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Short Communication

Frequency analysis of annular plates with elastic concentric supports by Green's function method

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Abstract

The paper concerns the free axisymmetric vibration of an annular plate with concentric circular elastic supports. The formulation of the problem concerns plates with free outer and inner edges and with an arbitrary number of circular supports. The exact solution is obtained by applying the Green's function method. The Green's function corresponding to the free annular plate is determined in an analytical form. Numerical examples are presented.

1. Introduction

Annular plates are used as components in various mechanical and structural constructions. The vibrational characteristics of plates with internal supports are of practical interest in civil and aerospace engineering. The analysis of free vibrations of annular plates with concentric circular supports is the subject, among others, of papers [1–4]. In Refs. [1–3] the solution to the vibration problem was obtained by using the Rayleigh–Ritz method. The results presented in paper [1] concern the free vibration of plates with different combinations of outer and inner support conditions. The authors of the paper [2] deal with the vibration of a free plate with two intermediate concentric supports. In Ref. [3] a variant of the Rayleigh–Ritz method was used to solve the free vibration problem of annular plates with concentric supports. The numerically obtained values of the frequency parameter are presented. Paper [4] is devoted to vibration analysis of annular and circular plates by an axisymmetric finite-element method.

This paper presents the solution to the problem of free axisymmetric vibration of annular plates with concentric elastic supports using the Green's function method. The Green's function corresponding to a free annular plate is determined by solving an auxiliary problem. The exact solution to the vibration problem is obtained for annular plates with an arbitrary number of concentric supports.

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2. Theory

Consider an annular plate with uniform thickness (Fig. 1). The axisymmetric vibration of the plate is governed by the differential equation:

$$D\frac{\partial}{\partial r}\left\{r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)\right]\right\} = qr - \bar{\rho}hr\frac{\partial^2 w}{\partial t^2}, \tag{1}$$

where w is the transverse displacement of the plate, r is the radial variable, t is the time variable, D is the bending rigidity of the plate, $\bar{\rho}$ is the mass per unit volume, q is the load per unit area and h is the plate thickness. For a plate with elastic concentric supports with radii r_j ($j = 1, 2, \ldots, n$), the function q takes the form:

$$q(r,t) = -\sum_{j=1}^{n} k_j w(r,t)\frac{1}{r}\delta(r - r_j), \tag{2}$$

where δ is the Dirac delta function and k_j's are the stiffness coefficients of the supports.

In case of free vibration one assumes: $w(r,t) = \bar{w}(r)\,\mathrm{e}^{\mathrm{i}\omega t}$. Introducing simultaneously the non-dimensional variables: $\bar{r} = r/a$, $\bar{r}_j = r_j/a$, $W = \bar{w}/a$ and quantities: $\Omega = \sqrt[4]{(a^4\bar{\rho}h\omega^2)/D}$, $K_j = a^2k_j/D$, where a is the radius of the outer edge of the plate, Eq. (1) can be rewritten in the form (the dashes over r are omitted):

$$\frac{\mathrm{d}}{\mathrm{d}r}\left\{r\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}W}{\mathrm{d}r}\right)\right]\right\} - \Omega^4 rW = -\sum_{j=1}^{n} K_j W(r_j)\delta(r - r_j). \tag{3}$$

Eq. (3) is completed by boundary conditions corresponding to the free annular plate. The conditions at the inner ($r = \beta$, where $\beta = b/a$) and outer ($r = 1$) edges are:

$$\left.\frac{\mathrm{d}^2W}{\mathrm{d}r^2} + v\frac{1}{r}\frac{\mathrm{d}W}{\mathrm{d}r}\right|_{r=1} = 0, \quad \left.\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}W}{\mathrm{d}r}\right)\right]\right|_{r=1} = 0, \tag{4a,b}$$

$$\left.\frac{\mathrm{d}^2W}{\mathrm{d}r^2} + v\frac{1}{r}\frac{\mathrm{d}W}{\mathrm{d}r}\right|_{r=\beta} = 0, \quad \left.\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}W}{\mathrm{d}r}\right)\right]\right|_{r=\beta} = 0, \tag{5a,b}$$

where v is the Poisson ratio.

The solution to the problem is obtained by using the Green's function method. The Green's function $G(r, \rho)$ corresponding to the free annular plate is a solution to the equation

$$\frac{\mathrm{d}}{\mathrm{d}r}\left\{r\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}G(r,\rho)\right)\right]\right\} - \Omega^4 rG(r,\rho) = \delta(r - \rho) \tag{6}$$

and satisfies, with respect to variable r, the boundary conditions (4) and (5). This function may be written in the form

$$G(r,\rho) = G_0(r,\rho) + G_1(r,\rho)H(r - \rho), \tag{7}$$

where $G_0(r, \rho)$ is a general solution to the homogeneous equation obtained from Eq. (6) and $G_1(r,\rho)H(r-\rho)$ is a particular solution to Eq. (6). It may be proved that the function $G_1(r,\rho)$ is a solution to the homogenous version of Eq. (6) which satisfies the following conditions:

$$G_1|_{r=\rho} = \left.\frac{\partial G_1}{\partial r}\right|_{r=\rho} = \left.\frac{\partial^2 G_1}{\partial r^2}\right|_{r=\rho} = 0 \quad \text{and} \quad \left.\frac{\partial^3 G_1}{\partial r^3}\right|_{r=\rho} = \frac{1}{\rho}. \tag{8}$$

The general solution to the homogeneous Eq. (6) has the form

$$G_1(r,\rho) = c_1J_0(r\Omega) + c_2I_0(r\Omega) + c_3Y_0(r\Omega) + c_4K_0(r\Omega). \tag{9}$$

The constants c_1–c_4 are determined by using the conditions (8). After transformations, the function $G_1(r,\rho)$ can be written in the form

$$G_1(r,\rho) = \frac{1}{2\Omega^2}\Big(I_0(r\Omega)K_0(\rho\Omega) - I_0(\rho\Omega)K_0(r\Omega) + \frac{\pi}{2}(J_0(r\Omega)Y_0(\rho\Omega) - J_0(\rho\Omega)Y_0(r\Omega))\Big). \tag{10}$$

Function G_0, as a solution to the homogeneous version of Eq. (6), can also be presented in the form:

$$G_0(r,\rho) = C_1 J_0(r\Omega) + C_2 I_0(r\Omega) + C_3 Y_0(r\Omega) + C_4 K_0(r\Omega). \tag{11}$$

The constants C_1–C_4 are determined by taking into account Eqs. (10) and (11) into Eq. (7) and by using boundary conditions (4) and (5). In order to present function G_0 in a simple form, the following functions are introduced:

$$\Phi_1(z) = z(J_0(z)I_1(z) + J_1(z)I_0(z)) - 2(1-\nu)J_1(z)I_1(z),$$

$$\Phi_2(z) = -z(J_0(z)K_1(z) - J_1(z)K_0(z)) + 2(1-\nu)J_1(z)K_1(z),$$

$$\Phi_3(z) = z(Y_0(z)I_1(z) + Y_1(z)I_0(z)) - 2(1-\nu)Y_1(z)I_1(z),$$

$$\Phi_4(z) = -z(Y_0(z)K_1(z) - Y_1(z)K_0(z)) + 2(1-\nu)Y_1(z)K_1(z),$$

$$\Psi_1(u,z) = -(J_0(z\Omega) + K_0(z\Omega)\Phi_1(u\Omega) - I_0(z\Omega)\Phi_2(u\Omega)),$$

$$\Psi_2(u,z) = -(Y_0(z\Omega) + K_0(z\Omega)\Phi_3(u\Omega) - I_0(z\Omega)\Phi_4(u\Omega)),$$

$$\Psi_3(u,z) = I_0(z\Omega) + \frac{\pi}{2}(Y_0(z\Omega)\Phi_1(u\Omega) - J_0(z\Omega)\Phi_3(u\Omega)),$$

$$\Psi_4(u,z) = K_0(z\Omega) + \frac{\pi}{2}(Y_0(z\Omega)\Phi_2(u\Omega) - J_0(z\Omega)\Phi_4(u\Omega))$$

and

$$d = \frac{4}{\pi} + \Phi_1(\Omega)\Phi_4(\beta\Omega) + \Phi_4(\Omega)\Phi_1(\beta\Omega) - \Phi_2(\Omega)\Phi_3(\beta\Omega) - \Phi_3(\Omega)\Phi_2(\beta\Omega).$$

Finally, function G_0 can be written as follows:

$$G_0(r,\rho) = \frac{1}{2\Omega^2 d}\left(\Psi_2(1,r)\Psi_1(\beta,\rho) - \Psi_1(1,r)\Psi_2(\beta,\rho) + \frac{2}{\pi}(\Psi_4(1,r)\Psi_3(\beta,\rho) - \Psi_3(1,r)\Psi_4(\beta,\rho))\right). \tag{12}$$

After using the properties of the Green's function $G(r, \rho)$, the solution to boundary problem (3)–(5) can be presented in the following form:

$$W(r) = -\sum_{j=1}^{n} K_j W(r_j) G(r, r_j). \tag{13}$$

Substituting $r = r_j, j = 1, 2, \ldots, n$ into Eq. (13), a set of n equations is obtained. The system of the equations can be written in a matrix form

$$\boldsymbol{A}\boldsymbol{W} = \boldsymbol{0}, \tag{14}$$

where $\boldsymbol{A} = [a_{ij}]_{1\leqslant i,j\leqslant n}$, $\boldsymbol{W} = [W(r_1)\ldots W(r_n)]^{\mathrm{T}}$ and $a_{ij} = K_i G(r_i, r_j) + \delta_{ij}$, where δ_{ij} is the Kronecker delta. A nontrivial solution of Eq. (14) exists if and only if

$$\det \boldsymbol{A} = 0, \tag{15}$$

This equation is a frequency equation of the vibration problem under consideration. The frequency equation is then solved numerically with respect to the frequency parameter Ω.

3. Numerical examples

In the case of annular plate with two elastic concentric supports ($n = 2$) Eq. (15) takes the form

$$\left[G(r_1, r_1) + \frac{1}{K_1}\right]\left[G(r_2, r_2) + \frac{1}{K_2}\right] - G(r_1, r_2)G(r_2, r_1) = 0. \tag{16}$$

If $K_1 \to \infty$ and/or $K_2 \to \infty$ in Eq. (16), then the frequency equation of the annular plate simply supported at $r = r_1$ and/or $r = r_2$ is obtained. For example, the frequency equation for the annular plate simply supported at the outer edge ($r_2 = 1$) and along an intermediate circle ($r = r_1$, where $\beta \leqslant r_1 < 1$), can be written in the form

$$G(1,1)G(r_1,r_1) - (G(1,r_1))^2 = 0. \tag{17}$$

The first three values of frequency parameters, Ω_i ($i = 1, 2, 3$), obtained as a numerical solution to Eq. (17) for various values of ratio $\beta = b/a$, and various radii of the circular support ($\beta \leqslant r_1 < 1$), are presented in Table 1. The fundamental frequency parameters Ω_1, obtained by using the present method are compared with

Table 1
Frequency parameter values for annular plates free at the inner edge, simply supported at the outer edge and along the intermediate circle with radius r_1 for various values of r_1 and ratio $\beta = b/a$

$\beta = b/a$	r_1								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1		*4.2708*	*4.7897*	*5.3246*	*5.2594*	*4.6568*	*4.1343*	*3.7342*	*3.4270*
	3.8059	4.2708	4.7897	5.3246	5.2594	4.6568	4.1343	3.7342	3.4270
	7.1960	8.0741	8.6300	7.4238	7.2764	8.3803	8.3234	7.4716	6.7894
	10.6296	11.8250	10.5877	11.0828	11.8055	10.3900	11.4374	11.4298	10.3251
0.2			*4.6938*	*5.3407*	*5.5430*	*4.8880*	*4.2748*	*3.8217*	*3.4842*
		4.0963	4.6938	5.3407	5.5430	4.8880	4.2748	3.8217	3.4842
		7.9606	9.0152	8.5808	7.5974	8.6079	9.0703	8.0102	7.1608
		11.8575	13.0443	11.4576	12.9066	11.7136	11.8222	12.6976	11.2797
0.3				*5.3540*	*6.0550*	*5.5765*	*4.7241*	*4.1187*	*3.6925*
			4.5912	5.3540	6.0550	5.5765	4.7241	4.1188	3.6925
			9.0408	10.3056	8.8403	8.9394	10.3667	9.1616	7.9879
			13.5104	14.3676	13.4294	14.4354	12.5928	14.7072	12.8930
0.4					*6.3228*	*6.8126*	*5.6438*	*4.7134*	*4.1068*
				5.3031	6.3228	6.8126	5.6438	4.7134	4.1068
				10.5147	11.9564	9.7886	11.3765	11.0464	9.3118
				15.7385	15.7573	16.4584	15.0911	16.7947	15.2644
0.5						*7.7639*	*7.4200*	*5.7994*	*4.8242*
					6.3280	7.7639	7.4200	5.7994	4.8242
					12.5953	13.7848	12.1058	14.0360	11.3771
					18.8702	17.9966	20.4560	18.0667	18.8464
0.6							*9.9608*	*7.9330*	*6.0802*
						7.8838	9.9608	7.9330	6.0802
						15.7275	15.6278	17.0405	14.8327
						23.5759	22.7524	22.3806	24.6966
0.7								*12.8767*	*8.5636*
							10.4911	12.8767	8.5636
							20.9566	19.3748	21.5711
							31.4249	32.9117	33.5256
0.8									*15.3466*
								15.7191	15.3466
								31.4233	34.0955
								47.1292	44.5148
0.9									31.4209
									62.8351
									94.2501

Frequency parameters in the table written in italic are square roots of the given in Ref. [2].

the results given in Ref. [2]. The frequencies Ω_1 for $r_1 = \beta$, relate to the annular plates with both edges simply supported. All numerical calculations presented here have been performed for the Poisson ratio $\nu = 0.3$.

Assuming $K_2 \to \infty$ in Eq. (16), the frequency equation for an annular plate free inside and simply supported outside with intermediate elastic support is obtained. The first two frequency parameter values, Ω_i ($i = 1, 2$), as functions of the intermediate support location $\xi = (r_1 - \beta)/(1 - \beta)$, were calculated using this equation. The results for various values of the stiffness coefficient K_1 and for $\beta = 0.2$, 0.4, 0.6 and 0.8, are presented in Figs. 2–5. All $\Omega_1(\xi)$ functions assume one local maximum and all $\Omega_2(\xi)$ functions assume two

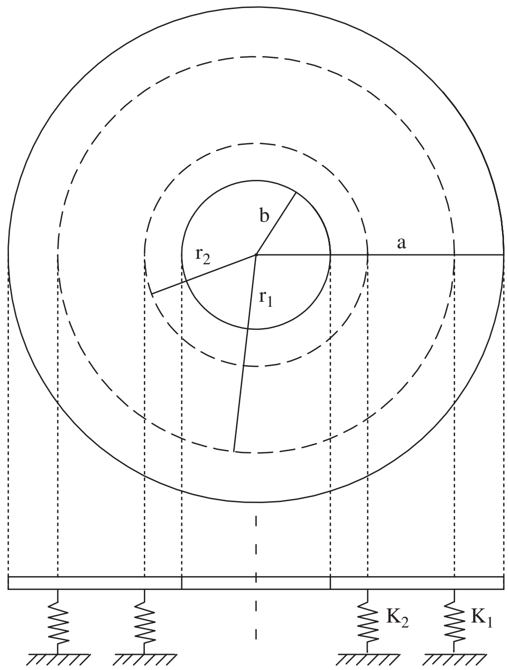

Fig. 1. Annular plate with free edges and two concentric circular supports.

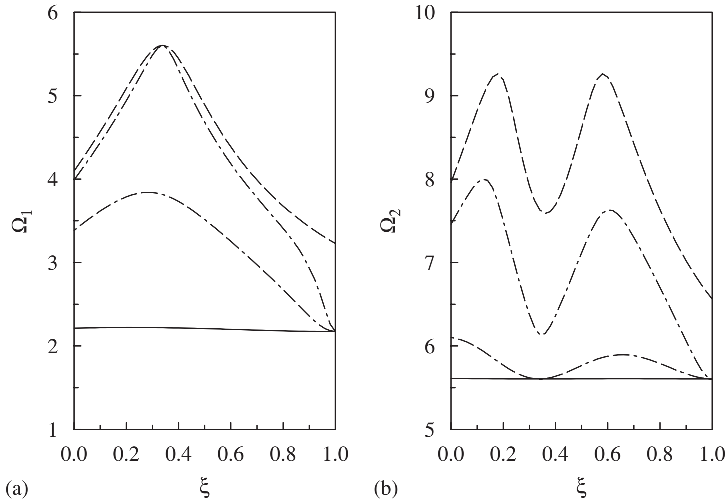

Fig. 2. Frequency parameter values Ω_n for the first two modes of axisymmetric vibration of annular plates simply supported at the outer edge, free at the inner edge and with an intermediate circular elastic support as functions of the ratio $\xi = (r_1 - \beta)/(1 - \beta)$ for $\beta = b/a = 0.2$ and various values of the stiffness coefficient of the support: (———) $K_1 = 1$, (-——) $K_1 = 100$, (-—) $K_1 = 1000$, (--—) $K_1 = 5000$, (---—) $K_1 = 10\,000$, (---) $K_1 = 50\,000$, (———) $K_1 = \infty$.

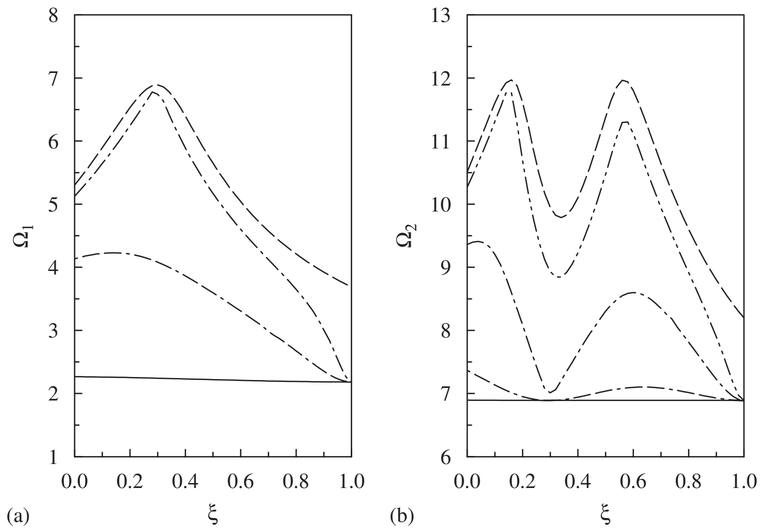

Fig. 3. As Fig. 2, but for $\beta = 0.4$.

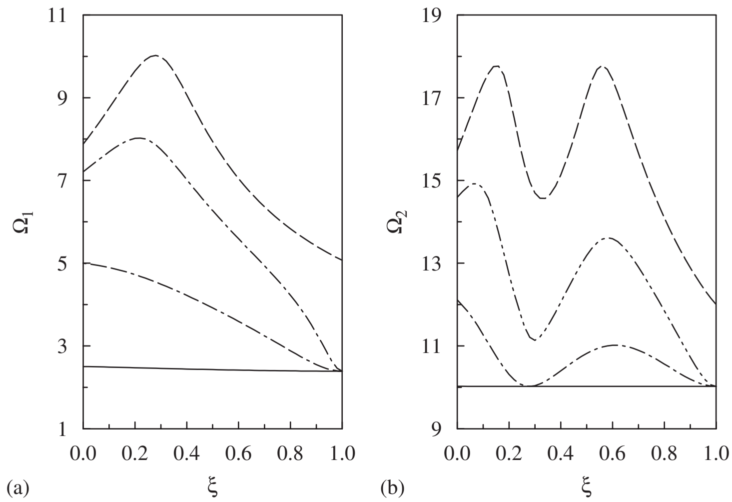

Fig. 4. As Fig. 2, but for $\beta = 0.6$.

local maximum in interval $(0, 1)$, independent of the values of the stiffness coefficient K_1 and ratio b/a. The maximum values of the dimensionless frequencies are greatest for the greatest values of the ratio b/a. The eigenfrequencies increase when the stiffness of the supporting rings increases. The courses of the curves in Figs. 2–4 also show that for some values of stiffness coefficient K_1, the eigenfrequencies Ω_i can be the same as for the plates with various supporting rings. From the numerical investigations it follows that the stiffness coefficients K_1, as well as the radius of the supporting ring r_1, have a significant effect on the free vibration frequencies of the annular plate. The effect is observed for annular plates with various values of the ratio $\beta = b/a$.

4. Conclusions

An exact solution to the problem of free axisymmetric vibration of annular plates with elastic or rigid intermediate circular supports has been presented. The frequency equation for the axisymmetric vibrations of

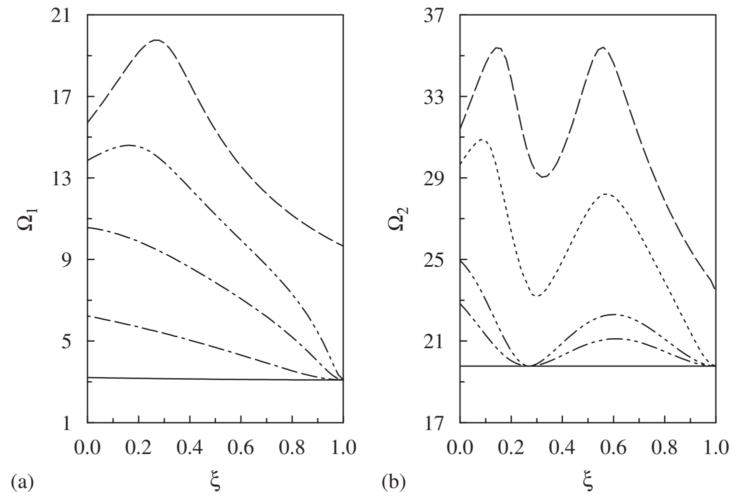

Fig. 5. As Fig. 2, but for $\beta = 0.8$.

the plate was obtained with the application of the Green's function method. The Green's function corresponding to the free annular plate was determined. Although the presented numerical results concern a free plate with two rigid or elastic supports, the solution includes the free vibrations of annular plates with an arbitrary number of supports. An analogous approach can be made for the vibration problems of annular plates with one or two clamped edges.

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