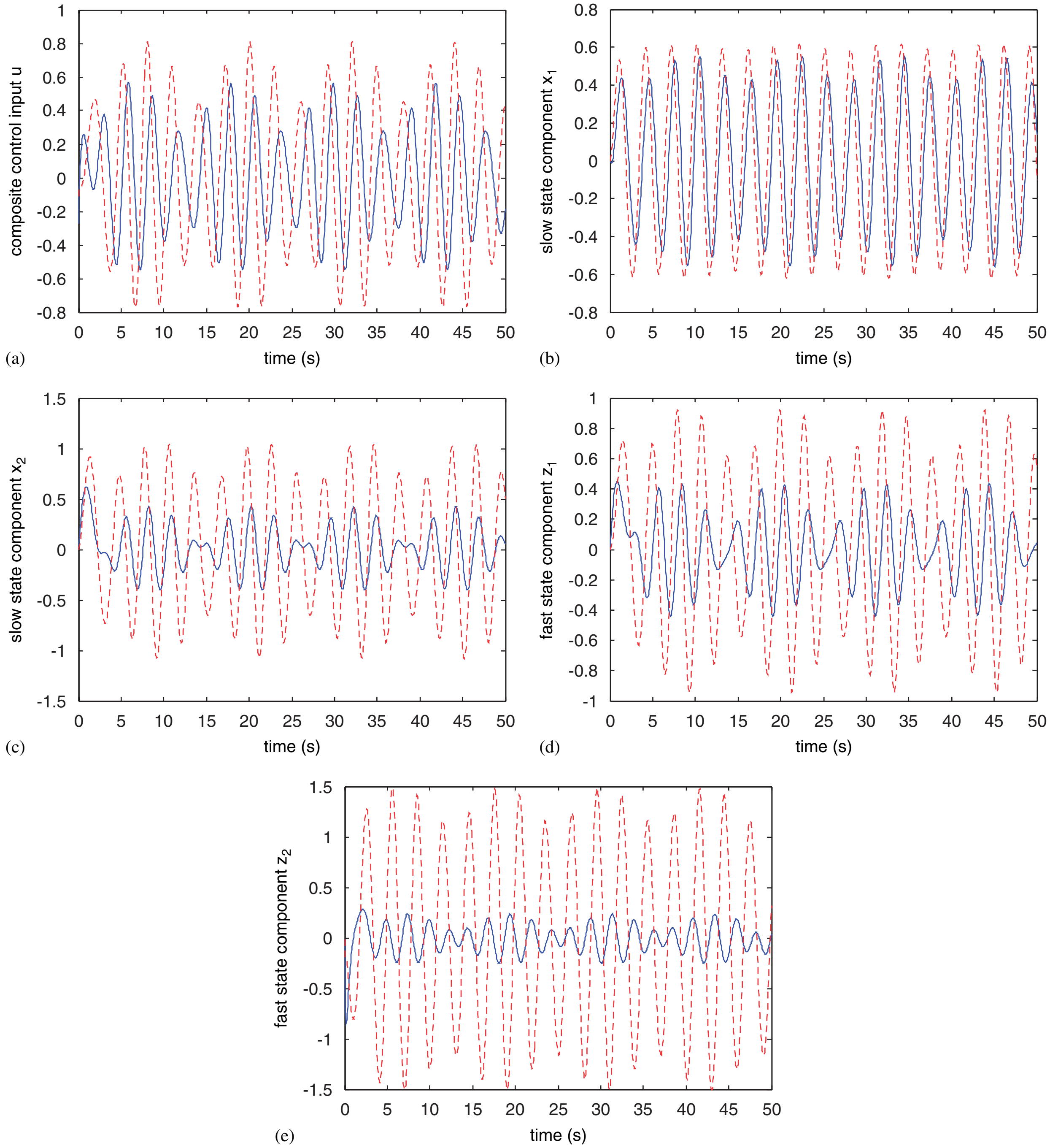

Fig. 2. Disturbances damping comparison curves of the system as $\tau = 0.2$: (a) curves of composite control input u_c; (b) curves of slow state component x_1; (c) curves of slow state component x_2; (d) curves of fast state component z_1; and (e) curves of fast state component z_2: (——) FFCC and (- - - - - -) FCC.

feedback optimal control technique is proposed to reject the external sinusoidal disturbances. This method avoids ill-defined numerical TPBV problem and reduces the size of computations. On the other hand, it is shown that the FFCC laws proposed in this paper are effective and easy to implement, and more effective about external sinusoidal disturbances.

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