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Free vibration analysis of rectangular plates with variable thickness and point supports

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Abstract

A discrete method is developed for analyzing the free vibration problem of rectangular plates with point supports. The fundamental differential equations involving Dirac's delta function are established for the bending problem of the plate with point supports. By transforming these differential equations into integral equations and using numerical integration, the solution of these equations is obtained and used as Green function to obtain the characteristic equation of the free vibration. The effects of the point support, the boundary condition, the variable thickness and aspect ratio on the frequencies are considered. By comparing the numerical results obtained by the present method with those previously published, the efficiency and accuracy of the present method are investigated.

1. Introduction

The plates with point supports play an important role in engineering fields, such as slabs on columns in civil engineering and printed circuit board in electronic engineering. Due to their practical importance, the free vibration analysis of these plates has received considerable attention. According to the positions of the point supports, two kinds of the plates with point supports have been studied. One is the plate with point supports along the edges. Another is the

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plate with interior point supports. Saliba [1] used the superposition method to analyze the free vibration of rectangular cantilever plates with symmetrically distributed point supports along the free edges. After 4 years, Saliba [2] used the same method to analyze the free vibration of rectangular cantilever plates with symmetrically distributed interior point support. Bapat et al. investigated the free vibration of rectangular plates with symmetrical point supports along the edges [3] and with asymmetrical point supports along the edges [4]. The flexibility function approach and the impulse function approach were used to simulate the point supports. A comparison of these two methods was also given and the advantage of the flexibility function method was shown. By using the same methods, Bapat and Suryanarayan [5] also studied the free vibration of rectangular plates with interior point supports. By dividing the plate into two sub-plates and satisfying the continuity conditions along the partition line and the compatibility condition of zero deflection at the point support, a set of equations were obtained. By utilizing the set of equations and the equivalent equations, the characteristic equation was obtained. Huang and Thambiratnam [6] used the finite strip element method to study the free vibration analysis of plates on elastic intermediate supports. The spring system was employed to simulate elastic intermediate supports. Kim and Dickinson [7] employed the Lagrangian multiplier method to analyze the free vibration of rectangular plates with arbitrarily located point supports. All of the above studies are limited to the plates with uniform thickness.

Plates with variable thickness are also extensively used in engineering fields. The free vibration analyses of these plates, such as plates with linearly varying thickness in one direction [8], plates with variable thickness in one or two directions [9], plates with bidirectional thickness variation [10] and plates with stepped thickness [11], have been studied.

In this paper, a discrete method is proposed for analyzing the free vibration of rectangular plates with point supports. No prior assumption of shape of deflection, such as shape functions used in the finite element method, is employed. The fundamental differential equations of a plate with point supports involving Dirac'delta functions are established and satisfied exactly throughout the whole plate. By transforming these equations into integral equations and using numerical integration, the solutions are obtained at the discrete points. The Green function, which is the solution for deflection, is used to obtain the characteristic equation of the free vibration. The proposed method is a general method. It can be used to analyze the free vibration of rectangular plates with arbitrarily located point support, various aspect ratio, variable thickness and general boundary conditions. The purpose of the paper is (1) to investigate the efficiency and accuracy of the present method for the rectangular plates with uniform thickness and point supports by comparing the present results with those reported early, and (2) to investigate the effect of the point support on the frequency parameter of plate with variable thickness.

2. Fundamental differential equations

Consider a rectangular plate of length a, width b, density ρ. An xyz coordinate system is used in the present study with its x–y plane contained in the middle plane of the rectangular plate, the z-axis perpendicular to the middle plane of the plate and the origin at one of the corners of the plate.

In this paper, the concentrated loads with Dirac'delta functions are used to simulate the point supports which limit the displacements of the plate but do not offer constraint on the slopes.

Considering the equations of equilibrium, the strain–displacement relations, the stress–strain relations and the load–stress relations, the fundamental differential equations of the plate having a concentrated load $\overline{P}$ at a point (x_q, y_r) and the point supports $\overline{P}_{cd}$ at each discrete point (x_c, y_d) are as follows:

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \overline{P}\delta(x - x_q)\delta(y - y_r) + \sum_{c=0}^{m}\sum_{d=0}^{n}\overline{P}_{cd}\delta(x - x_c)\delta(y - y_d) = 0,$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0,$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0,$$

$$\frac{\partial \theta_x}{\partial x} + v\frac{\partial \theta_y}{\partial y} = \frac{M_x}{D},$$

$$\frac{\partial \theta_y}{\partial y} + v\frac{\partial \theta_x}{\partial x} = \frac{M_y}{D},$$

$$\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} = \frac{2}{(1 - v)}\frac{M_{xy}}{D},$$

$$\frac{\partial w}{\partial x} + \theta_x = \frac{Q_x}{Gt_s},$$

$$\frac{\partial w}{\partial y} + \theta_y = \frac{Q_y}{Gt_s}, \tag{1}$$

where Q_x, Q_y are the shearing forces, M_{xy} the twisting moment, M_x, M_y the bending moments, θ_y, θ_x the rotations of the normal to the middle plane in the x- and y-directions, w the deflection, $D = Eh^3/(12(1 - v^2))$ the bending rigidity, E, G modulus, shear modulus of elasticity, respectively, v Poisson's ratio, h the thickness of plate, $t_s = h/1.2$; $\delta(x - x_q)$, $\delta(x - x_r)$, $\delta(x - x_c)$ and $\delta(x - x_d)$ Dirac's delta functions.

By introducing the non-dimensional expressions,

$$[X_1, X_2] = \frac{a^2}{D_0(1 - v^2)}[Q_y, Q_x], \quad [X_3, X_4, X_5] = \frac{a}{D_0(1 - v^2)}[M_{xy}, M_y, M_x],$$

$$[X_6, X_7, X_8] = [\theta_y, \theta_x, w/a].$$

Eq. (1) is rewritten as the following non-dimensional forms:

$$\mu\frac{\partial X_2}{\partial \eta} + \frac{\partial X_1}{\partial \zeta} + P\delta(\eta - \eta_q)\delta(\zeta - \zeta_r) + \sum_{c=0}^{m}\sum_{d=0}^{n} P_{cd}\delta(\eta - \eta_c)\delta(\zeta - \zeta_d) = 0,$$

$$\mu\frac{\partial X_3}{\partial\eta}+\frac{\partial X_4}{\partial\zeta}-\mu X_1=0,$$

$$\mu\frac{\partial X_5}{\partial\eta}+\frac{\partial X_3}{\partial\zeta}-\mu X_2=0,$$

$$\mu\frac{\partial X_7}{\partial\eta}+v\frac{\partial X_6}{\partial\zeta}-IX_5=0,$$

$$v\mu\frac{\partial X_7}{\partial\eta}+\frac{\partial X_6}{\partial\zeta}-IX_4=0,$$

$$\mu\frac{\partial X_6}{\partial\eta}+\frac{\partial X_7}{\partial\zeta}-JX_3=0,$$

$$\frac{\partial X_8}{\partial\eta}+X_7-HX_2=0,$$

$$\frac{\partial X_8}{\partial\zeta}+\mu X_6-\mu HX_1=0, \tag{2}$$

where $\mu=b/a$, $I=\mu(1-v^2)(h_0/h)^3$, $J=2\mu(1+v)(h_0/h)^3$, $H=((1+v)/5)(h_0/a)^2(h_0/h)$, $P=\overline{P}a/(D_0(1-v^2))$, $P_{cd}=\overline{P}_{cd}a/(D_0(1-v^2))$, $D_0=Eh_0^3/(12(1-v^2))$ is the standard bending rigidity, h_0 the standard thickness of the plate, $k=\frac{5}{6}$ the shear correction factor, $\delta(\eta-\eta_q)$, $\delta(\zeta-\zeta_r)$, $\delta(\eta-\eta_c)$ and $\delta(\zeta-\zeta_d)$ Dirac's delta functions.

In the above equation, the variable quantity h_0/h has been separated and expressed only in the quantities I, J and H so that the equation can be used for the plate with continuously variable thickness or stepped thickness.

Eq. (2) can also be expressed as the following simple form:

$$\sum_{s=1}^{8}\left\{F_{1ts}\frac{\partial X_s}{\partial\zeta}+F_{2ts}\frac{\partial X_s}{\partial\eta}+F_{3ts}X_s\right\}+P\delta(\eta-\eta_q)\delta(\zeta-\zeta_r)\delta_{1t}$$
$$+\sum_{c=0}^{m}\sum_{d=0}^{n}P_{cd}\delta(\eta-\eta_c)\delta(\zeta-\zeta_d)\delta_{1t}=0, \tag{3}$$

where $t=1$–8; δ_{1t} is Kronecker's delta; $F_{111}=F_{124}=F_{133}=F_{156}=F_{167}=F_{188}=1$; $F_{146}=v$; $F_{212}=F_{223}=F_{235}=F_{247}=F_{266}=\mu$; $F_{257}=\mu v$; $F_{278}=1$; $F_{321}=F_{332}=-\mu$; $F_{345}=F_{354}=-I$; $F_{363}=-J$; $F_{372}=-H$; $F_{377}=1$; $F_{381}=-\mu H$; $F_{386}=\mu$; other $F_{kts}=0$.

3. Discrete green function

As given in Ref. [9], by dividing a rectangular plate vertically into m equal-length parts and horizontally into n equal-length parts as shown in Fig. 1, the plate can be considered as a group of discrete points which are the intersections of the $(m+1)$-vertical and $(n+1)$-horizontal dividing lines. To describe the present method conveniently, the rectangular area, $0\leqslant\eta\leqslant\eta_i$, $0\leqslant\zeta\leqslant\zeta_j$,

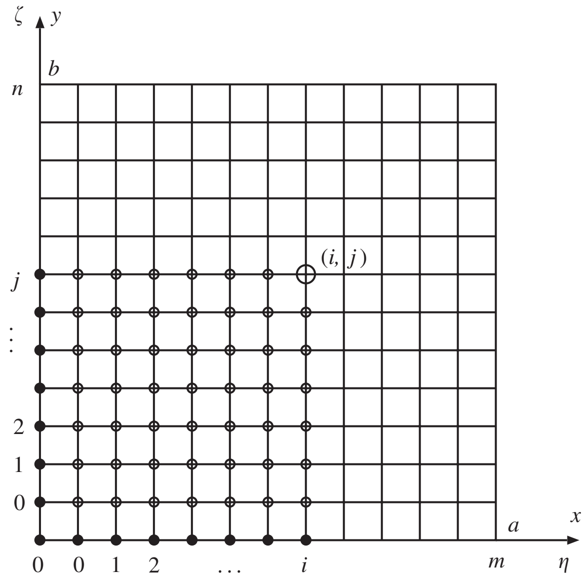

Fig. 1. Discrete points on a rectangular plate.

corresponding to the arbitrary intersection (i,j) as shown in Fig. 1 is denoted as the area $[i,j]$, the intersection (i,j) denoted by $\bigcirc$ is called the main point of the area $[i,j]$, the intersections denoted by $\circ$ are called the inner dependent points of the area, and the intersections denoted by $\bullet$ are called the boundary-dependent points of the area.

By integrating Eq. (3) over the area $[i,j]$, the following integral equation is obtained:

$$\sum_{s=1}^{8}\left\{F_{1ts}\int_0^{\eta_i}[X_s(\eta,\zeta_j)-X_s(\eta,0)]\,\mathrm{d}\eta+F_{2ts}\int_0^{\zeta_j}[X_s(\eta_i,\zeta)-X_s(0,\zeta)]\,\mathrm{d}\zeta\right.$$
$$\left.+F_{3ts}\int_0^{\eta_i}\int_0^{\zeta_j}X_s(\eta,\zeta)\,\mathrm{d}\eta\,\mathrm{d}\zeta\right\}+Pu(\eta-\eta_q)u(\zeta-\zeta_r)\delta_{1t}$$
$$+\sum_{c=0}^{m}\sum_{d=0}^{n}P_{cd}u(\eta-\eta_c)u(\zeta-\zeta_d)\delta_{1t}=0, \tag{4}$$

where $u(\eta-\eta_q)$, $u(\zeta-\zeta_r)$, $u(\eta-\eta_c)$ and $u(\zeta-\zeta_d)$ are the unit step functions.

Next, by applying the trapezoidal rule of the approximate numerical integration over the area $[i,j]$, the simultaneous equation for the unknown quantities $X_{sij}=X_s(\eta_i,\zeta_j)$ at the main point (i,j) of the area $[i,j]$ is obtained directly from Eq. (4) as follows:

$$\sum_{s=1}^{8}\left\{F_{1ts}\sum_{k=0}^{i}\beta_{ik}(X_{skj}-X_{sk0})+F_{2ts}\sum_{l=0}^{j}\beta_{jl}(X_{sil}-X_{s0l})+F_{3ts}\sum_{k=0}^{i}\sum_{l=0}^{j}\beta_{ik}\beta_{jl}X_{skl}\right\}$$
$$+Pu_{iq}u_{jr}\delta_{1t}+\sum_{c=0}^{m}\sum_{d=0}^{n}P_{cd}u_{ic}u_{jd}\delta_{1t}=0, \tag{5}$$

where $\beta_{ik} = \alpha_{ik}/m$; $\beta_{jl} = \alpha_{jl}/n$; $\alpha_{ik} = 1 - (\delta_{0k} + \delta_{ik})/2$; $\alpha_{jl} = 1 - (\delta_{0l} + \delta_{jl})/2$; $t = 1$–8; $i = 1$–m; $j = 1$–n; $u_{iq} = u(\eta_i - \eta_q)$; $u_{jr} = u(\zeta_j - \zeta_r)$; $u_{ic} = u(\eta_i - \eta_c)$; $u_{jd} = u(\zeta_j - \zeta_d)$.

By retaining the quantities at main point (i,j) on the left-hand side of the equation, putting other quantities on the right-hand side, the following equation can be obtained:

$$\sum_{s=1}^{8}\{(F_{1ts}\beta_{ii} + F_{2ts}\beta_{jj} + F_{3ts}\beta_{ii}\beta_{jj})X_{sij}\}$$
$$= \sum_{s=1}^{8}\left\{F_{1ts}\sum_{k=0}^{i}\beta_{ik}[X_{sk0} - X_{skj}(1-\delta_{ik})] + F_{2ts}\sum_{l=0}^{j}\beta_{jl}[X_{s0l} - X_{sil}(1-\delta_{jl})]\right.$$
$$\left. -F_{3ts}\sum_{k=0}^{i}\sum_{l=0}^{j}\beta_{ik}\beta_{jl}X_{skl}(1-\delta_{ik}\delta_{jl})\right\} - Pu_{iq}u_{jr}\delta_{1t} - \sum_{c=0}^{m}\sum_{d=0}^{n}P_{cd}u_{ic}u_{jd}\delta_{1t}. \quad (6)$$

By using the matrix transition, the solution X_{pij} of the above Eq. (6) is obtained as follows:

$$X_{pij} = \sum_{t=1}^{8}\left\{\sum_{k=0}^{i}\beta_{ik}A_{pt}[X_{tk0} - X_{tkj}(1-\delta_{ik})] + \sum_{l=0}^{j}\beta_{jl}B_{pt}[X_{t0l} - X_{til}(1-\delta_{jl})]\right.$$
$$\left. + \sum_{k=0}^{i}\sum_{l=0}^{j}\beta_{ik}\beta_{jl}C_{ptkl}X_{tkl}(1-\delta_{ik}\delta_{jl})\right\} - A_{p1}Pu_{iq}u_{jr} - \sum_{c=0}^{m}\sum_{d=0}^{n}A_{p1}P_{cd}u_{ic}u_{jd}, \quad (7)$$

where $p = 1$–8, A_{pt}, B_{pt} and C_{ptkl} are given in Appendix A.

In Eq. (7), the quantity X_{pij} is not only related to the quantities X_{tk0} and X_{t0l} at the boundary dependent points but also the quantities X_{tkj}, X_{til} and X_{tkl} at the inner dependent points. The maximal number of the unknown quantities is $6(m-1)(n-1) + 3(m+n+1)$. In order to reduce the unknown quantities, the area $[i,j]$ is spread according to the regular order as $[1,1]$, $[1,2], \ldots, [1,n]$, $[2,1]$, $[2,2], \ldots, [2,n], \ldots, [m,1]$, $[m,2], \ldots, [m,n]$. With the spread of the area according to the above-mentioned order, the quantities X_{tkj}, X_{til} and X_{tkl} at the inner dependent points can be eliminated by substituting the obtained results into the corresponding terms of the right-hand side of Eq. (7). By repeating this process, the quantity X_{pij} at the main point is only related to the quantities X_{rk0} $(r = 1,3,4,6,7,8)$ and X_{s0l} $(s = 2,3,5,6,7,8)$ at the boundary dependent points. The maximal number of the unknown quantities is reduced to $3(m+n+1)$. It can be noted the number of the unknown quantities of the present method is fewer than that of the finite element method for the same divisional number $m(\geqslant 3)$ and $n(\geqslant 3)$. Based on the above consideration, Eq. (7) is rewritten as follows:

$$X_{pij} = \sum_{d=1}^{6}\left\{\sum_{f=0}^{i}a_{pijfd}X_{rf0} + \sum_{g=0}^{j}b_{pijgd}X_{s0g}\right\} + \overline{q}_{pij}P + \sum_{c=0}^{m}\sum_{d=0}^{n}\overline{q}_{pijcd}P_{cd}, \quad (8)$$

where a_{pijfd}, b_{pijgd}, $\overline{q}_{pij}$ and $\overline{q}_{pijcd}$ are given in Appendix B.

Eq. (8) gives the discrete solution of the fundamental differential Eq. (3) of the bending problem of a plate with a concentrated load and point supports, and the discrete Green function is chosen as $X_{8ij}a^2/[PD_0(1-v^2)]$, that is $w(x_0, y_0, x, y)/\overline{P}$.

4. Characteristic equation

During the free vibration, for the harmonic displacement, there is inertial force $\rho h\omega^2\hat{w}(x,y)\,\mathrm{d}x\,\mathrm{d}y$ at every point (x,y), in which ρ is the mass density of the plate material, ω is the circular frequency and $\hat{w}(x,y)$ is the displacement at point (x,y). By applying the Green function $w(x_0,y_0,x,y)/\overline{P}$, which is the displacement of point (x_0,y_0) of a plate with a unite concentrated load at point (x,y), the displacement of point (x_0,y_0) of a plate with inertial force $\rho h\omega^2\hat{w}(x,y)\,\mathrm{d}x\,\mathrm{d}y$ at point (x,y) is $\rho h\omega^2\hat{w}(x,y)[w(x_0,y_0,x,y)/\overline{P}]\,\mathrm{d}x\,\mathrm{d}y$. Therefore, by using the method of superposition, the displacement amplitude $\hat{w}(x_0,y_0)$ of point (x_0,y_0) of the rectangular plate during the free vibration is given as follows:

$$\hat{w}(x_0,y_0)=\int_0^a\int_0^b \rho h\omega^2\hat{w}(x,y)[w(x_0,y_0,x,y)/\overline{P}]\,\mathrm{d}x\,\mathrm{d}y. \tag{9}$$

By using the numerical integration method and the following non-dimensional expressions:

$$\lambda^4=\frac{\rho_0 h_0\omega^2a^4}{D_0(1-v^2)},\quad k=1/(\mu\lambda^4),\quad H(\eta,\zeta)=\frac{\rho(x,y)}{\rho_0}\frac{h(x,y)}{h_0},$$

$$W(\eta,\zeta)=\frac{\hat{w}(x,y)}{a},\quad G(\eta_0,\zeta_0,\eta,\zeta)=\frac{w(x_0,y_0,x,y)}{a}\frac{D_0(1-v^2)}{\overline{P}a},$$

where ρ_0 is the standard mass density, the characteristic equation is obtained from Eq. (9) as

$$\begin{vmatrix} \mathbf{K}_{00} & \mathbf{K}_{01} & \mathbf{K}_{02} & \cdots & \mathbf{K}_{0m} \\ \mathbf{K}_{10} & \mathbf{K}_{11} & \mathbf{K}_{12} & \cdots & \mathbf{K}_{1m} \\ \mathbf{K}_{20} & \mathbf{K}_{21} & \mathbf{K}_{22} & \cdots & \mathbf{K}_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{K}_{m0} & \mathbf{K}_{m1} & \mathbf{K}_{m2} & \cdots & \mathbf{K}_{mm} \end{vmatrix}=0, \tag{10}$$

where

$$\mathbf{K}_{ij}=\beta_{mj}\begin{bmatrix} \beta_{n0}H_{j0}G_{i0j0}-k\delta_{ij} & \beta_{n1}H_{j1}G_{i0j1} & \beta_{n2}H_{j2}G_{i0j2} & \cdots & \beta_{nn}H_{jn}G_{i0jn} \\ \beta_{n0}H_{j0}G_{i1j0} & \beta_{n1}H_{j1}G_{i1j1}-k\delta_{ij} & \beta_{n2}H_{j2}G_{i1j2} & \cdots & \beta_{nn}H_{jn}G_{i1jn} \\ \beta_{n0}H_{j0}G_{i2j0} & \beta_{n1}H_{j1}G_{i2j1} & \beta_{n2}H_{j2}G_{i2j2}-k\delta_{ij} & \cdots & \beta_{nn}H_{jn}G_{i2jn} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_{n0}H_{j0}G_{inj0} & \beta_{n1}H_{j1}G_{inj1} & \beta_{n2}H_{j2}G_{inj2} & \cdots & \beta_{nn}H_{jn}G_{injn}-k\delta_{ij} \end{bmatrix}.$$

5. Numerical results

To investigate the validity of the proposed method, the frequency parameters are given for rectangular plates with arbitrarily located point supports, various aspect ratios, general boundary conditions and variable thickness. In this paper, three kinds of plates with variable thickness are

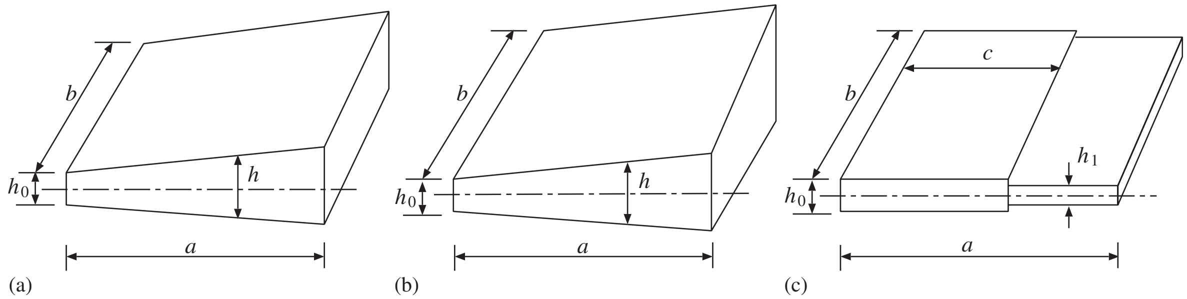

Fig. 2. Plates with variable thickness: (a) variable thickness in one direction $h = h_0(1 + \alpha x/a)$; (b) variable thickness in two directions $h = h_0(1 + \alpha x/a)(1 + \beta y/b)$ and (c) stepped thickness in one direction.

studied and they are shown in Fig. 2. The ratio of the thickness and length $h_0/a = 0.001$ is adopted. In all tables and figures, the symbols F, S, and C denote free, simply supported and clamped edges. Four symbols such as SCFC delegate the boundary conditions of the plate, the first indicating the conditions at $x = 0$, the second at $y = 0$, the third at $x = a$ and the fourth at $y = b$. All the convergent values of the frequency parameters are obtained for the plates by using Richardson's extrapolation formula for two cases of divisional numbers m $(= n)$. Some of the results are compared with those reported previously.

5.1. Rectangular plates with a central point support

In order to examine the convergency, numerical calculation is carried out by varying the number of divisions m and n for a SSSS square plate with a central point support. The lowest 6 natural frequency parameters of the plate are shown in Fig. 3. It shows a good convergency of the numerical results by the present method. After studying the figure, it is decided to obtain the convergent results of frequency parameter by using Richardson's extrapolation formula for two cases of divisional numbers m $(= n)$ of 14 and 16. By the same method, the suitable number of divisions m $(= n)$ can be determined for the other plates.

Table 1 shows the numerical values for the lowest 6 natural frequency parameter λ of SSSS rectangular plates with a central point support. The aspect ratios $b/a = 0.5, 1.0, 2.0$ are considered. The results obtained by Huang and Thambiratnam [6] and Kim and Dickinson [7] are also shown in the table. The nodal lines of 6 modes of free vibration of the plates with $b/a = 0.5, 1.0$ are shown in Fig. 4, which are identical to those obtained in Ref. [6] for the square plate. The mode with no nodal line is the third one for the square plate, but the second mode for plates with $b/a = 0.5$.

Table 2 shows the numerical values for the lowest 6 natural frequency parameter λ of CCCC rectangular plates with a central point support and aspect ratios $b/a = 0.5, 1.0, 2.0$. The present results of CCCC square plate are in good agreement with those obtained by Kim and Dickinson [7]. The nodal lines of 6 modes of the plates with $b/a = 0.5, 1.0$ are shown in Fig. 5. It can be observed for the modes with nodal lines passing through the center of the plate, the frequency parameters and mode shapes are the same for the plates with or without a central point support.

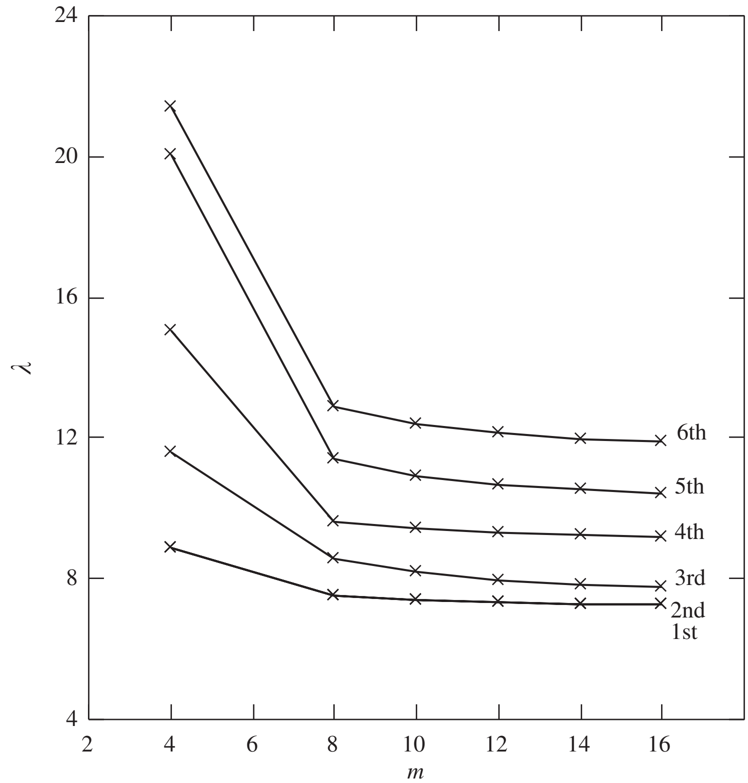

Fig. 3. The natural frequency parameter λ versus the divisional number m ($= n$) for the SSSS square plate with a central point support.

The phenomenon occurred in CCCC plate is as the same as that in SSSS plate. For the ratio $b/a = 1.0$, the nodal lines of modes of CCCC plates are as the same as those of SSSS plates. For the ratio $b/a = 0.5$, there are some changes of mode order in the third, fourth, fifth and sixth modes. From Tables 1 and 2, it can be noted the boundary conditions affect the frequency parameters considerately.

5.2. Square plates with a point support on a corner

Table 3 shows the numerical values for the lowest 6 natural frequency parameter λ of square plates with a point support on the corner (a, b). Three kinds of boundary conditions are considered. Table 3 involves the other values obtained by Kim and Dickinson [7] and it shows satisfactory accuracy of the numerical results by the present method. Fig. 6 shows the effect of the aspect ratio on the frequency parameters. It can be noted the frequency parameters decrease with the increase of the aspect ratio for all of these plates. Due to the similarity between the mode shapes of SSFF and CCFF plates, the mode shapes of CCFF plates are omitted. Only the mode shapes of SCFF and SSFF plates with $b/a = 1.0, 1.5$ are shown in Fig. 6.

Table 1
Natural frequency parameter λ for SSSS rectangular plates with a central point support

b/a	References	Mode sequence number					
		1st	2nd	3rd	4th	5th	6th
0.5	Present						
	14 × 14	9.195	10.132	13.479	15.242	14.631	16.086
	16 × 16	9.172	10.055	13.427	15.027	14.572	15.768
	Ex.*	9.096	9.802	13.259	14.325	14.381	14.728
1	Present						
	14 × 14	7.297	7.297	7.828	9.254	10.539	11.977
	16 × 16	7.272	7.272	7.743	9.216	10.449	11.885
	Ex.	7.191	7.191	7.466	9.095	10.158	11.585
	Ref. [6]	7.19	7.19	7.44	9.10	—	—
	Ref. [7]	7.192	7.192	7.466	9.098	10.172	11.597
2	Present						
	14 × 14	4.598	5.066	6.740	7.621	7.316	8.043
	16 × 16	4.586	5.027	6.714	7.514	7.286	7.884
	Ex.	4.548	4.901	6.629	7.162	7.195	7.364

Ex.*: The convergent values of the frequency parameters obtained by using Richardson's extrapolation formula.

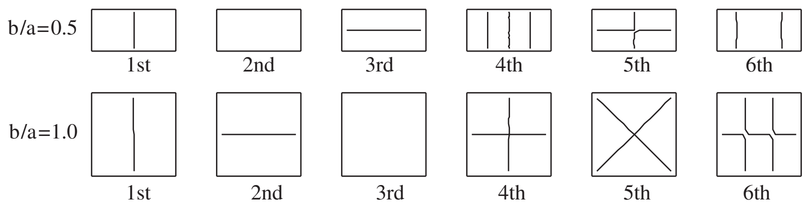

Fig. 4. Nodal patterns for SSSS rectangular plates with a central point support.

5.3. CFFF square plates with two arbitrarily located point supports

Table 4 shows the numerical values for the lowest 6 natural frequency parameter λ of the CFFF square plate with two point supports. The points $(a/2, 0), (a/2, b)$, the points $(a/2, b/4), (a/2, 3b/4)$ and points $(a/4, 3b/4), (3a/4, b/4)$ are chosen as the positions of the two supports, respectively. In order to compare with the results obtained by Saliba [1,2] and Kim and Dickinson [7], two kinds of Poisson's ratio $v = 0.333$ and $v = 0.3$ are used. Table 4 shows satisfactory accuracy of the numerical results by the present method and the fundamental frequency parameter of plate with two point supports along the edge is lower than those of corresponding plates with interior two point supports. The optimal location of the point support in increasing the fundamental frequency parameter discussed here is at point $(a/2, b/4), (a/2, 3b/4)$.

Table 2
Natural frequency parameter λ for CCCC rectangular plates with a central point support

b/a	References	Mode sequence number					
		1st	2nd	3rd	4th	5th	6th
0.5	Present						
	14×14	11.712	12.636	16.804	17.458	17.699	18.874
	16×16	11.674	12.534	16.702	17.160	17.595	18.391
	Ex.	11.549	12.200	16.368	16.185	17.253	16.813
1	Present						
	14×14	8.966	8.966	9.675	10.905	12.325	13.718
	16×16	8.919	8.919	9.545	10.844	12.182	13.579
	Ex.	8.766	8.766	9.121	10.645	11.714	13.125
	Ref. [7]	8.771	8.771	9.175	10.651	11.745	13.152
2	Present						
	14×14	5.856	6.318	8.402	8.729	8.850	9.438
	16×16	5.837	6.267	8.351	8.580	8.798	9.196
	Ex.	5.774	6.100	8.184	8.093	8.627	8.407

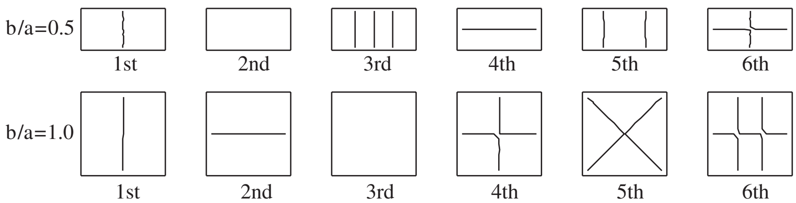

Fig. 5. Nodal patterns for CCCC rectangular plates with a central point.

5.4. SSSS square plates with variable thickness in one direction

Table 5 presents the numerical values for the lowest 6 natural frequency parameter λ of the SSSS square plates with variable thickness in one direction shown in Fig. 2(a). For the plate with a central point support, three cases of $\alpha = 0.1, 0.8, 1.2$ are considered. The numerical values for the lowest 6 natural frequency parameter λ of the SSSS square plates without point support are also shown and compared with the results of Appl and Byers [8]. The nodal lines of 6 modes of free vibration of these plates are shown in Fig. 7. For the plates without point supports, the vertical nodal lines move to the thinner part with increase of the value of α. The changes can be found in the third, the fourth and the sixth mode shapes. For the plates with a central point support, the straight lines change to curve lines with increase of the value of α. The changes can be found in the first and fifth mode shapes. The obvious change can be also found in the third mode.

Table 3
Natural frequency parameter λ for square plates with a point support on the corner (a, b)

BC	References	Mode sequence number					
		1st	2nd	3rd	4th	5th	6th
SSFF	Present						
	14×14	3.274	4.285	5.814	6.958	7.465	8.467
	16×16	3.260	4.279	5.790	6.920	7.429	8.417
	Ex.	3.213	4.260	5.712	6.794	7.311	8.255
	Ref. [7]	3.174	4.261	5.663	6.765	7.315	8.214
SCFF	Present						
	14×14	3.637	4.763	6.224	7.221	8.041	8.860
	16×16	3.623	4.752	6.196	7.184	7.989	8.809
	Ex.	3.576	4.717	6.108	7.065	7.820	8.644
	Ref. [7]	3.538	4.711	6.059	7.049	7.807	8.617
CCFF	Present						
	14×14	4.120	5.048	6.603	7.788	8.315	9.314
	16×16	4.100	5.038	6.574	7.745	8.265	9.251
	Ex.	4.035	5.005	6.478	7.563	8.102	9.047
	Ref. [7]	3.988	5.008	6.426	7.535	8.117	9.009

5.5. SSSS square plates with variable thickness in two directions

Table 6 presents the numerical values for the lowest 6 natural frequency parameter λ of the SSSS square plates with variable thickness in two directions shown in Fig. 2(b). For the plate with a central point support, three kinds of combination of α and β are considered. The numerical values for the lowest 6 natural frequency parameter λ of the SSSS square plates without point support are also shown and compared with the results of Singh and Saxena [10]. Table 6 shows the frequency parameter λ increases with the increase of α or β for both the plates with and without point support.

5.6. SSSS square plates with stepped thickness in one direction

Table 7 presents the numerical values for the lowest 6 natural frequency parameter λ of the SSSS square plates with stepped thickness in one direction shown in Fig. 2(c). For the plate with a central point support, two kinds of the ratios c/a and h_1/h_0 are considered. The numerical values for the lowest 6 natural frequency parameter λ of the SSSS square plates without point support are also shown and compared with the exact solutions obtained by Xiang and Wang [11]. It is noted the present results agree well with these exact solutions of plates without point support even for the higher frequency parameters. The frequency parameters increase with the increase of the ratio of c/a or h_1/h_0. The effects of the ratios c/a and h_1/h_0 on the frequency parameters are significant.

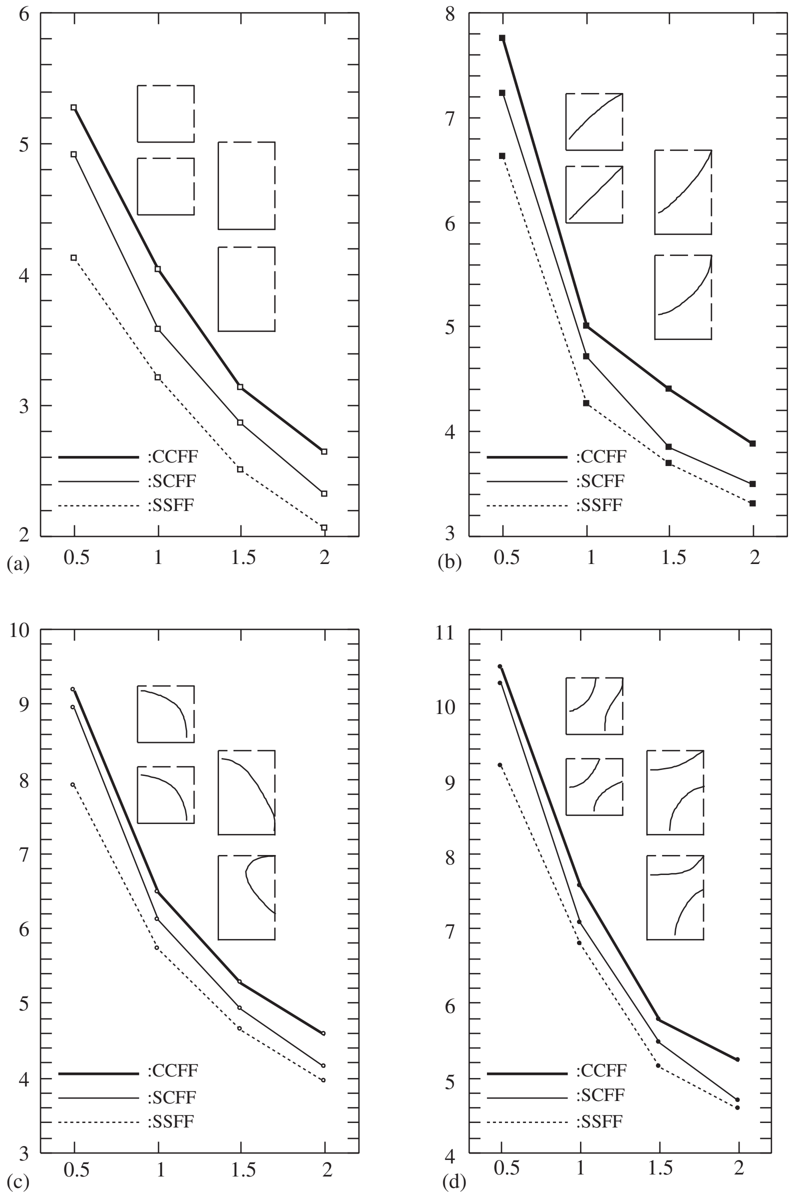

Fig. 6. The natural frequency parameter λ and mode shape versus the aspect ratio for the rectangular plates with a point support on the corner (a, b): (a) the first frequency parameters and mode shapes; (b) the second frequency parameters and mode shapes; (c) the third frequency parameters and mode shapes and (d) the fourth frequency parameters and mode shapes.

6. Conclusions

A discrete method is extended for analyzing the free vibration problem of rectangular plate with point supports. No prior assumption of shape of deflection, such as shape functions used in the

Table 4
Natural frequency parameter λ for CFFF rectangular plates with two point supports

Position of point supports	v	References	Mode sequence number					
			1st	2nd	3rd	4th	5th	6th
$(a/2,0),(a/2,b)$	0.333	Present						
		Ex.	2.570	4.154	5.205	6.451	7.440	8.037
		Ref. [1]	2.525	4.099	5.162	6.364	7.295	—
		Ref. [7]	2.538	4.127	5.166	6.420	7.394	—
$(a/2,b/4),(a/2,3b/4)$	0.333	Present						
		Ex.	3.225	4.118	5.184	7.054	7.561	8.011
		Ref. [2]	3.173	4.086	5.228	6.999	7.424	—
$(a/2,0),(a/2,b)$	0.3	Present						
		Ex.	2.586	4.173	5.166	6.467	7.397	7.998
		Ref. [7]	2.562	4.172	5.158	6.473	7.392	—
$(a/2,b/4),(a/2,3b/4)$	0.3	Present						
		Ex.	3.133	4.221	5.280	7.474	6.831	8.016
$(a/4,3b/4),(3a/4,b/4)$	0.3	Present						
		Ex.	2.908	5.045	5.363	6.015	6.521	7.506

Table 5
Natural frequency parameter λ for SSSS square plates with variable thickness in one direction

Cases of point support	α	References	Mode sequence number					
			1st	2nd	3rd	4th	5th	6th
Plate with no point support	0.1	Present						
		Ex.	4.660	7.367	7.367	9.318	10.402	10.406
		Ref. [8]	4.661	—	—	—	—	—
	0.8	Present						
		Ex.	5.356	8.412	8.466	10.698	11.767	11.914
		Ref. [8]	5.355	—	—	—	—	—
Plate with a central point support	0.1	Present						
		Ex.	7.360	7.367	7.657	9.318	10.405	11.858
	0.8	Present						
		Ex.	8.241	8.412	9.022	10.698	11.835	13.269
	1.2	Present						
		Ex.	8.658	8.917	9.731	11.384	12.502	13.920

finite element method, is employed in this method. A concentrated load with Dirac's delta function is used to simulate the point support. The characteristic equation of the free vibration is gotten by using the Green function. The effects of positions of point supports, the variable

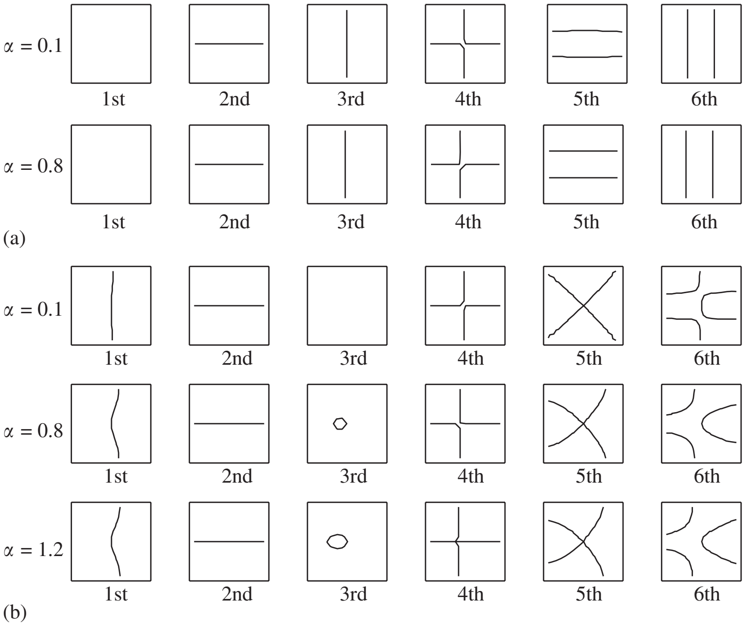

Fig. 7. Nodal patterns for SSSS rectangular plates with variable thickness in one direction: (a) plates without point support and (b) plates with a central point support.

Table 6
Natural frequency parameter λ for SSSS square plates with variable thickness in two directions

Cases of point support	α	β	References	Mode sequence number					
				1st	2nd	3rd	4th	5th	6th
Plate with no point support	0.5	−0.5	Present						
			Ex.	4.365	6.806	6.880	8.705	9.520	9.620
			Ref. [10]	4.365	6.810	6.882	—	—	—
	0.5	0.5	Present						
			Ex.	5.659	8.880	8.935	11.302	12.515	12.515
			Ref. [10]	5.659	8.885	8.938	—	—	—
Plate with a central point support	−0.5	−0.5	Present						
			Ex.	4.995	5.302	5.776	6.702	7.319	8.005
	0.5	−0.5	Present						
			Ex.	6.583	6.869	7.431	8.706	9.566	10.573
	0.5	0.5	Present						
			Ex.	8.684	8.945	9.510	11.303	12.515	13.972

thickness, the aspect ratio and the boundary conditions on the frequencies are considered. Some results by the present method have been compared with those previously reported. It shows that the present results have a good convergence and satisfactory accuracy.

Table 7
Natural frequency parameter λ for SSSS square plates with stepped thickness in one direction

Cases of point support	c/a	h_1/h_0	References	Mode sequence number					
				1st	2nd	3rd	4th	5th	6th
Plate with no point support	0.25	0.5	Present						
			Ex.	3.647	5.463	5.464	7.115	7.470	7.705
			Ref. [11]	3.658	5.451	5.477	7.136	7.485	7.666
	0.25	0.8	Present						
			Ex.	4.198	6.578	6.593	8.365	9.221	9.359
			Ref. [11]	4.199	6.582	6.589	8.367	9.239	9.368
	0.75	0.8	Present						
			Ex.	4.411	6.947	7.018	8.827	9.837	9.939
			Ref. [11]	4.421	6.966	7.035	8.844	9.862	9.980
Plate with a central point support	0.25	0.5	Present						
			Ex.	5.227	5.473	5.754	7.115	7.582	8.614
	0.25	0.8	Present						
			Ex.	6.547	6.578	6.833	8.365	9.290	10.598
	0.75	0.5	Present						
			Ex.	6.223	6.695	7.351	8.410	9.423	10.419
	0.75	0.8	Present						
			Ex.	6.923	7.032	7.392	8.843	9.904	11.207

Acknowledgment

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Appendix A

$$A_{p1} = \gamma_{p1}, \quad A_{p2} = 0, \quad A_{p3} = \gamma_{p2}, \quad A_{p4} = \gamma_{p3}, \quad A_{p5} = 0,$$

$$A_{p6} = \gamma_{p4} + v\gamma_{p5}, \quad A_{p7} = \gamma_{p6}, \quad A_{p8} = \gamma_{p7},$$

$$B_{p1} = 0, \quad B_{p2} = \mu\gamma_{p1}, \quad B_{p3} = \mu\gamma_{p3}, \quad B_{p4} = 0,$$

$$B_{p5} = \mu\gamma_{p2}, \quad B_{p6} = \mu\gamma_{p6}, \quad B_{p7} = \mu(v\gamma_{p1} + \gamma_{p5}), \quad B_{p8} = \gamma_{p8},$$

$$C_{p1kl} = \mu(\gamma_{p3} + k_{kl}\gamma_{p7}), \quad C_{p2kl} = \mu\gamma_{p2} + k_{kl}\gamma_{p8},$$

$$C_{p3kl} = J\gamma_{p6}, \quad C_{p4kl} = I_{kl}\gamma_{p4}, \quad C_{p5kl} = I_{kl}\gamma_{p5},$$

$$C_{p6kl} = -\mu\gamma_{p7}, \quad C_{p7kl} = -\gamma_{p8}, \quad C_{p8kl} = 0, \quad [\gamma_{pk}] = [\bar{\gamma}_{pk}]^{-1},$$

$$\overline{\gamma}_{11} = \beta_{ii}, \quad \overline{\gamma}_{12} = \mu\beta_{jj}, \quad \overline{\gamma}_{22} = -\mu\beta_{ij}, \quad \overline{\gamma}_{23} = \beta_{ii}, \quad \overline{\gamma}_{25} = \mu\beta_{jj},$$

$$\overline{\gamma}_{31} = -\mu\beta_{ij}, \quad \overline{\gamma}_{33} = \mu\beta_{jj}, \quad \overline{\gamma}_{34} = \beta_{ii}, \quad \overline{\gamma}_{44} = -I_{ij}\beta_{ij}, \quad \overline{\gamma}_{46} = \beta_{ii},$$

$$\overline{\gamma}_{47} = \mu v\beta_{jj}, \quad \overline{\gamma}_{55} = -I_{ij}\beta_{ij}, \quad \overline{\gamma}_{56} = v\beta_{ii}, \quad \overline{\gamma}_{57} = \mu\beta_{jj}, \quad \overline{\gamma}_{63} = -J_{ij}\beta_{ii},$$

$$\overline{\gamma}_{66} = \mu\beta_{jj}, \quad \overline{\gamma}_{67} = \beta_{ii}, \quad \overline{\gamma}_{71} = -\mu k_{ij}\beta_{ij}, \quad \overline{\gamma}_{76} = \mu\beta_{ij}, \quad \overline{\gamma}_{78} = \beta_{ii}, \quad \overline{\gamma}_{82} = -H_{ij}\beta_{ij},$$

$$\overline{\gamma}_{87} = \beta_{ij}, \quad \overline{\gamma}_{88} = \beta_{jj}, \qquad \text{other } \overline{\gamma}_{pk} = 0, \quad \beta_{ij} = \beta_{ii}\beta_{jj}.$$

Appendix B

$$a_{1i0i1} = a_{3i0i2} = a_{4i0i3} = 1, \quad a_{6i0i4} = a_{7i0i5} = a_{8i0i6} = 1,$$

$$b_{20jj1} = b_{30jj2} = b_{50jj3} = 1, \quad b_{60jj4} = b_{70jj5} = b_{80jj6} = 1, \quad b_{30002} = 0,$$

$$\begin{aligned} a_{pijfd} = \sum_{t=1}^{8} \Bigg\{ & \sum_{k=0}^{i} \beta_{ik} A_{pt}[a_{tk0fd} - a_{tkjfd}(1-\delta_{ki})] + \sum_{l=0}^{j} \beta_{jl} B_{pt}[a_{t0lfd} - a_{tilfd}(1-\delta_{lj})] \\ & + \sum_{k=0}^{i}\sum_{l=0}^{j} \beta_{ik}\beta_{jl} C_{ptkl} a_{tklfd}(1-\delta_{ki}\delta_{lj}) \Bigg\}, \end{aligned}$$

$$\begin{aligned} b_{pijfd} = \sum_{t=1}^{8} \Bigg\{ & \sum_{k=0}^{i} \beta_{ik} A_{pt}[b_{tk0gd} - b_{tkjgd}(1-\delta_{ki})] + \sum_{l=0}^{j} \beta_{jl} B_{pt}[b_{t0lgd} - b_{tilgd}(1-\delta_{lj})] \\ & + \sum_{k=0}^{i}\sum_{l=0}^{j} \beta_{ik}\beta_{jl} C_{ptkl} b_{tklgd}(1-\delta_{ki}\delta_{lj}) \Bigg\}, \end{aligned}$$

$$\begin{aligned} \overline{q}_{pij} = \sum_{t=1}^{8} \Bigg\{ & \sum_{k=0}^{i} \beta_{ik} A_{pt}[\overline{q}_{tk0} - \overline{q}_{tkj}(1-\delta_{ki})] + \sum_{l=0}^{j} \beta_{jl} B_{pt}[\overline{q}_{t0l} - \overline{q}_{til}(1-\delta_{lj})] \\ & + \sum_{k=0}^{i}\sum_{l=0}^{j} \beta_{ik}\beta_{jl} C_{ptkl} - A_{p1} u_{iq} u_{jr} \Bigg\}, \end{aligned}$$

$$\begin{aligned} \overline{q}_{fpijcd} = \sum_{e=1}^{8} \Bigg\{ & \sum_{k=0}^{i} \beta_{ik} A_{pe}[\overline{q}_{fek0cd} - \overline{q}_{fekjcd}(1-\delta_{ki})] + \sum_{l=0}^{j} \beta_{jl} B_{pe}[\overline{q}_{fe0lcd} - \overline{q}_{feilcd}(1-\delta_{lj})] \\ & + \sum_{k=0}^{i}\sum_{l=0}^{j} \beta_{ik}\beta_{jl} C_{pekl} \overline{q}_{feklcd}(1-\delta_{ki}\delta_{lj}) \Bigg\} - \gamma_{pf} u_{ik} u_{jr} \overline{u}_{fkl}. \end{aligned}$$

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