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Short Communication

Effect of element thickness on the eigenvalues of beams

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Abstract

The sensitivities of eigenvalues to the change of element thickness have been calculated for beams with various end conditions. For each beam, the sensitivities along the beam length fluctuate more for higher modes and the number of maxima increases linearly with the mode number. The sensitivities are high and positive at clamped ends, negative at free ends, and very low near simply supported ends. For a cantilever beam there exist positions where the eigenvalue sensitivities to element thickness are zero, which means that the eigenvalues are not affected by the thickness at those points. For a simply supported beam the sensitivities are always positive for all modes.

1. Introduction

Finite element (FE) analysis is widely used to predict the dynamic responses of mechanical systems and structures subject to dynamic loading. The predicted responses may differ from the experimentally measured ones and there have been active researches on FE model updating [1] so that the predicted responses based on the model agree with the measured ones. The related researches are surveyed [2] and summarized [3] in references. One of the approaches to model updating is the sensitivity analysis [4]. In this approach the sensitivities of the model responses, for example, eigenvalues (natural frequencies) and eigenvectors (mode shapes) of the FE model, to changes in the updating parameters are calculated, and the updating parameters of the model are modified according to the sensitivities. Material properties, physical dimensions, joint parameters [5], and element correction factors [6,7] can be selected as updating parameters.

This paper investigates some characteristics of the eigenvalue sensitivities to element thickness for beams with various boundary conditions. These characteristics can be used in modifying beam-like structures so that the structures have desired eigenvalues.

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2. Sensitivities of the eigenvalues

Using the FE analysis [8], the stiffness matrix of the jth element of a beam is

$$[K_{ej}] = \frac{Ebh_j^3}{12l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}, \tag{1}$$

where l, b, and h_j represent the length, width, and thickness of the element, respectively, and E, Young's modulus of the material. The mass matrix of the element becomes

$$[M_{ej}] = \frac{\rho bh_j l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}, \tag{2}$$

where ρ represents the density of the material. It is assumed that each element has the same length and width, but varying thickness. Letting the element matrices $[M_{ej}]$ and $[K_{ej}]$ have the same sizes as the whole system matrices, with zeros outside the corresponding positions, and rows and columns deleted for fixed boundary conditions, the system mass and stiffness matrices are expressed by the summation of element matrices as follows:

$$[M] = \sum_{j=1}^{N}[M_{ej}], \tag{3}$$

$$[K] = \sum_{j=1}^{N}[K_{ej}], \tag{4}$$

where N is the number of elements.

It is known that the sensitivity of the eigenvalue (square of the natural frequency) λ_i of mode i to change in the updating parameter θ_j is expressed by the following equation [9]:

$$\frac{\partial \lambda_i}{\partial \theta_j} = \phi_i^{\mathrm{T}}\left(\frac{\partial [K]}{\partial \theta_j} - \lambda_i \frac{\partial [M]}{\partial \theta_j}\right)\phi_i, \tag{5}$$

where ϕ_i represents the mass normalized eigenvector of mode i. If we take element thickness h_j as updating parameters, we obtain

$$\frac{\partial [K]}{\partial h_j} = \frac{Ebh_j^2}{4l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}, \tag{6}$$

$$\frac{\partial [M]}{\partial h_j} = \frac{\rho bl}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}. \tag{7}$$

The eigenvalue λ_i and the eigenvector ϕ_i are calculated by the FE analysis. If we insert Eqs. (6) and (7) into Eq. (5), the sensitivities of eigenvalues are obtained.

3. Results

3.1. Cantilever beam

The element mass and stiffness matrices were formed and the eigenvalues and eigenvectors were calculated for a cantilever beam with length 270 mm, width 35 mm, thickness 1.5 mm, Young's modulus $175 \times 10^9\,\mathrm{N/m^2}$, and density $7850\,\mathrm{kg/m^3}$. The beam is composed of five beam elements with equal length and is shown in Fig. 1. The sensitivities of eigenvalues to the element thickness in Eq. (5) were calculated and are listed in Table 1.

The natural frequencies were calculated for the above cantilever beam. Then the thickness of one of the 5 elements was increased by 1% with the thickness of the other elements unchanged, and the natural frequencies were calculated for each case. The calculated natural frequencies for the 6 cases are listed in Table 2. Observing the variation of the natural frequencies, it can be found that the variation agrees with the sensitivities in Table 1.

The above cantilever beam was divided into 20 elements with equal length and the sensitivities of eigenvalues to each element thickness were calculated in a similar manner. Fig. 2 shows the calculated sensitivities of each eigenvalue. In the figure, the horizontal axis represents the location of the element whose thickness is changed. The figure shows that the sensitivities fluctuate more for higher modes and the number of maxima increases linearly with the mode number. When the thickness of an element near the clamped end increases, the eigenvalues increase for all modes. On the other hand, increasing the thickness of an element near the tip decreases the eigenvalues for all modes. For each mode there exist positions where the eigenvalue is not affected by the thickness at those points. If we choose different numbers of elements, the magnitudes of the sensitivities of eigenvalues will vary. However, similar figures will be resulted.

From a dimensional analysis it can be shown that an eigenvalue of the beam in Fig. 3 is expressed by the following equation.

$$\lambda = \frac{E}{\rho L^2} F\left(\frac{L_1}{L}, \frac{L_2}{L}, \frac{h}{L}, \frac{h_j}{L}\right), \tag{8}$$

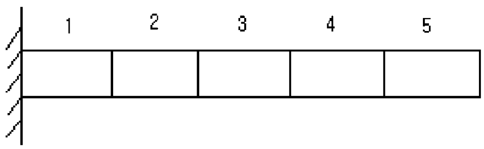

Fig. 1. Cantilever beam composed of five elements.

Table 1
Sensitivities of the eigenvalues to the element thickness for a cantilever beam (units: $\mathrm{rad^2/s^2m}$)

Element j	1	2	3	4	5
$\partial\lambda_1/\partial h_j$	1.166e7	5.424e6	1.256e6	−1.505e6	−3.878e6
$\partial\lambda_2/\partial h_j$	2.105e8	2.026e7	1.933e8	1.444e8	−5.883e7
$\partial\lambda_3/\partial h_j$	9.676e8	8.743e8	2.948e8	1.772e9	1.121e8
$\partial\lambda_4/\partial h_j$	3.449e9	2.193e9	4.189e9	3.366e9	2.494e9

Table 2
Natural frequencies of the cantilever beam when the thickness of one element is increased by 1% (units: Hz)

Position of element whose thickness is increased	None	1	2	3	4	5
f_1	15.69	15.83	15.76	15.71	15.68	15.65
f_2	98.40	98.81	98.44	98.77	98.67	98.29
f_3	276.4	277.0	277.0	276.6	277.6	276.4
f_4	546.0	547.2	546.7	547.4	547.1	546.8

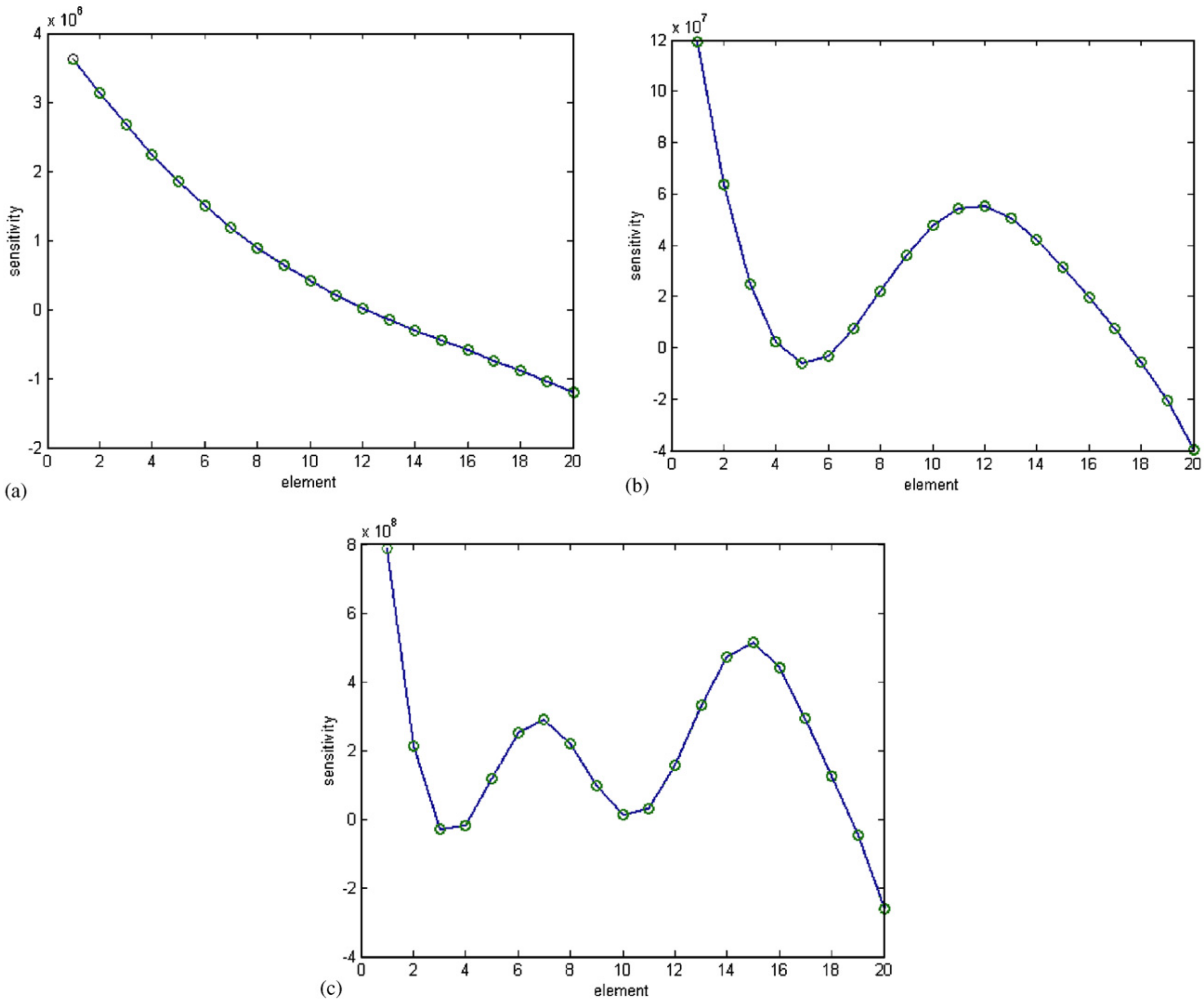

Fig. 2. Sensitivities of the eigenvalues to the element thickness for a cantilever beam: (a) 1st eigenvalue, (b) 2nd eigenvalue and (c) 3rd eigenvalue.

where F is some function which is unknown from an dimensional analysis. If the ratios, L_1/L and L_2/L, are fixed and the sensitivity, $\partial\lambda/\partial h_j$, is calculated when $h_j = h$, it becomes

$$\frac{\partial\lambda}{\partial h_j} = \frac{E}{\rho L^2} G\left(\frac{h}{L}\right), \tag{9}$$

where G is a function different from F. The above equation explains the effects of some parameters on the sensitivities of eigenvalues.

3.2. Simply supported beam

The sensitivities of eigenvalues to the change in the element thickness were calculated for a simply supported beam with the same material properties and dimensions as the previous cantilever beam. The beam was divided into 20 elements with equal length and the sensitivities of eigenvalues to each element thickness were calculated in a similar manner. Fig. 4 shows the calculated sensitivities of each eigenvalue. In the figure, the horizontal axis represents the location of the element whose thickness is changed. As expected, the sensitivities

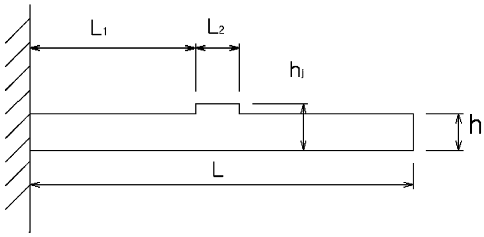

Fig. 3. Cantilever beam having an element with different thickness.

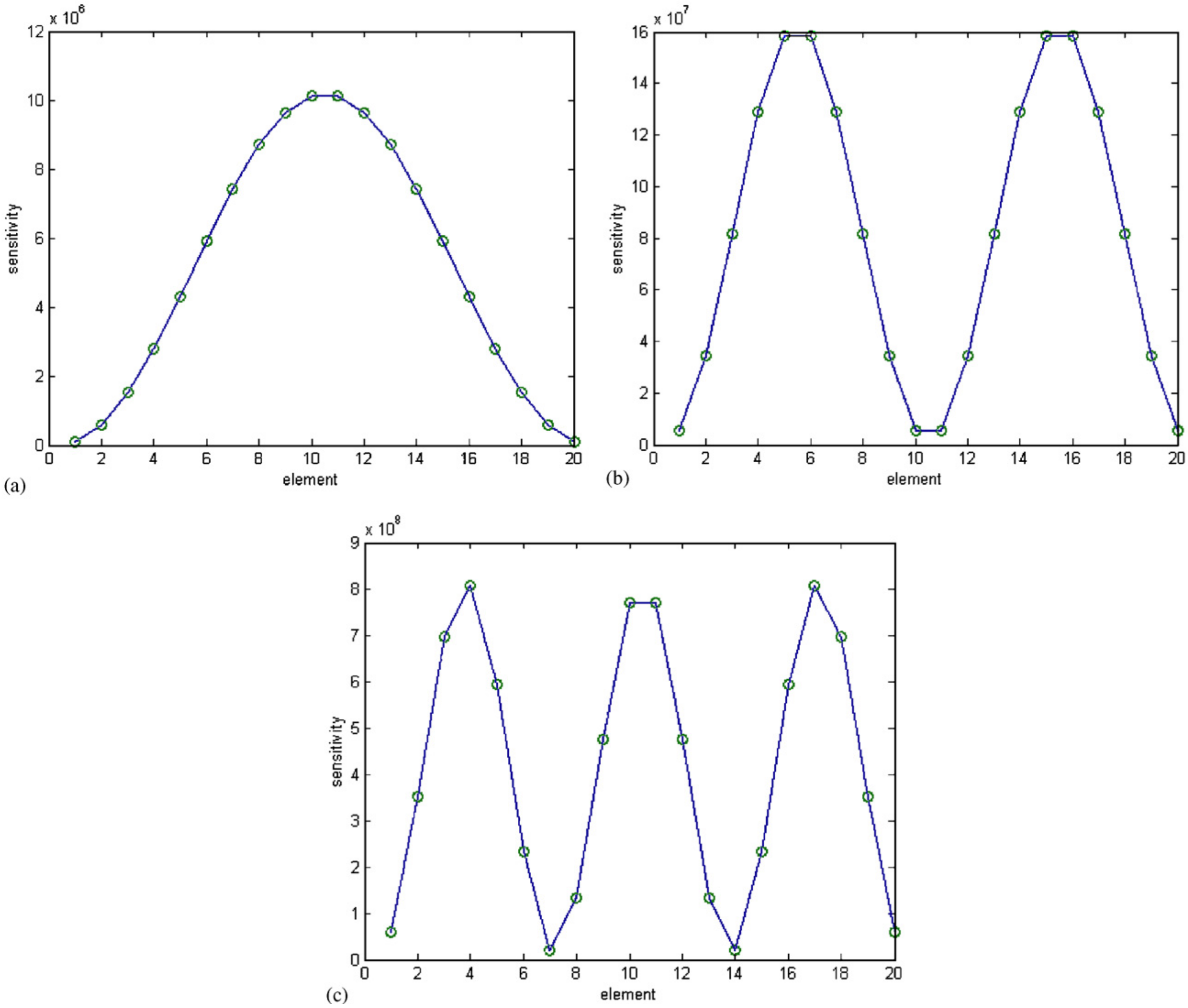

Fig. 4. Sensitivities of the eigenvalues to the element thickness for a simply supported beam: (a) 1st eigenvalue, (b) 2nd eigenvalue and (c) 3rd eigenvalue.

show symmetry. The figure shows that the sensitivities fluctuate more for higher modes and the number of maxima increases linearly with the mode number, which is the same phenomenon as for a cantilever beam. The sensitivities near simply supported ends are very low. The sensitivities are always positive for all modes. It means that increasing the thickness of any element results in increase of the eigenvalues of all modes.

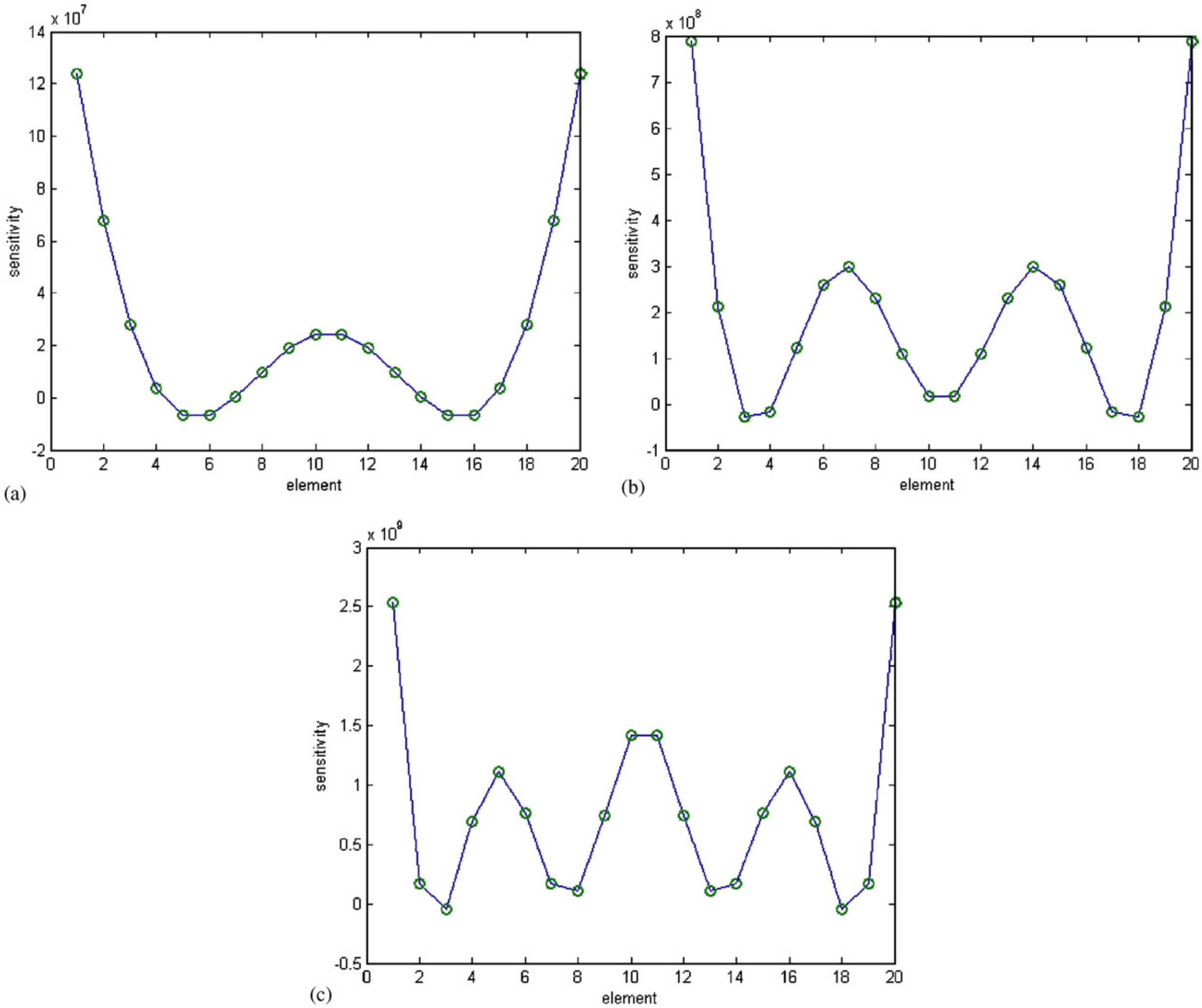

Fig. 5. Sensitivities of the eigenvalues to the element thickness for a beam with clamped–clamped ends: (a) 1st eigenvalue, (b) 2nd eigenvalue and (c) 3rd eigenvalue.

3.3. Other types of beam

The sensitivities of eigenvalues were calculated for the same beam with clamped–clamped ends and clamped–simply supported ends. The results are shown in Figs. 5 and 6. Fig. 5 shows that the sensitivities are high and positive at clamped ends. Fig. 6 shows that the sensitivities are high and positive at clamped ends and very low near simply supported ends. Both figures show the same phenomenon concerning the fluctuation of sensitivities along beam length.

4. Conclusions

The sensitivities of eigenvalues to the change of element thickness were calculated for beams with various end conditions. For each beam, the sensitivities along the beam length fluctuate more for higher modes and the number of maxima increases linearly with the mode number. When the thickness of an element near the clamped end of a cantilever beam increases, the eigenvalues for all modes increase. On the

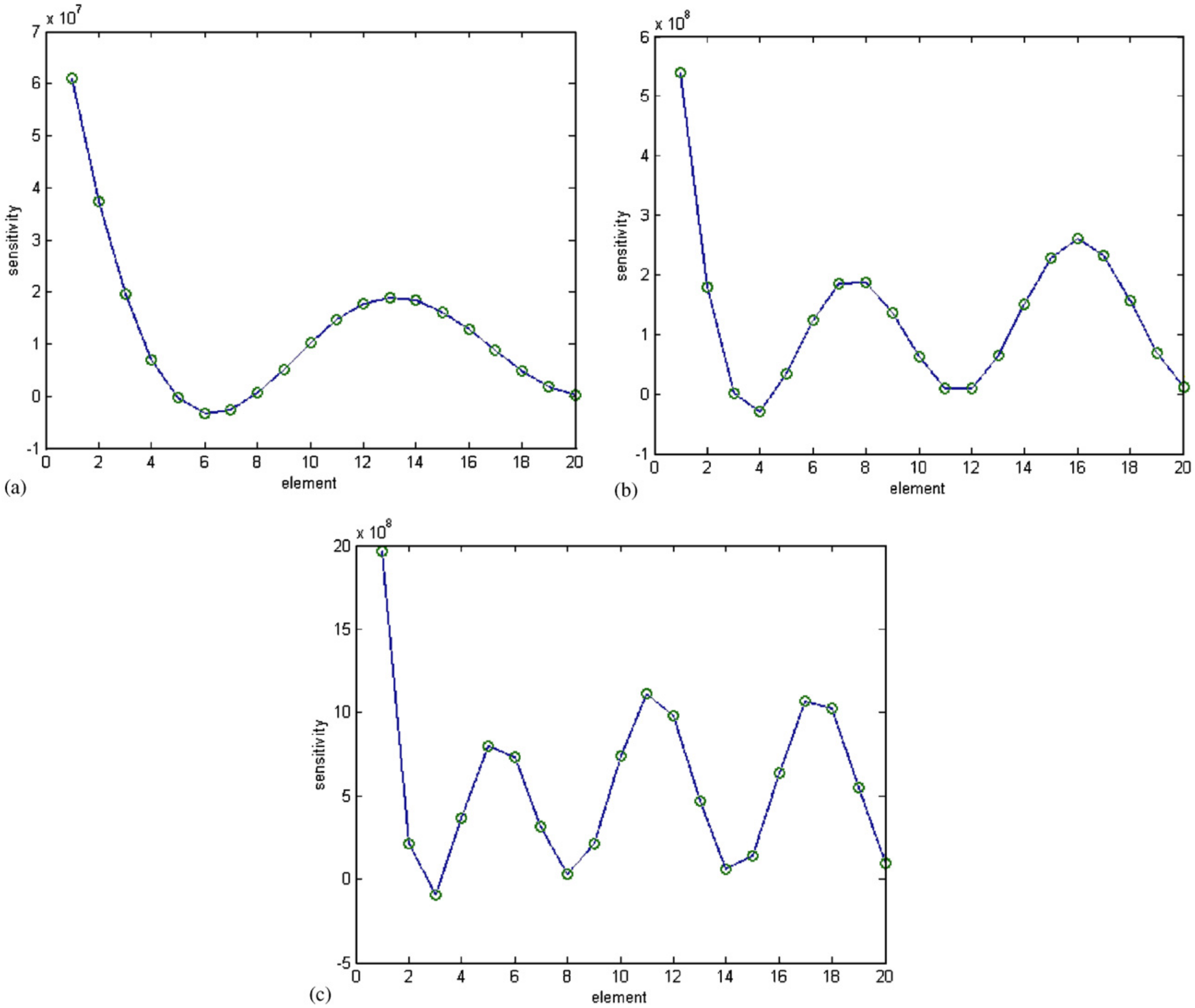

Fig. 6. Sensitivities of the eigenvalues to the element thickness for a beam with clamped–simply supported ends: (a) 1st eigenvalue, (b) 2nd eigenvalue and (c) 3rd eigenvalue.

other hand, increase of the thickness of an element near the tip decreases the eigenvalues for all modes. For each mode there exist positions where the eigenvalue sensitivities to element thickness are zero, which means that the eigenvalues are not affected by the thickness at those points. For a simply supported beam the sensitivities near simply supported ends are very low. The sensitivities are always positive for all modes. Calculating the eigenvalue sensitivities for a clamped–clamped beam and a clamped–simply supported beam, it was found that the sensitivities are high and positive at clamped ends, and very low near simply supported ends. These results agree with those obtained for a cantilever beam and a simply supported beam.

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