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Short Communication

# A new method for reducing the natural frequency of single degree of freedom systems

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**Abstract**

In isolators of single degree of freedom (SDF), the natural frequency must be as low as possible to increase the effectiveness of the anti-vibration. In each isolator, factors influencing the natural frequency such as the length of a pendulum in a simple pendulum, stiffness of a spring in the mass–spring system, etc., are different. In this paper, it is shown that the stored potential energy is a factor which has the biggest influence on the natural frequency. From this factor, a general method to reduce the natural frequency is then introduced.


## 1. Introduction

Many isolators can be modeled as single degree of freedom (SDF) systems. One of the most important factors of an isolator is the natural frequency [1–3]. The lower the natural frequency is, the more quickly the vibration energy will be attenuated at higher frequencies. However, reducing the natural frequency is not a simple work. There are many different factors influencing the natural frequency.

Let us consider four examples of SDF systems. First, in a simple pendulum, the pendulum length and the acceleration of gravity are two factors influencing the natural frequency. Second, stiffness of the spring and the mass are two factors influencing the natural frequency in a mass–spring system. Third, in Ref. [4], the author shows that the lower the stored energy in a horizontal isolator is, the lower the natural frequency is. In this isolator whose structure is a straight line linkage, the mass will move in a flat horizontal plane. So, the stored energy is approximately equal to zero, and the natural frequency is approximately zero. In this example, the stored energy is obviously an important factor. Finally, an isolator which is introduced in Ref. [1] is made by combining a common spring with negative stiffness elements. Stiffness of this isolator can reach zero and the natural frequency is approximately zero.

Here questions are why the combination of the common spring and negative stiffness elements makes the natural frequency reach zero, and why factors which influence the natural frequency are different, even though the simple pendulum, the spring–mass system and the straight line structure are all SDF systems. What are the common factors influencing the natural frequency? If the factors are discovered, a general method to reduce the natural frequency can be found.

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The rest of this paper is organized as following: In Section 2, the general form of the linearized equation of motion of the SDF systems is shown. Then, two basic SDF systems, a pendulum system and a spring–mass system, are introduced as examples of the general form of the equations of motion of the SDF systems. This section also shows how the natural frequency is calculated and the stored potential energy is the main factor which influences the natural frequency. Section 3 is devoted to show the general method which reduces the natural frequency of the SDF system. The main idea of the paper is summarized in Section 4.

## 2. General linearized equation of motion of SDF systems and the main factor influencing the natural frequency

### 2.1. General linearized equation of motion of SDF systems

Since the effort of this paper only concentrates on reducing the natural frequency, the damper of the isolation system is ignored. The potential function of an SDF system depends only on the position of the mass in the potential field [5]. Moreover, the potential function is an even function because most of the systems with SDF vibrate symmetrically about the equilibrium point [6]. Let us call the potential function $U(x)$ where $x$ is a generalized displacement coordinate. The kinetic function of the systems is written as

$$T = \tfrac{1}{2}mv_c^2 + \tfrac{1}{2}I_c\omega^2, \tag{1}$$

where $m$ is the mass, $I_c$ is the moment of inertia, $v_c$ is the velocity of the center of gravity, and $\omega$ is the angular velocity.

In general, the SDF systems consist of lots of elements. From Eq. (1), the kinetic energy function can be shown in either Eqs. (2) or (3):

$$T = \sum_i m_i \left[\sum_i f_i(x) g_i(\dot{x})\right]^2 + \sum_i I_i \left[\sum_i h_i(x) k_i(\dot{x})\right]^2, \tag{2}$$

$$T = \sum_i m_i \left[\sum_i f_i(x) g_i^2(\dot{x})\right] + \sum_i I_i \left[\sum_i h_i(x) k_i^2(\dot{x})\right], \tag{3}$$

where the functions $g_i(\dot{x})$ and $k_i(\dot{x})$ consist of the first or higher degree terms in $\dot{x}$, $m_i$ is the mass of the element $i$, and $I_i$ is the moment of inertia of the element $i$.

Applying the Lagrange's law into a holonomic conservative system with SDF, the equation of motion can be derived in two forms of either Eqs. (4) or (5):

$$\begin{aligned}
&2\left\{\sum_i m_i \left[\sum_i g_i' \left(\sum_i f_i g_i'\right) + \sum_i g_i'' \left(\sum_i f_i g_i'\right)\right] f_i + \sum_i I_i \left[\sum_i k_i' \left(\sum_i h_i k_i'\right) + \sum_i k_i'' \left(\sum_i h_i k_i'\right)\right] h_i \right\} \ddot{x} \\
&\quad + 2\left\{\sum_i m_i f_i' \left[\sum_i g_i \left(\sum_i f_i g_i'\right) + \sum_i g_i' \left(\sum_i f_i g_i\right)\right] + \sum_i I_i h_i' \left[\sum_i k_i \left(\sum_i h_i k_i'\right) + \sum_i k_i' \left(\sum_i h_i k_i\right)\right]\right\} \dot{x} \\
&\quad - 2\left[\sum_i m_i \sum_i f_i g_i \left(\sum_i f_i' g_i\right) + \sum_i I_i \sum_i h_i k_i \left(\sum_i h_i' k_i\right)\right] \\
&\quad + \frac{\partial U}{\partial x} = 0,
\end{aligned} \tag{4}$$

$$\begin{aligned}
&2\left[\sum_i m_i f_i \left(\sum_i \left(g_i'^2 + g_i g_i''\right)\right) + \sum_i I_i h_i \left(\sum_i \left(k_i'^2 + k_i k g_i''\right)\right)\right] \ddot{x} \\
&\quad + \left[2\sum_i m_i \sum_i f_i g_i g_i' + 2\sum_i I_i \sum_i h_i k_i k_i'\right] \dot{x} - \left[\sum_i m_i \left(f_i' g_i^2\right) + \sum_i I_i \left(h_i' k_i^2\right)\right] \\
&\quad + \frac{\partial U}{\partial x} = 0,
\end{aligned} \tag{5}$$

where

$$f_i \text{ is } f_i(x), \quad f_i' = \frac{\partial f_i(x)}{\partial x}, \quad f_i'' = \frac{\partial^2 f_i(x)}{\partial x^2},$$

$$g_i \text{ is } g_i(\dot{x}) \quad g_i'' = \frac{\partial g_i(\dot{x})}{\partial \dot{x}}, \quad g_i'' = \frac{\partial^2 g_i(\dot{x})}{\partial \dot{x}^2},$$

$$h_i \text{ is } h_i(x) \quad h_i' = \frac{\partial h_i(x)}{\partial x}, \quad h_i'' = \frac{\partial^2 h_i(x)}{\partial x^2}$$

and

$$k_i \text{ is } k_i(\dot{x}) \quad k_i'' = \frac{\partial k_i(\dot{x})}{\partial \dot{x}}, \quad k_i'' = \frac{\partial^2 k_i''(\dot{x})}{\partial \dot{x}^2}.$$

In a SDF system [6], the general form of a linearized equation of motion must be

$$\ddot{y} + a\dot{y} + by = 0, \tag{6}$$

where $a$ and $b$ are constants. The linearized equation is derived by considering small motions about equilibrium position and by ignoring any non-linear terms of $\dot{x}$ and $x$ which are very small when compared with linear terms $\dot{x}$ and $x$, respectively.

In Eq. (4), the second and third terms are ignored because they are small non-linear terms of $\dot{x}$. Similarly, the second and third terms in Eq. (5) are ignored. In Eqs. (4) and (5), the coefficient of the acceleration $\ddot{x}$ must be linearized to become a constant by ignoring non-linear terms or by considering non-linear terms as constants because of small motions. At that time, the linearized equation of motion becomes

$$A\ddot{x} + \frac{\partial U}{\partial x} = 0, \tag{7}$$

where $A$ is a constant which consists of inertia terms.

Let us look at two examples to illustrate more clearly the general form of the equation of motion. Firstly, in the simple pendulum of Fig. 1, when horizontal displacement $x$ $(= l\theta)$ is chosen as a new generalized displacement coordinate, it is easy to recognize that the potential function is written in the new coordinate as follows:

$$U = mgl(1 - \cos\,\theta) \approx \frac{mgx^2}{2l}, \tag{8}$$

where $m$ is the bob mass and $l$ is the length of the pendulum.

Then, the equation of motion now becomes

$$m\ddot{x} + \frac{\partial U}{\partial x} = 0. \tag{9}$$

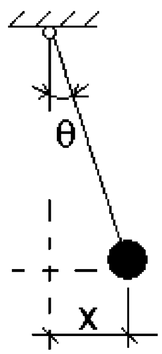


Fig. 1. A simple pendulum.

Secondly, the potential function of the spring–mass system is given by

$$U = \tfrac{1}{2}Kx^2, \tag{10}$$

where $K$ is stiffness of the spring, $x$ is the displacement from the equilibrium point. The equation of motion is

$$M\ddot{x} + \frac{\partial U}{\partial x} = 0, \tag{11}$$

where $M$ is the mass. It is obvious that the forms of Eq. (9) and Eq. (11) are similar with that of Eq. (7).

### 2.2. Main factor influencing the natural frequency

After some algebraic manipulations of Eq. (7), the period is given as

$$T = \int_{x_{\min}}^{x_{\max}} \sqrt{\frac{2A}{U(x_{\max}) - U(x)}}, \tag{12}$$

where $x_{\max}$ and $x_{\text{mix}}$ are two extreme points. It is noted that the natural frequency is the inverse of the period. From Eq. (12), the natural frequency depends on the constant $A$ consisting of inertia terms. When the constant $A$ increases, the natural frequency will reduce. However, this is not good for an isolation system because it makes the isolation system heavy. Besides, in some systems such as the pendulum, inertia has no influence on the natural frequency.

When an isolation system is stable, the potential function is a convex one [6]. Moreover, in linearized systems of SDF, the potential function is an even one and it achieves the lowest value at the equilibrium point. Thus, $U(x)$ is less than or equal to $U(x_{\max})$.

Let us consider two SDF systems I and II. They have the same value of the constant $A$ but their potential functions $U_{\text{I}}(x)$ and $U_{\text{II}}(x)$are different. Let us say a function $D(x)$ satisfying with

$$D(x) = U_{\text{I}}(x) - U_{\text{II}}(x). \tag{13}$$

Eq. (12) is applied to systems I and II and the difference between two periods of two systems is calculated, the result becomes:

$$T_{\text{II}} - T_{\text{I}} = \int_{x_{\min}}^{x_{\max}} \frac{\sqrt{2A}[D(x_{\max}) - D(x)]}{\sqrt{(U_{\text{I}}(x_{\max}) - U_{\text{I}}(x))(U_{\text{II}}(x_{\max}) - U_{\text{II}}(x))}\left[\sqrt{U_{\text{I}}(x_{\max}) - U_{\text{I}}(x)} + \sqrt{U_{\text{II}}(x_{\max}) - U_{\text{II}}(x)}\right]}, \tag{14}$$

where $T_{\text{I}}$ is the period of system I, $T_{\text{II}}$ is the period of system II.

Since the denominator is positive in Eq. (14), the difference of the two periods is positive if the numerator is positive. In other words, in two systems having the same inertias and the same working ranges, if $D(x)$ is a convex function, the natural frequency of system I will be higher than that of system II. Note that $D(x)$ represents the difference of two stored potential energies of two system I and II. When the difference increases, the convexity of $D(x)$ increases. As the result, the difference of two periods will increase. Therefore, it is concluded that the stored potential energy is the main factor which influences the natural frequency. That is, the lower the stored potential energy is, the lower the natural frequency is. This conclusion is also mentioned in Ref. [4].

## 3. Method to reduce the natural frequency of SDF systems

The analyses in Eq. (14) can be explained in Fig. 2. When the convexity of $D(x)$ increases, the difference between two natural frequencies of the above two systems I and II will increase.

In Eq. (8), when the length of the pendulum is increased, the stored potential energy at any point $x$ in the working range will decrease. As a result, the convexity of $D(x)$ increases and it makes the natural frequency of the pendulum reduce. In Eq (10), it is clear that the stored potential energy will reduce when the stiffness of the

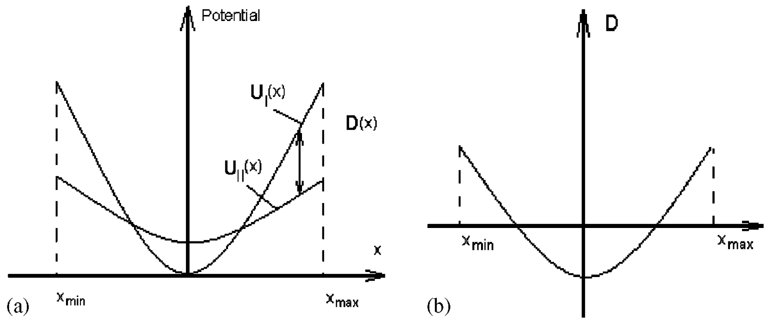


Fig. 2. (a) Difference $D(x)$ between the two potential energies of the two systems I and II. (b) Convexity of function $D(x)$.

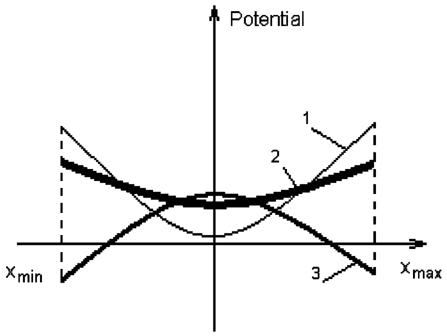


Fig. 3. A new concept to decrease the natural frequency.

spring decreases, and it makes the convexity of $D(x)$ increase. That means the reduction of the natural frequency.

Increasing the convexity of $D(x)$ by the length of the pendulum or the stiffness of the spring is unfeasible because this way will make the size of isolators unacceptably big. Thus, in order to reduce the natural frequency, a new method to increase the convexity of $D(x)$ is suggested, which is the goal of this paper. It is illustrated in Fig. 3.

In Fig. 3, line 1 illustrates the potential function of the original isolation system. Line 2 illustrates the potential function of the expected isolation system whose natural frequency is as low as expected. As the above analyses have shown, the natural frequency of the system illustrated by line 2 is smaller than that of the system illustrated by line 1. Line 3 illustrates the potential function of the system that is added to the original isolation system. When the added system is combined to the original isolation system, the sum of the two potential functions, one of the original isolation system and the other of the added system, equals that of the expected system. That means the sum value of lines 1 and 3 equals the value of line 2.

Here, questions are what the added system illustrated in line 3 is, and what the characteristic of the added system is. The potential function of the added system is a concave function and it achieves the maximum value at the equilibrium point. Note that the added system is an unstable one with the above requirements. In addition, the potential function of the added system must be the difference of two potential functions, one of the original isolation system and the other of the expected isolation system. However, in practice, the added system is sometimes chosen first and the expected isolation system will be determined later. The choice must satisfy that the potential function of the expected isolation system is a convex function for stability [7].

Three approaches to reduce the natural frequency are introduced in Fig. 4. In these approaches, the systems in the upper position of Fig. 4 are combined with the original isolation systems to become the systems in the lower position of Fig. 4. Through experiments and mathematical analyses, it is proved that the natural frequency remarkably reduces.

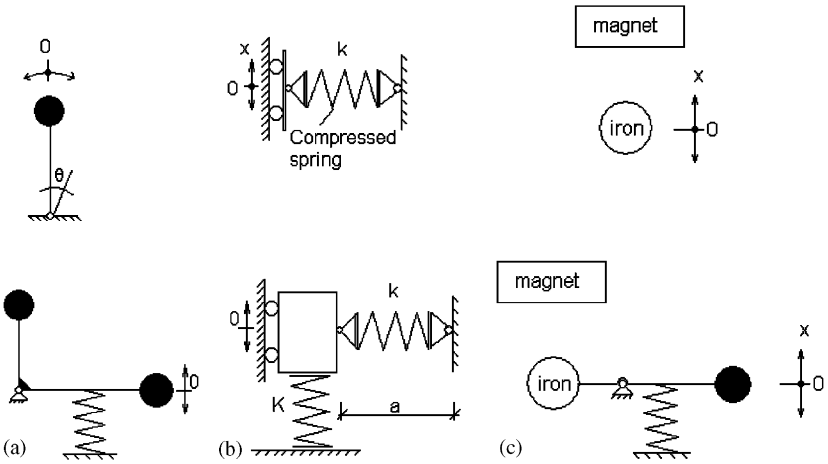


Fig. 4. Types of the added systems in the upper position. Expected isolation systems which consist of original and added systems in the lower position, (a) inverted pendulum and the combined system, (b) negative stiffness system and the combined system, (c) a steel mass in magnetic field and the combined system.

Let us consider more clearly these approaches. Firstly, the system in the upper position of Fig. 4a is an inverted pendulum and it is combined with an isolation system as illustrated in the lower position of Fig. 4a. In Ref. [8], the natural frequency of the combined system in Fig. 4a can achieve 0.3 Hz. Secondly, the system in the upper position of Fig. 4b is a negative stiffness element. This system is analyzed in Ref. [1]. Following mathematical analyses, when the negative stiffness element is combined with the original isolation system, the natural frequency of the combined system can reach 0 Hz. Finally, in Ref. [9], the system in the upper position of Fig. 4c is used to generate a negative stiffness that is combined with a system having a positive stiffness to make systems with zero stiffness. Consequently, the natural frequency reaches zero. However, the methods to reduce the natural frequency in the above isolation systems are different. So in this paper, a general method for all of these systems in Fig. 4 is proposed.

The potential functions of the unstable systems in the upper position of Fig. 4 satisfy the potential function illustrated as line 3 of Fig. 3. When these systems are connected to the original isolation system, they become the systems in the lower position of Fig. 4. These systems are the expected isolation systems which are illustrated as line 2 of Fig. 3. Then, the natural frequencies of the combined systems will reduce as shown above. The natural frequency of the expected system is determined from Eq. (12).

## 4. Conclusion

This paper shows that the stored potential energy is the most important common factor which influences the natural frequency of the SDF systems. Using this factor, the new method to reduce the natural frequency is introduced. The method can be more general one when compared with other methods to reduce the natural frequency.

The paper also shows that most of the unstable systems can be used as the added systems. Thus, by combining the added systems with the original isolation systems, building isolation systems having the low natural frequency becomes easier.

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