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Piecewise-linearized methods for oscillators with fractional-power nonlinearities

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Abstract

A piecewise linearization method based on the Taylor series expansion of nonlinear ordinary differential equation with respect to time, displacement and velocity is developed for the study of one-degree-of-freedom nonlinear oscillators with smooth and fractional-power nonlinearities. The method provides smooth solutions and explicit, nonstandard finite difference expressions for the displacement and velocity, and is exact for constant coefficients, linear ordinary differential equations with linear time-dependent forcings. The method is applied to ten oscillators with fractional-power nonlinearities and its results are compared with those of harmonic balance techniques, Ritz procedure and numerical solutions based on nonstandard finite difference methods, in terms of displacement, velocity, energy and angular frequency, when available. It is shown that the linearization method presented here provides slightly more (less) accurate results than those of nonstandard methods for fractional powers greater (smaller) than one, and both techniques are more accurate than those that freeze the nonlinearities at the previous time step or linearize the nonlinear terms with respect to only the displacement or the velocity. For the examples and time steps considered in this paper, the linearization method has been found to be more accurate than the harmonic balance and Ritz procedures.

1. Introduction

Nonlinear oscillators with fractional-power nonlinearities have been of great interest for many years [1] and have recently been the subject of active research by several investigators who have proposed to analyze them by means of perturbation methods, e.g., harmonic balance, slowly varying amplitude and phase, etc., and numerical techniques. For example, Mickens et al. [2] have studied a simple harmonic oscillator with fractional damping by the Krylov–Bogoliubov–Mitropolski or slowly varying amplitude and phase technique. Oyedeji [3] has analyzed a nonlinear elastic force van der Pol oscillator and employed the harmonic balance method to determine the first approximation to the solution. He has also used a nonstandard finite difference method [4] to determine its numerical solution. Hogan [5] has employed the method of multiple (time) scales to study the dynamics of a piecewise smooth van der Pol oscillator and determined the amplitude of the limit cycle.

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Several examples of oscillators with fractional-powers nonlinearities were presented by Mickens [6] who proposed their analysis by means of the combination of equivalent linearization and averaging techniques. Other authors have applied first linearization and then harmonic balance techniques to the analysis of nonlinear oscillators [7]. These techniques have the advantage that they result in simple algebraic equations, instead of the complicated nonlinear ones that arise upon the application of the harmonic balance method. Combined linearization/perturbation methods for perturbed smooth dynamical systems first linearize the ordinary differential equation by either a straightforward linearization or a minimization of the difference between the nonlinear terms and the linear ones that approximate them [8], and are approximation techniques that are usually limited to small values of the perturbation parameter.

In this paper, a piecewise linearized method for the solution of one-degree-of-freedom problems with smooth or fractional-power nonlinearities is presented. The method is not a perturbation one, but it is based on the piecewise time linearization of the nonlinear differential equation and provides smooth solutions since the displacement and the velocity are assumed to be continuous at the end of the intervals. The method is based on and is an extension of the time linearization technique developed by the author and co-workers [9–11] for nonlinear first-order ordinary differential equations and does not require the presence of either linear damping or linear stiffness terms in the governing equation. In addition, the linearization method presented here provides second-order accurate, nonstandard, explicit, finite difference equations which are regular for smooth vector fields with symmetric Jacobian matrices, preserves the linear stability of hyperbolic equilibrium points, exhibits invariant closed curves that converge to periodic hyperbolic trajectories, and can adequately approximate any trajectory near a stable periodic trajectory for sufficiently small time steps [12]. Furthermore, the technique presented here is different from that proposed by Dai and Singh [13] in that the latter requires the presence of either linear damping and/or linear stiffness terms, and is based on the Taylor series expansion of the nonlinearities in the time or independent variable and, therefore, requires stronger differentiability requirements than those of the linearization method presented in this paper, if linear, quadratic or cubic terms are kept in the Taylor series expansion of Dai and Singh's procedure [13].

The paper has been arranged as follows. In Section 2, the piecewise linearization method for nonlinear oscillators with smooth nonlinearities is first formulated and compared with that of Dai and Singh [13]. In that section, we show that, upon imposing continuity and smoothness requirements at the end of each interval, the method results in nonstandard, explicit finite difference formulae that depend on the derivatives of the nonlinearities with respect to time, displacement and velocity. In that section, we also analyze the linear stability and convergence of the piecewise linearization method. A variety of conservative and non-conservative oscillators with fractional-power nonlinearities are studied in Section 3, where the results of the piecewise linearization method presented here are compared with available exact, asymptotic and numerical solutions in terms of the predicted displacement, velocity, energy and angular frequency. The paper ends with a summary of the major findings.

2. Formulation

In this paper, we shall be concerned with the following single degree-of-freedom equation

$$\ddot{x} = f(t, x, \dot{x}), \quad t > 0, \tag{1}$$

subject to $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$, where t is time, x is the displacement, and the dot denotes differentiation with respect to t. Eq. (1) can be written as a system of two first-order ordinary differential equations upon defining $y \equiv \dot{x}$ and has a unique solution in the interval $[t_0, t_0 + \alpha]$ provided that f is continuous on the rectangle $Q \equiv \{(t, x) : t_0 \leqslant t \leqslant t_0 + a, \|X - X_0\| \leqslant b\}$ where $X \equiv (x, \dot{x})^{\mathrm{T}}$ and the superscript T denotes transpose, and satisfies a uniform Lipschitz condition with respect to both x and $\dot{x}$ on Q, where $\alpha = \min\{a, b/M\}$ with $M = \max \|f\| : (t, x, \dot{x}) \in Q$, according to the Picard–Lindelof's theorem.

Eq. (1) cannot, in general, be integrated analytically, and, therefore, its solution must be determined numerically. In this paper, we first decompose the interval of integration $0 \leqslant t \leqslant T$ $(T > 0)$ into non-overlapping intervals $t_n < t \leqslant t_{n+1}$ where n is a natural number including zero such that $t_0 = 0$. Furthermore, although we shall mainly be concerned with nonlinear oscillators with fractional-power nonlinearities where f is continuous and differentiable with respect t, and continuous but not necessarily differentiable with respect to x and/or $\dot{x}$, e.g., $f(t,x,\dot{x}) = Ax^p$ where p is a rational number such that $0<p<1$ and A is a constant, the piecewise linearization methods presented here are also valid for smooth $f(t,x,\dot{x})$, i.e., f is a continuous with continuous first-order derivatives with respect to t, x and $\dot{x}$. For this reason, the presentation that follows first considers the case that f is a smooth function of its three arguments. It must be emphasized that the examples considered in this paper, except Examples 7 and 9 with $n=0$, do not satisfy the Lipschitz condition and, therefore, the Picard–Lindelof's theorem does not guarantee the existence and uniqueness of their solutions, and, as a consequence, there is no guarantee that these examples have smooth solutions.

2.1. Smooth $f(t,x,\dot{x})$

For smooth $f(t,x,\dot{x})$ in $[t_n, t_{n+1}]$, Eq. (1) can be linearized with respect to the previous time level in the interval $t_n \leqslant t \leqslant t_{n+1}$, and the result of this linearization can be written as the following second-order, linear, ordinary differential equation

$$\ddot{x} = f_n + T_n(t-t_n) + J_n(x-x_n) + H_n(\dot{x}-\dot{x}_n), \quad t_n < t \leqslant t_{n+1}, \tag{2}$$

where $f_n \equiv f(t_n, x_n, \dot{x}_n)$, $T_n \equiv \partial f/\partial t(t_n, x_n, \dot{x}_n)$, $J_n \equiv \partial f/\partial x(t_n, x_n, \dot{x}_n)$ and $H_n \equiv \partial f/\partial \dot{x}(t_n, x_n, \dot{x}_n)$, and the right-hand side of Eq. (1) has been linearized with respect to its three arguments. Eq. (2) is subject to $x(t_n) = x_n$ and $\dot{x}(t_n) = \dot{x}_n$.

Eq. (2) satisfies the conditions of the Picard–Lindelof's theorem and, therefore, has a unique solution in $[t_n, t_{n+1}]$, can be solved analytically in each interval $t_n \leqslant t \leqslant t_{n+1}$ and the form of its solution depends on the roots of the characteristic equation $\lambda^2 - H_n\lambda - J_n = 0$ and on the values of H_n and J_n. For example, if $H_n^2/4 + J_n < 0$ and $J_n \neq 0$, then the solution to Eq. (2) can be written as

$$x(t) = \exp(\mu_n(t-t_n))(A_n\cos(\Omega_n(t-t_n)) + B_n\sin(\Omega_n(t-t_n))) + \alpha_n + \beta_n(t-t_n), \tag{3}$$

where $\mu_n = (1/2)H_n$, $\Omega_n^2 = -(H_n^2/4 + J_n)$, $\beta_n = -T_n/J_n$, $\alpha_n = -(H_n/J_n)\beta_n - P_n/J_n$, $P_n = f_n - J_nx_n - H_n\dot{x}_n$, and the values of A_n and B_n can be easily determined from the conditions $x(t_n) = x_n$ and $\dot{x}(t_n) = \dot{x}_n$ as

$$A_n = x_n - \alpha_n, \tag{4}$$

$$B_n = \frac{1}{\Omega_n}(\dot{x}_n - A_n\mu_n - \beta_n). \tag{5}$$

Eq. (3) yields the following explicit non-standard finite difference expressions for the conditions on the discriminant and J_n considered above

$$x_{n+1} = x(t_{n+1}) = \exp(\mu_nk_n)(A_n\cos(\Omega_nk_n) + B_n\sin(\Omega_nk_n)) + \alpha_n + \beta_nk_n, \tag{6}$$

$$\dot{x}_{n+1} = \dot{x}(t_{n+1}) = \exp(\mu_nk_n)((A_n\mu_n + B_n\Omega_n)\cos(\Omega_nk_n) + (B_n\mu_n - A_n\Omega_n)\sin(\Omega_nk_n)) + \beta_n, \tag{7}$$

where $k_n = t_{n+1} - t_n$ is the (possibly variable) time step.

Analytical solutions to Eq. (2) for other values of the discriminant, J_n and H_n can be found in any good textbook on linear ordinary differential equations, e.g. [14], and are not reported here.

In this paper, we shall refer to the finite difference formulae of Eq. (2) as the FL method (or simply FL) to signify that it corresponds to a piecewise linearized (L) technique where the linearization has been performed with respect to the three arguments of $f(t,x,\dot{x})$, i.e., with respect to t, x and $\dot{x}$. This method, i.e., Eqs. (6) and (7), provides the exact solution to Eq. (1) (in the absence of round-off errors) if $f(t,x,\dot{x})$ is a linear function of its three arguments, i.e., if $f(t,x,\dot{x}) = b + ct + dx + g\dot{x}$, where b, c, d and g are constants.

Since FL is based on the piecewise analytical solution of linear equations, it does not require the presence of linear damping and stiffness terms. This is to be contrasted with other linearization techniques which require the presence of linear damping and/or linear stiffness [13], i.e., Eq. (1) must be of the form

$$\ddot{x} + c\dot{x} + dx = F(t,x,\dot{x}), \quad t>0, \tag{8}$$

where c and d are constants, at least, one of which is different from zero, and use piecewise analytical integration of Eq. (8) by expanding $F(t,x,\dot{x})$ in Taylor series expansions in terms of $t-t_n$. If only the first (constant) term of the Taylor series expansion is retained, these techniques are identical to the one presented above if in Eq. (2) one sets $T_n = J_n = H_n = 0$. Moreover, the piecewise linearization method presented here results in non-standard finite difference equations, i.e., Eqs. (6) and (7), which are different from those obtained by Mickens [4]; it also provides analytical solutions in each (t_n, t_{n+1}) which are smooth in each $[t_n, t_{n+1}]$.

Eq. (2) can also be written as a system of first-order, linear, ordinary differential equations $\dot{X} = F$ where $X \equiv (x, \dot{x})^{\mathrm{T}}$ and $F \equiv (\dot{x}, f_n + T_n(t - t_n) + J_n(x - x_n) + H_n(\dot{x} - \dot{x}_n))^{\mathrm{T}}$ for $t_n < t \leqslant t_{n+1}$, and, for the scalar equation $\dot{x} = \lambda x$ where λ is a constant, it can be easily shown that the piecewise linearization method presented above is A-stable, i.e., it is stable for $\mathrm{Re}(\lambda) < 0$, because its amplification factor, for a constant step size equal to k, is equal to $\mathrm{e}^{\lambda k}$.

The accuracy of the piecewise linearization method presented above can be easily (but lengthly) deduced by expanding in Taylor's series the left- and right-hand sides of Eqs. (6) and (7). In order to illustrate the accuracy of the piecewise linearization method presented above while avoiding the lengthy algebra that the Taylor's series expansions of Eqs. (6) and (7) require, we consider the special case $f = f(x)$, $J_n \equiv -\lambda_n^2 < 0$ and constant step size, k. Under these conditions, Eqs. (6) and (7) can be written as

$$x_{n+1} = x_n + \frac{f_n}{\lambda_n^2}(1 - \cos(\lambda_n k)) + \frac{\dot{x}_n}{\lambda_n}\sin(\lambda_n k), \tag{9}$$

$$\dot{x}_{n+1} = \frac{f_n}{\lambda_n}\sin(\lambda_n k) + \dot{x}_n\cos(\lambda_n k), \tag{10}$$

which, upon expanding the right-hand sides in Taylor's series about t_n, can be written as

$$x_{n+1} = x_n + k\dot{x}_n + \frac{k^2}{2}f_n - \frac{k^3}{6}\lambda_n^2\dot{x}_n - \frac{k^4\lambda_n^2}{24}f_n + O(k^5), \tag{11}$$

$$\dot{x}_{n+1} = \dot{x}_n + kf_n - \frac{k^2\lambda_n^2}{2}\dot{x}_n - \frac{k^3\lambda_n^2}{6}f_n + O(k^4). \tag{12}$$

On the other hand, if f is sufficiently differentiable, it can be easily shown by means of Taylor's series expansion that the exact solution of Eq. (1) can be written as

$$x_{n+1}^e = x_n + k\dot{x}_n + \frac{k^2}{2}f_n - \frac{k^3}{6}\lambda_n^2\dot{x}_n + \frac{k^4}{24}(f_{xx}\dot{x}_n^2 - \lambda_n^2 f_n) + O(k^5), \tag{13}$$

$$\dot{x}_{n+1}^e = \dot{x}_n + kf_n - \frac{k^2\lambda_n^2}{2}\dot{x}_n + \frac{k^3\lambda_n^2}{6}(f_{xx}\dot{x}_n^2 - f_n) + O(k^4), \tag{14}$$

where $f_{xx} = \mathrm{d}^2f/\mathrm{d}x^2$.

Eqs. (11)–(14) imply that the differences in the displacement and velocity between the exact solution and that of the piecewise linearization method presented here are $x_{n+1}^e - x_{n+1} = O(k^4)$ and $\dot{x}_{n+1}^e - \dot{x}_{n+1} = O(k^3)$, respectively, if $f \in C^2$. Furthermore, for $f = f(x)$, Eq. (1) has a first integral which can be expressed as

$$\frac{1}{2}(\dot{x}_{n+1}^{e2} - \dot{x}_n^{e2}) = \int_{x_n}^{x_{n+1}} f(x)\,\mathrm{d}x, \tag{15}$$

whereas the piecewise linearization method presented here yields

$$\tfrac{1}{2}(\dot{x}_{n+1}^2 - \dot{x}_n^2) = f_n(x_{n+1} - x_n) + \tfrac{1}{2}J_n(x_{n+1} - x_n)^2, \tag{16}$$

which coincides with Eq. (15) if $f(x)$ is a linear function of x.

In Eq. (2), the right-hand side of Eq. (1) has been linearized with respect to the three arguments of the $f(t, x, \dot{x})$, i.e., a complete or full (F) linearization of $f(t, x, \dot{x})$ has been performed. Partial piecewise linearization methods (here referred to as PL) for the case of smooth $f(t, x, \dot{x})$ can also be developed; for e.g., one may choose to linearize f with respect to only t and x, t and $\dot{x}$, or x and $\dot{x}$, and obtain second-order linear ordinary differential equations similar to Eq. (2) which can be integrated analytically in $[t_n, t_{n+1}]$ and whose solutions can be easily derived from Eq. (2). For example, if the linearization of $f(t, x, \dot{x})$ is performed with respect to only t and x, t and $\dot{x}$, or x and $\dot{x}$, the corresponding analytical solutions can be obtained by setting H_n, J_n or T_n to zero, respectively, in Eqs. (2), (6) and (7). These partial piecewise linearization methods are less accurate than the FL technique presented in this section because the linearization of $f(t, x, \dot{x})$ is not performed with respect to its three arguments. A particularly interesting case arises when f in Eq. (1) is approximated as f_n, and will be considered in detail in the next subsection.

2.2. Continuous $f(t, x, \dot{x})$

If the $f(t, x, \dot{x})$ is a continuous but not differentiable function with respect to any of its three arguments, the formulation presented above cannot be applied, for that formulation requires that T, J and H be continuous functions. In this case, Eq. (1) can be approximated by

$$\ddot{x} = f_n, \quad t_n < t \leqslant t_{n+1}, \tag{17}$$

which implies that, in the interval $[t_n, t_{n+1}]$, the acceleration is constant and, therefore, the velocity and displacement are linear and quadratic functions, respectively, of time, i.e.,

$$x_{n+1} = x_n + k_n\dot{x}_n + \frac{k_n^2}{2} f_n, \tag{18}$$

$$\dot{x}_{n+1} = \dot{x}_n + k_n f_n, \tag{19}$$

which can also be obtained by taking the limits $T_n \to 0$, $J_n \to 0$ and $H_n \to 0$ in Eqs. (6) and (7). Eq. (19) corresponds to the first-order accurate explicit Euler method whose linear stability demands that $|\lambda k| \leqslant 1$ for the scalar equation $\dot{x} = \lambda x$ where λ is a constant. Furthermore, a comparison between the exact value of x_{n+1} determined from Eq. (1) by assuming that f is sufficiently differentiable and Eqs. (18) and (19) indicate that $x^e_{n+1} - x_{n+1} = \frac{1}{6} k_n^3 \, \mathrm{d}f/\mathrm{d}t(\phi)$ and $\dot{x}^e_{n+1} - \dot{x}_{n+1} = \frac{1}{2} k_n^2 \, \mathrm{d}f/\mathrm{d}t(\theta)$, respectively, where the superscript e denotes the exact solution, $\phi \in [t_n, t_{n+1}]$ and $\theta \in [t_n, t_{n+1}]$. Therefore, Eqs. (18) and (19) are second- and first-order accurate, respectively, for smooth f. For non-smooth f, there is no guarantee that a unique and smooth solution exists, and $\mathrm{d}f/\mathrm{d}t = \partial f/\partial t + (\partial f/\partial x)\dot{x} + f\,\partial f/\partial \dot{x}$ may not exist at some points in the interval of integration because f may not be differentiable with respect to t, x and/or $\dot{x}$ and, therefore, the above error estimates are not valid. This is a consequence of the lack of differentiability of f and the fact that estimates of the local truncation errors of finite difference methods for Eq. (1) make use of Taylor series expansions; it is also the reason of why most of the numerical methods that have been used to-date in non-smooth mechanics problems have ignored differentiability and employed the explicit Euler method given by Eqs. (18) and (19) [1]. The few attempts that have been made to deal with non-smooth dynamics problems by means of non-dissipative second-order accurate operator-splitting techniques, e.g. [15], for smooth problems have shown that these techniques exhibit only first-order accuracy in non-smooth problems, i.e., their order of accuracy drops from two to one. Although the analysis presented above is not valid for non-smooth f, it indicates that, for non-smooth f, the order of the errors in x_{n+1} and $\dot{x}_{n+1}$ drops by, at least, one in both cases, i.e., the order of accuracy for the displacement and velocity are expected to be, at most, 2 and 1, respectively, for non-smooth f. This will be verified at the end of next section.

Since Eqs. (18) and (19) can also be used as approximations when $f(t, x, \dot{x})$ is smooth in $[t_n, t_{n+1}]$, one may state that the approximation of f in $[t_n, t_{n+1}]$ by f_n results in a conditionally stable method whose accuracy is lower than that of the piecewise linearization method presented in the previous subsection. Moreover, for the case that $f = f(x)$, Eq. (17) provides the following integral of motion:

$$\frac{1}{2}(\dot{x}^2_{n+1} - \dot{x}^2_n) = f_n(x_{n+1} - x_n), \tag{20}$$

which differs from that of Eq. (16) in the term proportional to J_n.

Eq. (1) can be integrated in $[t_n, t_{n+1}]$ to yield

$$\dot{x} = \dot{x}_n + \int_{t_n}^{t} f(\tau, x, \dot{x})\,\mathrm{d}\tau, \tag{21}$$

which, upon approximating the integrand by f_n, yields

$$\dot{x} = \dot{x}_n + f_n(t - t_n), \tag{22}$$

that, in turn, results in Eq. (19). Upon integration of Eq. (22), one obtains Eq. (18). Therefore, Eqs. (18) and (19) are identical to those obtained by quadratures when f is approximated by f_n in $[t_n, t_{n+1}]$. However, we have not yet been able to find any relationship between FL and numerically techniques based on quadrature, presumably because of the third and fourth terms in the right-hand side of Eq. (2) that result in the exponential and trigonometric, explicit, non-standard difference Eqs. (6) and (7).

The above discussion indicates that, for smooth $f(t, x, \dot{x})$, the piecewise linearization method FL should be used because of its absolute stability and accuracy. When the nonlinearities are not differentiable with respect to t, x or $\dot{x}$ and, therefore, FL is not applicable, one can use either the piecewise partial linearization techniques discussed above or the piecewise constant acceleration approximation employed in this subsection. Alternatively, one can use FL when $f(t, x, \dot{x})$ is piecewise smooth and either the constant acceleration method discussed in this subsection or PL; as indicated above, in partial piecewise linearization techniques, the linearization is only performed with respect to the arguments of $f(t, x, \dot{x})$ for which this function is smooth. The combination of FL and PL for the study of nonlinear oscillators when $f(t, x, \dot{x})$ is non-smooth, is here referred to as FL.

As indicated previously, the lack of differentiability of f has been the reason for the wide use of the explicit Euler method given by Eqs. (18) and (19) [1]. This method is really a first-order accurate technique for non-smooth f, does not require the evaluation of derivatives and is conditionally stable. By way of contrast, the PL technique proposed in this paper offers an alternative to the widely used explicit Euler method and uses the derivatives when they do exist. As a consequence, it is expected to be more accurate than the explicit Euler method for both smooth and non-smooth problems. On the other hand, FL is second-order accurate and A-stable, but is only applicable to smooth problems. The verification of these statements is carried out in the next section.

3. Results

In this section, we present some sample results which have been obtained with the piecewise linearization method presented in Section 2 for some oscillators with fractional-power nonlinearities. The examples include both conservative and non-conservative single degree-of-freedom problems, and problems with unknown free (or natural) frequencies, and have been chosen to assess the accuracy of the piecewise linearization method in terms of the displacement, velocity, energy conservation/dissipation, and amplitude and frequency of the limit cycle (when applicable) as a function of the time step. The accuracy of the piecewise linearization method presented here is assessed by comparing its numerical solutions with analytical or asymptotic ones, when available.

Although the piecewise linearization method presented in this paper can use a variable time step that may be adapted to the evolution of the solution, all the results presented here have been obtained with a constant time step equal to k.

Example 1. This example corresponds to a fractional van der Pol oscillator governed by

$$\ddot{y} + y = \varepsilon(1 - y^2)(\dot{y})^{2/3}, \tag{23}$$

$$y(0) = 3, \quad \dot{y}(0) = 1, \tag{24}$$

where $\varepsilon = 0.1$. Note that the right-hand side of Eq. (23) is a continuous function of both y and $\dot{y}$, but it is not differentiable with respect to $\dot{y}$ at $\dot{y} = 0$.

Eq. (23) was solved by means of a frozen or a piecewise constant acceleration method, F, where the nonlinear terms were evaluated at the previous time step, i.e., Eqs. (18) and (19), a partial linearization method, PL, where only the linearization was performed with respect to y, and the full linearization technique, FL, described in Section 2 where the linearization was performed with respect to y and $\dot{y}$, except when $\dot{y}_n = 0$ where the linearization was only performed with respect to y because in this case the right-hand side of Eq. (23) is not differentiable with respect to $\dot{y}$ when $\dot{y} = 0$.

For $k = 0.001$, few differences in the solutions obtained with FL, PL and F were observed as indicated in Table 1 which shows the maximum and minimum values of y and $\dot{y}$ obtained with these methods as functions of the time step for $80 \leqslant t \leqslant 100$. This table clearly illustrates the deterioration of the accuracy of F, PL and FL as k is increased; this deterioration is larger for F than for PL and FL as observed for $k = 0.1$, and may be attributed to the first-order accuracy of F. Although not shown here, FL, PL and F resulted in (unphysical) damped solutions for $k = 1$, and FL was found more accurate than PL which, in turn, was more accurate than F, for $k = 1$.

Table 1 also shows that the differences between the results obtained with F, PL and FL increase as k is increased. For $k = 0.01$, F resulted in a broader phase diagram than PL and FL due to a phase shift, as illustrated in Fig. 1. The broadness of the phase diagram and the phase shift of F were found to increase as k was increased from 0.01 to 0.1. For the latter value of k, it was also observed that PL resulted in a broader phase diagram than FL; this broadness was similar to that of F for $k = 0.01$ (cf. Fig. 1).

Fig. 1 also shows that the displacement obtained with F is indistinguishable to the naked eye from that of PL, and this displacement lags behind that obtained with FL.

The angular frequencies predicted by PL are equal to 1.1147, 1.1143, 1.1184 and 1.0383 for $k = 0.001$, 0.01, 0.1 and 1, respectively; those of F are 1.1145, 1.1101, 0.8703 and 1.0281 for $k = 0.001$, 0.01, 0.1 and 1, respectively; and, those of FL are 1.1161, 1.1143, 1.1157 and 1.0239 for $k = 0.001$, 0.01, 0.1 and 1, respectively, which indicate that, for this example, the frequencies calculated with PL are comparable to those of FL and do not exhibit a monotonic behavior as functions of the time step. On the other hand, the frequencies predicted by F are in good accord with those of PL for $k = 0.001$ and 0.01, but not for $k = 0.1$. This result is consistent with the accuracy study presented in Section 2.

The differences observed in Fig. 1 and Table 1 are associated with the non-differentiability of the right-hand side of Eq. (23) with respect to $\dot{y}$ when $\dot{y} = 0$, and the fact that F treats this term as a constant in each interval $[t_n, t_{n+1}]$, whereas PL approximates this term by a constant plus a linear term in $(y - y_n)$, as discussed in Section 2. On the other hand, FL uses a full linearization (with respect to y and $\dot{y}$ whenever $\dot{y} \neq 0$) and a linearization with respect to only y whenever the right-hand side of Eq. (23) is not differentiable with respect to this variable. As shown in Section 2, FL is more accurate than PL and F, and the results presented in Fig. 1 indicate that the displacement predicted by PL is nearly identical to that of F, whereas the phase diagram of the former is more accurate than the latter. Therefore, for $k = 0.01$, the velocity predicted by PL is more accurate than that of F as one should have expected because PL accounts for the variation of the right-hand side of Eq. (23) with respect to y, whereas F freezes this term at t_n.

Calculations performed with an equation similar to Eq. (23) but where the power of $\dot{y}$ is $\frac{1}{3}$ instead of $\frac{2}{3}$ show that the amplitude of the limit cycle predicted by FL with $k = 0.01$ is equal to 1.8254, and this value is in good accord with the amplitude of 1.82574 obtained by Mickens [16] by means of the harmonic balance method. These calculations also show similar trends to those described above.

The results shown above and others not presented here indicate that FL is more accurate than F and PL; therefore, unless stated otherwise, the calculations in the examples reported below have been carried out with FL.

Table 1
Maximum and minimum values of y and $\dot{y}$ for F, PL and FL as functions of the time step for $80 \leqslant t \leqslant 100$ for Example 1

k	$(\max(y), \min(y))_{PL}$	$(\max(y), \min(y))_{F}$	$(\max(y), \min(y))_{FL}$
0.001	(3.1337, −4.2548)	(3.1309, −4.2443)	(3.1507, −4.3118)
0.01	(3.1391, −4.2721)	(3.0911, −4.1142)	(3.1366, −4.2637)
0.1	(3.1756, −4.3797)	(2.7982, −3.3587)	(3.1416, −4.2770)
k	$(\max(\dot{y}), \min(\dot{y}))_{PL}$	$(\max(\dot{y}), \min(\dot{y}))_{F}$	$(\max(\dot{y}), \min(\dot{y}))_{FL}$
0.001	(3.4544, −3.4539)	(3.4498, −3.4507)	(3.4778, −3.4778)
0.01	(3.4640, −3.4592)	(3.3865, −3.9992)	(3.4582, −3.4582)
0.1	(3.5349, −3.4868)	(2.9697, −3.0340)	(3.4675, −3.4688)

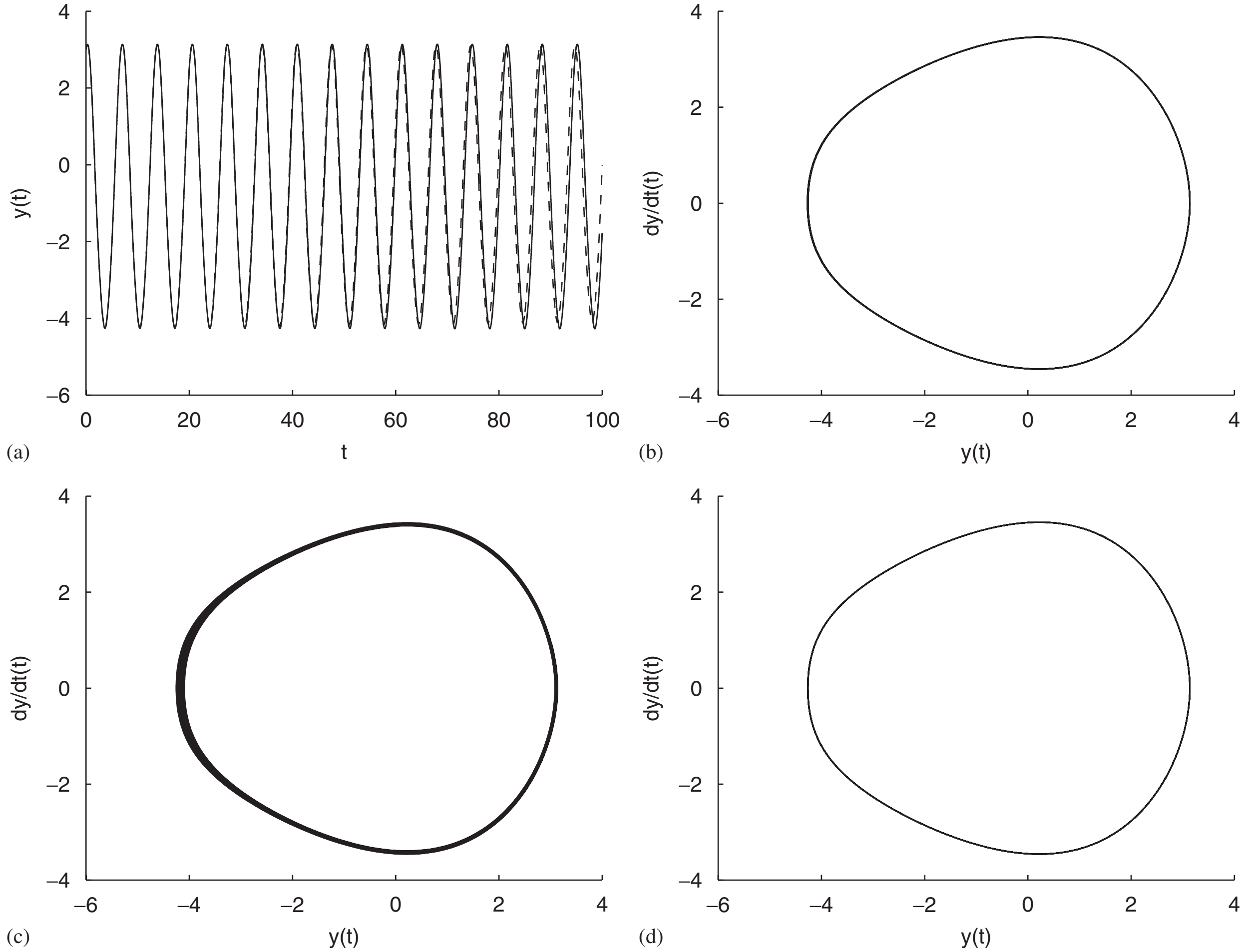

Fig. 1. y as a function of t (a), and phase diagrams obtained with PL (b), F (c) and FL (d) for Example 1 with $k = 0.01$. (The solid, dashed and dashed-dotted lines in (a) correspond to the solutions obtained with PL, F and FL, respectively.)

Example 2. This example corresponds to

$$\ddot{y} + y^{1/3} + \varepsilon\dot{y} = 0, \tag{25}$$

$$y(0) = 0.5, \quad \dot{y}(0) = 0, \tag{26}$$

and $\varepsilon = 0.1$. The total energy of the system, i.e., $E(t) = \frac{1}{2}\dot{y}^2 + \frac{3}{4}y^{4/3}$, decreases with time, and the second term in the left-hand side of Eq. (25) is not differentiable with respect to y at $y = 0$. For this example, PL coincides with FL.

Fig. 2 illustrates the results obtained with FL and $k = 0.01$. This figure shows the damping of both the displacement and velocity as well as the smoothness of the solution at $y = 0$ because the piecewise linearization method provides analytical solutions in (t_n, t_{n+1}) which are smooth in $[t_n, t_{n+1}]$.

Calculations performed with $k = 0.001$ and 0.01 were found to yield results that they were indistinguishable to the naked eye, i.e., $k = 0.001$ yielded $y(100) = -0.0009137420$ and $\dot{y}(100) = -0.008520428$, whereas $k = 0.01$ resulted in $y(100) = -0.001452578$ and $\dot{y}(100) = 0.004207011$, but those corresponding to $k = 0.1$ indicate that the numerical solution settles down into a (non-existing) cycle for t greater than approximately 45 and the maximum values of y and $\dot{y}$ in this cycle were about 0.03 and 0.12, respectively. The value of

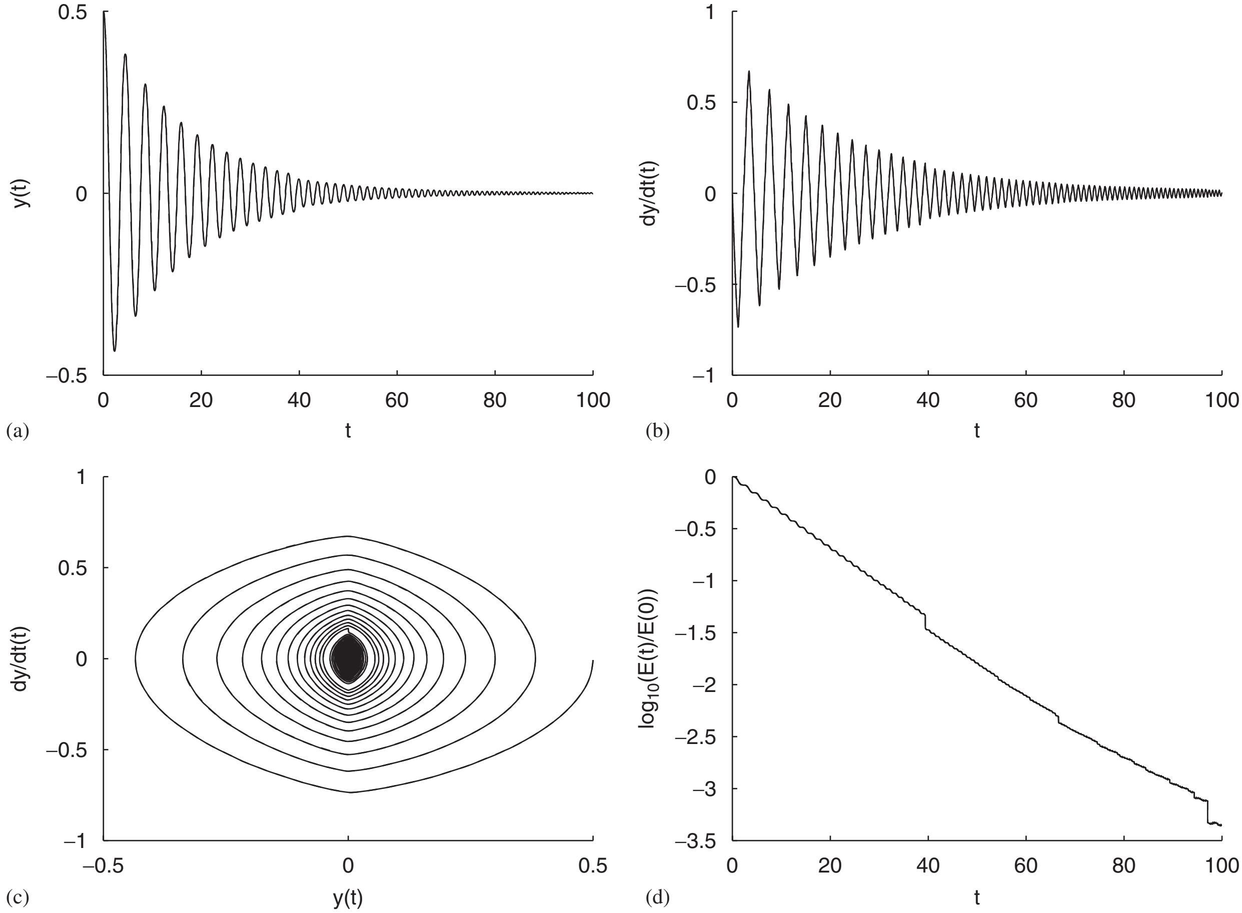

Fig. 2. y (a), $\dot{y}$ (b) and $E(t)/E(0)$ (d) as functions of t and phase diagram (c) obtained with FL and $k = 0.01$ for Example 2.

$E(100)/E(0)$ was found to be equal to 3.4538×10^{-4}, 4.3691×10^{-4}, and 0.0203 for $k = 0.001$, 0.01, and 0.1, respectively, thus indicating that the energy at $t = 100$ increases as k is increased.

Example 3. This example corresponds to

$$\ddot{y} + y^{1/3} + \varepsilon y^3 = 0, \tag{27}$$

$$y(0) = 1, \quad \dot{y}(0) = 0, \tag{28}$$

and $\varepsilon = 0.5$. The total energy of the system, i.e., $E(t) = \frac{1}{2}\dot{y}^2 + \frac{3}{4}y^{4/3} + \frac{1}{4}\varepsilon y^4 = E(0)$ is conserved, and the second term is not differentiable with respect to y at $y = 0$. Since the total energy is conserved, Eq. (27) has an infinite number of periodic solutions. For this example, PL coincides with FL.

The results presented in Fig. 3 indicate that, for the initial conditions given above, the solution is a limit cycle, and the energy is almost conserved for $k = 0.01$. The maximum and minimum values of $E(t)/E(0)$ for $0 \leqslant t \leqslant 100$ are 1.0013 and 0.9997, respectively, for $k = 0.001$, and 1.0159 and 0.9990, respectively, for $k = 0.01$, and 1.7464 and 0.9988, respectively, for $k = 0.1$, thus indicating that the larger the value of k, the larger is the energy conservation violation. In fact, the energy was found to increase in an oscillatory manner for $k = 0.1$ at a faster rate than that for $k = 0.01$, and the phase diagram for this time step was very broad.

The angular frequencies predicted by FL are 1.2385, 1.2375, 1.2652 and 1.1365 for $k = 0.001$, 0.01, 0.1 and 1, respectively; therefore, the frequencies predicted by FL are not monotonic functions of k.

Although not shown here, FL was found to result in overflow for $k = 0.1$.

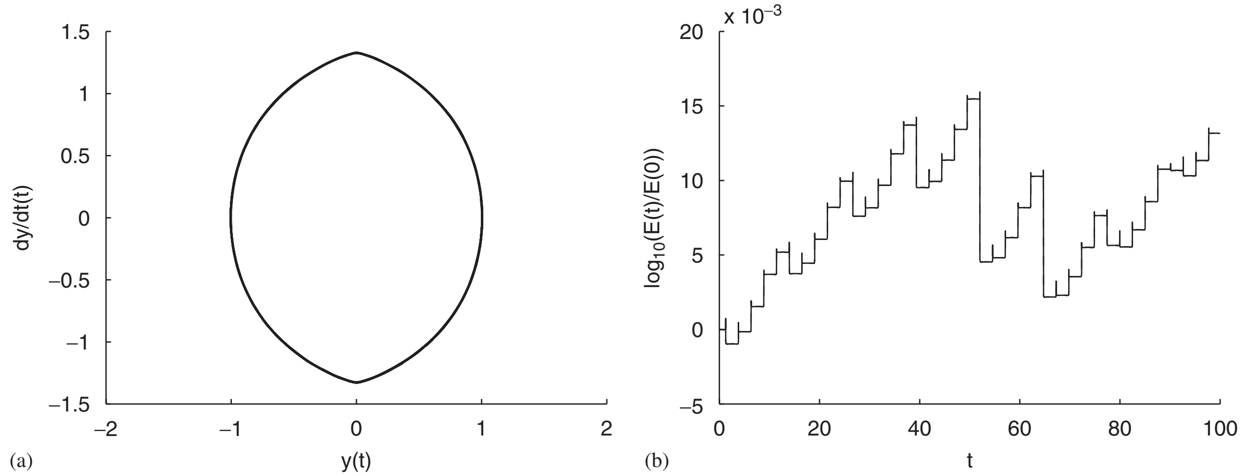

Fig. 3. Phase diagram (a) and energy (b) obtained with FL and $k = 0.01$ for Example 3.

Example 4. This example corresponds to

$$\ddot{y} + y^{1/3} = \varepsilon(1 - y^2)\dot{y}, \tag{29}$$

$$y(0) = 0.5, \quad \dot{y}(0) = 0, \tag{30}$$

and $\varepsilon = 0.1$. The second term in the left-hand side of Eq. (29) is continuous but not differentiable with respect to y at $y = 0$. For this example, PL coincides with FL.

The maximum values of $E(t)/E(0)$ for $0 \leqslant t \leqslant 100$ where $E(t) = \frac{1}{2}\dot{y}^2 + \frac{3}{4}y^{4/3}$, are 7.0245, 7.0430, 7.4913 and 13.7474 for $k = 0.001$, 0.01, 0.1 and 1, respectively, thus indicating that the accuracy of FL degrades as the time step is increased. However, the differences between the results obtained with $k = 0.001$ and 0.01 are very small, e.g., the relative errors in the energy at $t = 100$ and the angular frequency are less than 0.3% and 0.05%, respectively, for $k = 0.01$, and the results presented in Fig. 4 indicate that, after an initial transient, y, $\dot{y}$ and E become periodic functions of time for $k = 0.01$. Note that the phase diagram is smooth at $y = 0$ because the piecewise linearization method provides analytical solutions in (t_n, t_{n+1}) which are smooth in $[t_n, t_{n+1}]$. No periodic solution was found numerically for $k = 0.1$.

The angular frequencies predicted by FL are 0.8411, 0.8407, 0.8397 and 0.7854 for $k = 0.001$, 0.01, 0.1 and 1, respectively; therefore, the angular frequency decreases as k is increased, and the relative errors in the angular frequency (measured with respect to the angular frequency obtained with $k = 0.001$) are approximately equal to 0.05%, 0.17% and 6.62% for $k = 0.01$, 0.1 and 1, respectively.

Example 5. This example corresponds to

$$\ddot{y} + y^3 = \varepsilon(1 - y^2)\dot{y}^{1/3}, \tag{31}$$

$$y(0) = 0.5, \quad \dot{y}(0) = 0, \tag{32}$$

and $\varepsilon = 0.1$. Note that the term in the right-hand side of Eq. (31) is not differentiable with respect to $\dot{y}$ at $\dot{y} = 0$. For this example, PL coincides with FL.

The solution of this problem exhibits an initial transient before settling down into a periodic motion characterized by a limit cycle elongated in the y direction and a periodic energy as illustrated in Fig. 5 which corresponds to $k = 0.01$. A similar periodic motion was also observed even with $k = 0.1$; in fact, the angular frequency was found to be equal to 1.5020, 1.5020, 1.4936 and 0.7605 for $k = 0.001$, 0.01, 0.1 and 1, respectively, whereas the maximum values of where $E(t) = \frac{1}{2}\dot{y}^2 + \frac{1}{4}y^4$, for $0 \leqslant t \leqslant 100$ are 164.1928, 164.2022,

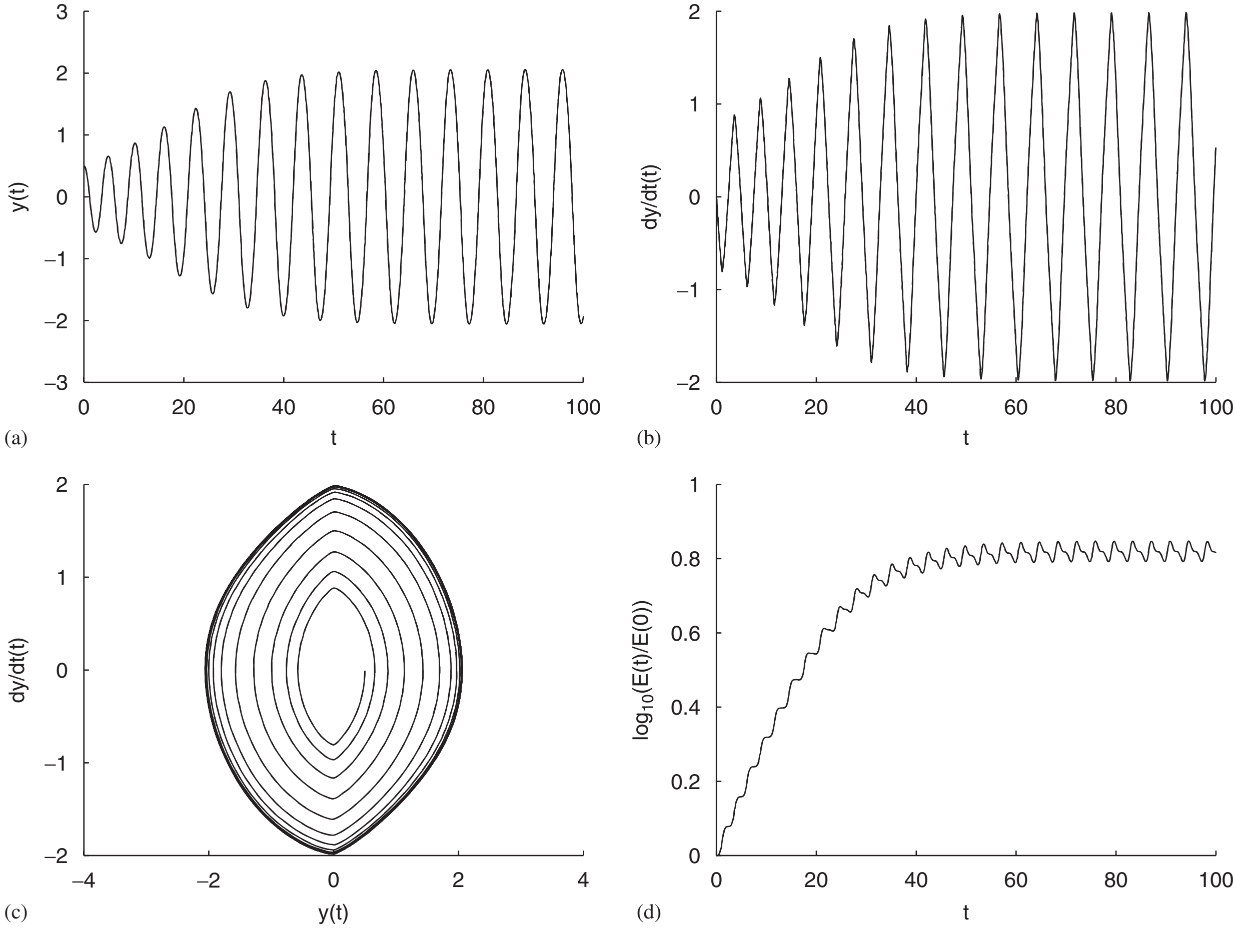

Fig. 4. y (a), $\dot{y}$ (b) and $E(t)/E(0)$ (d) as functions of t and phase diagram (c) obtained with FL and $k = 0.01$ for Example 4.

and 162.0134 for $k = 0.001$, 0.01 and 0.1, respectively. Therefore, both the frequency and energy decrease as k is increased.

Example 6. This example corresponds to

$$\ddot{y} + y^{3/5} = \varepsilon(1 - y^2)\dot{y}, \tag{33}$$

$$y(0) = 0.5, \quad \dot{y}(0) = 0, \tag{34}$$

and $\varepsilon = 0.1$. Note that the second term in the left-hand side of Eq. (33) is not differentiable with respect to y at $y = 0$. For this example, PL coincides with FL.

The solution of this problem exhibits an initial transient before settling down into a periodic motion characterized by a limit cycle and a periodic energy as illustrated in Fig. 6 which corresponds to $k = 0.01$. A similar periodic motion was also observed even with $k = 0.1$; in fact, the angular frequency was found to be equal to 0.9026, 0.9026, 0.9035 and 0.8403 for $k = 0.001$, 0.01, 0.1 and 1, respectively, and the maximum values of $E(t) = \frac{1}{2}\dot{y}^2 + \frac{5}{8}y^{8/5}$ for $0 \leqslant t \leqslant 100$ are 10.0542, 10.0523 and 10.2092 for $k = 0.001$, 0.01 and 0.1, respectively; therefore, for this example, the angular frequency and the energy are not monotonic functions of k. Note that the phase diagram is smooth everywhere and, in particular, at $y = 0$, because the piecewise linearization method provides analytical solutions in (t_n, t_{n+1}) which are smooth in $[t_n, t_{n+1}]$.

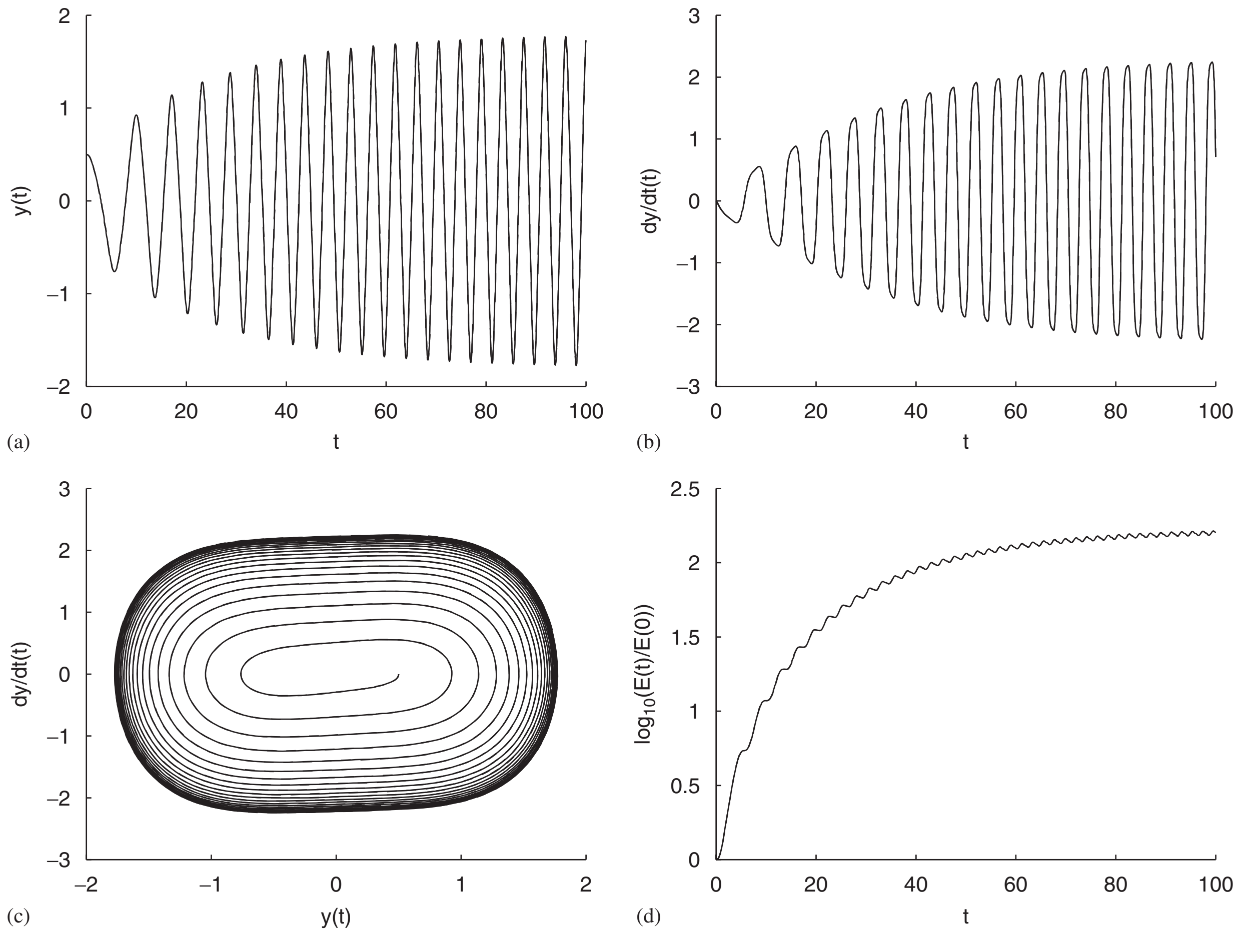

Fig. 5. y (a), $\dot{y}$ (b) and $E(t)/E(0)$ (d) as functions of t and phase diagram (c) obtained with FL and $k = 0.01$ for Example 5.

Example 7. This example corresponds to the following nonlinear oscillator

$$\ddot{y} + y^{5/3} = 0, \tag{35}$$

$$y(0) = 100, \quad \dot{y}(0) = 0, \tag{36}$$

which has a constant energy, i.e., $E(t) = \frac{1}{2}\dot{y}^2 + \frac{3}{8}y^{8/3} = E(0)$, and the second term is differentiable with respect to y at $y = 0$. For this example, PL coincides with FL.

Eq. (35) has been previously analyzed by Hu and Xiong [17] who used the harmonic balance method and a non-standard finite difference scheme due to Mickens [4].

Fig. 7 illustrates the phase diagram and the normalized energy obtained with FL and M, for $k = 0.01$. The M method corresponds to the non-standard finite difference technique developed by Mickens [4] and can be written as

$$\frac{y_{n+1} - y_n}{\sin(k)} = z_n, \quad \frac{z_{n+1} - z_n}{\sin(k)} = -(y_{n+1})^{5/3}. \tag{37}$$

M provides discrete solutions, whereas FL provides analytical solutions in each (t_n, t_{n+1}) which are smooth in $[t_n, t_{n+1}]$.

Both M and FL are non-standard finite difference methods and are here compared in terms of the phase diagram and energy conservation as shown in Fig. 7. This figure clearly illustrates that FL preserves more

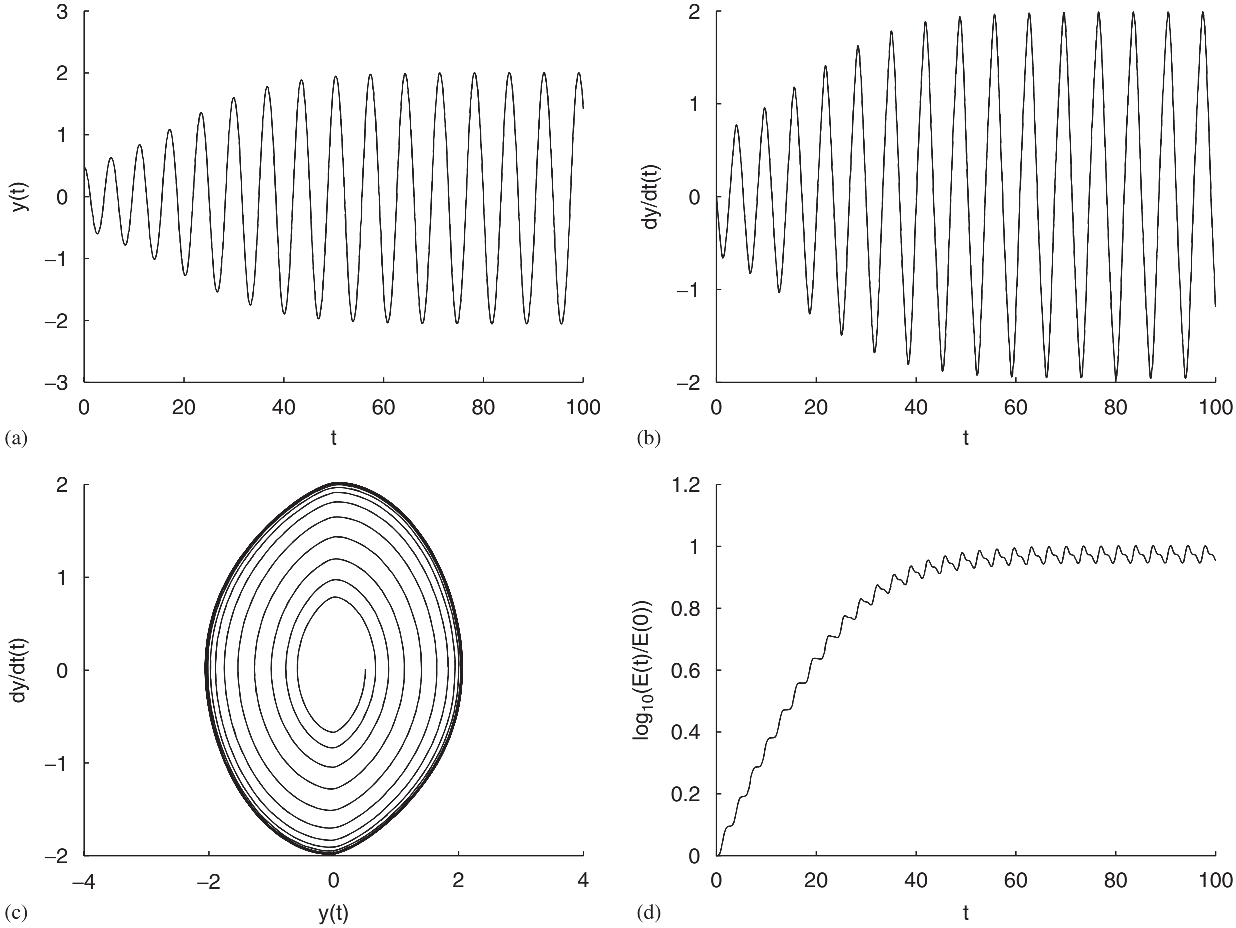

Fig. 6. y (a), $\dot{y}$ (b) and $E(t)/E(0)$ (d) as functions of t and phase diagram (c) obtained with FL and $k = 0.01$ for Example 6.

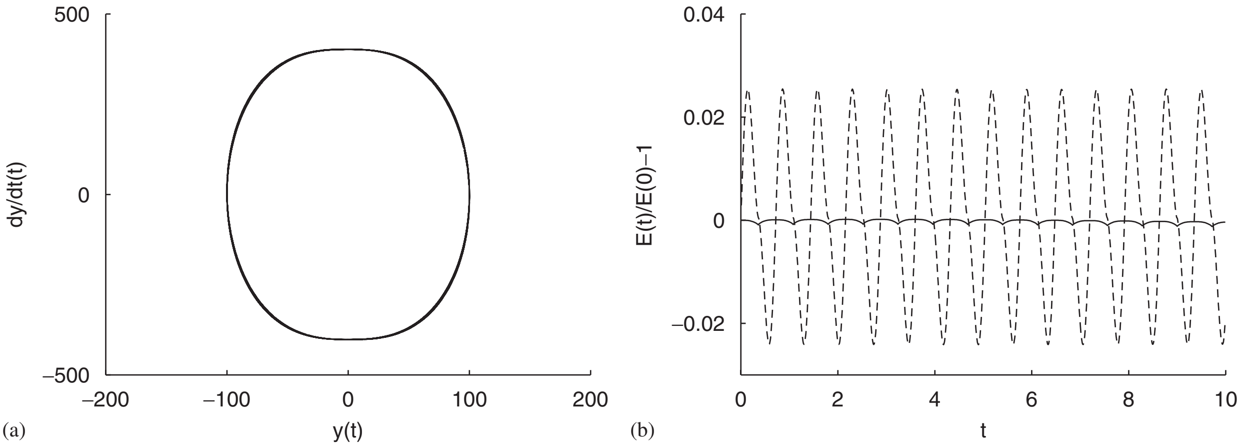

Fig. 7. Phase diagrams (a) and energy (b) obtained with FL (solid line) and M (dashed line) and $k = 0.01$ for Example 7.

accurately the energy than M, for $k = 0.01$. The maximum and minimum energies predicted by FL are 1 and 1, respectively, for $k = 0.001$, 1.0002 and 0.9988, respectively, for $k = 0.01$, and 0.9994 and 0.7332, respectively, for $k = 0.1$, whereas the maximum and minimum energies predicted by M are 1.0025 and 0.9975, respectively, for $k = 0.001$, 1.0255 and 0.9759, respectively, for $k = 0.01$, and 1.3251 and 0.7953, respectively, for $k = 0.1$. Therefore, FL preserves the energy more accurately than M as k is varied. In addition, the maximum energy predicted by FL is not a monotonic function of k, whereas the minimum energy predicted by this method is a decreasing function of the time step. The maximum and minimum energies predicted by M increase and decrease, respectively, as k is increased.

The angular frequencies predicted by FL are 4.3651, 4.3661 and 4.1948 for $k = 0.001$, 0.01 and 0.1, respectively, whereas M predicts 4.3651, 4.3674 and 4.4236, respectively. Therefore, the angular frequencies predicted by FL are not monotonic functions of the time step, whereas those of M increase as k is increased.

Example 8. This example corresponds to the following oscillator

$$\ddot{y} + y^{1/3} = 0, \tag{38}$$

$$y(0) = 1, \quad \dot{y}(0) = 0, \tag{39}$$

which has a constant energy, i.e., $E(t) = \frac{1}{2}\dot{y}^2 + \frac{3}{4}y^{4/3} = E(0)$, and the second term in the left-hand side of Eq. (38) is not differentiable with respect to y at $y = 0$. For this example, PL coincides with FL.

Eq. (38) has been previously analyzed by Cooper and Mickens [18] by means of a generalized harmonic balance method combined with a numerical technique that yielded an angular frequency equal to $1.054/y_0^{1/3}$, whereas that predicted by the harmonic balance method was equal to $1.049/y_0^{1/3}$, where $y_0 = y(0)$.

Compared with the previous example, here M preserves the energy more accurately than FL. In fact, the results presented in Fig. 8 that corresponds to $k = 0.01$ indicate that the energy predicted by M oscillates in an almost erratic fashion about 1, whereas that of FL increases in an oscillatory growing manner. Similar results to those shown in Fig. 8 have also been observed for $k = 0.001$ and 0.1, although the maximum value of the amplitude of the energy oscillations predicted by both FL and M increases as k is increased.

The maximum and minimum energies predicted by FL are 1.0007 and 0.9998, respectively, for $k = 0.001$, 1.0338 and 0.9964, respectively, for $k = 0.01$, and 1.7275 and 0.9755, respectively, for $k = 0.1$, whereas the maximum and minimum energies predicted by M are 1.0055 and 0.9994, respectively, for $k = 0.001$, 1.0059 and 0.9944, respectively, for $k = 0.01$, and 1.0801 and 0.9515, respectively, for $k = 0.1$. Therefore, both FL and M predict a maximum (minimum) energy that increases (decreases) as k is increased.

A plausible explanation of the differences between the results presented in Figs. 7 and 8 is that the nonlinearity of Example 7 is differentiable at $y = 0$ and the Jacobian of this nonlinearity is nil at $y = 0$. On the other hand, the nonlinearity of Example 8 is not differentiable at $y = 0$, and, therefore, FL makes use of a first-order accurate piecewise constant acceleration method, i.e., Eqs. (18) and (19), when $y = 0$ and the second-order accurate Eqs. (6) and (7), otherwise. This implies that the numerical approximation provided by FL whenever $y \approx 0$ is not a very good one because this method freezes the nonlinearity of Example 8 to that at the

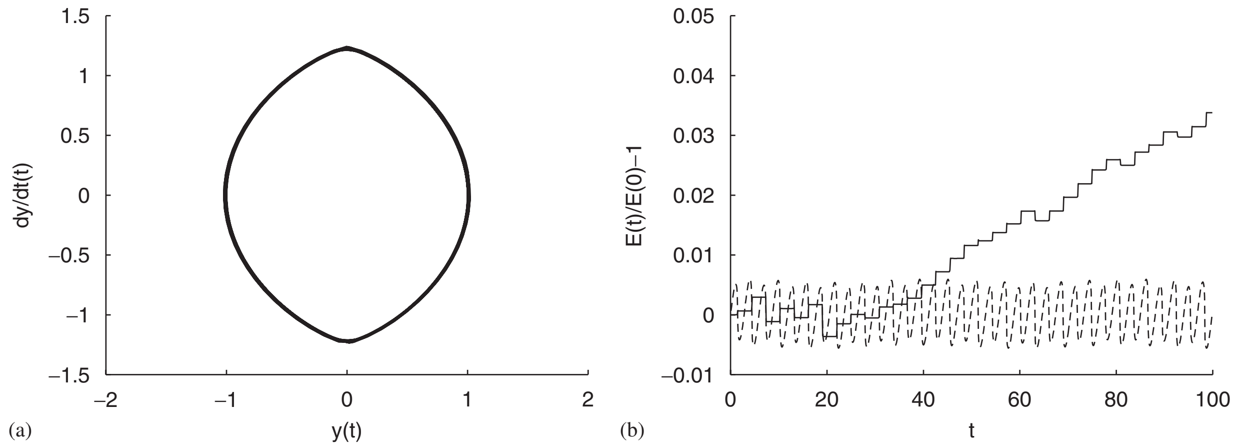

Fig. 8. Phase diagrams (a) and energy (b) obtained with FL (solid line) and M (dashed line) and $k = 0.01$ for Example 8.

previous time level whenever $y=0$. On the other hand, M does not make use of the Jacobian of the nonlinearity at all.

Example 9. Example 8 is a special case of the following oscillator:

$$\ddot{y}+y^{1/(2n+1)}=0, \tag{40}$$

$$y(0)=y_0, \quad \dot{y}(0)=0, \tag{41}$$

which has a constant energy, i.e., $E(t)=\frac{1}{2}\dot{y}^2+[(2n+1)/(2n+2)]y^{(2n+2)/(2n+1)}=E(0)$, and n is a natural number including zero. $n=0$ and 1 correspond to the linear harmonic oscillator and Example 8, respectively, whereas $n\to\infty$ implies that $y^{1/(2n+1)}\to \mathrm{sign}(y)$. For $n=0$, FL provides the exact solution of Eqs. (40) and (41) in the absence of round-off errors, and, for this example, PL coincides with FL.

Eq. (40) has been studied previously by van Horssen [19] that presented its solution in terms of an integral that had to be performed numerically, tabulated the angular frequency as a function of $y_0=y(0)$, and compared his results with those obtained by Mickens [20] who employed a generalized harmonic balance method. The results shown by van Horssen indicate that the harmonic balance technique underpredicts the angular frequency and that this underprediction increases as n is increased.

Eq. (40) has also been studied by Swamy et al. [21] who applied the harmonic balance method and a Ritz procedure, and found that the latter predicts more accurate angular frequencies than the former, and by Awrejcewicz and Andrianov [22] who considered the limit $n\to\infty$ using a (perturbative) δ-expansion technique.

Eqs. (40) and (41) were transformed into

$$\ddot{z}+z^{1/(2n+1)}=0, \tag{42}$$

$$z(0)=1, \quad \dot{z}(0)=0, \tag{43}$$

where the dots in Eqs. (42) and (43) denote differentiation with respect to τ, $y=y_0 z$ and $t=y_0^{n/(2n+1)}\tau$.

Table 2 shows a comparison between the exact frequencies determined by van Horssen [19], those of a harmonic balance procedure [21] and those of FL with $k=0.001$. The frequencies of FL were calculated by means of a bisection technique.

The results presented in Table 2 indicate that FL predicts angular frequencies in good accord with the exact ones obtained by van Horssen and these frequencies are more accurate than those based on a harmonic balance/numerical procedure [19,21]. Table 2 also shows that M predicts slightly more accurate frequencies

Table 2
Angular frequencies calculated by van Horssen (vH) [19], harmonic balance (HB) [21], FL and M with $k=0.001$ for Example 9

n	$\omega_{\mathrm{vH}}(n)y_0^{n/(2n+1)}$	$\omega_{\mathrm{FL}}(n)y_0^{n/(2n+1)}$	$\omega_{\mathrm{M}}(n)y_0^{n/(2n+1)}$	$\omega_{\mathrm{H}B}(n)y_0^{n/(2n+1)}$
0	1.00000	1.0005481	1.0005477	
1	1.07045	1.0709420	1.0710573	1.049115
2	1.08613	1.0848049	1.0866512	1.048122
3	1.09302	1.1085051	1.0937899	1.044052
4	1.09689	1.0910252	1.0975536	1.040169
5	1.09938	1.0925251	1.0997605	1.036840
6	1.10110	1.0887105	1.1017105	
7	1.10238	1.0906835	1.1028751	
8	1.10335	1.0909885	1.1039369	
9	1.10412	1.0918665	1.1045795	
10	1.10474	1.0932994	1.1052844	1.026280
50	1.10947	1.0874369	1.1098155	1.009188
500	1.11059	1.0906087	1.1112629	
∞, 10,000	$\frac{\pi}{2\sqrt{2}}$	1.2219744	1.1114939	

than FL. As explained before in Example 8, this result is to be expected for $n>0$ since the nonlinear restoring function is not differentiable when $y=0$ and the piecewise linearization method presented here is second-order accurate when $y\neq 0$, and first-order accurate otherwise.

Example 10. This example corresponds to the following oscillator:

$$\ddot{y} + \mathrm{sign}\,(y)|y|^q = 0, \tag{44}$$

$$y(0) = y_0, \quad \dot{y}(0) = 0, \tag{45}$$

where $q\geqslant 0$. Eq. (44) has the following integral of motion $E(t) = \frac{1}{2}\dot{y}^2 + \frac{1}{q+1}|y|^{q+1} = E(0)$ and its angular frequency can be calculated exactly in terms of the Gamma function [23]. For $q=0$, the exact angular frequency is $\omega_{\mathsf{EX}} = \pi/\sqrt{8y_0}$. For this example, PL coincides with FL.

Eq. (44) has been previously considered by Gottlieb [23] who manipulated it before applying a harmonic balance technique. Gottlieb also provided exact frequencies for positive q, used the harmonic balance method with only one Fourier coefficient and a modified harmonic balance procedure that accounts for even and odd values of $1/q$, and provided an extensive table of comparisons between the exact and approximate values of the angular frequency. Such a table indicates that the harmonic balance method with only one Fourier coefficient overestimates the frequency, whereas the same technique with some manipulations underpredicts it.

Eqs. (44) and (45) can also be written as

$$\ddot{z} + \mathrm{sign}(z)|z|^q = 0, \tag{46}$$

$$z(0) = 1, \quad \dot{z}(0) = 0, \tag{47}$$

where the dots in Eqs. (46) and (47) denote differentiation with respect to τ, $y = y_0 z$ and $t = y_0^{(1-q)/2}\tau$, and the angular frequencies reported below are those corresponding to Eqs. (46) and (47). The angular frequencies presented in Table 3 indicate that FL predicts slightly more accurate frequencies than M for a large range of values of q and $k = 0.0001$; the differences between the frequencies predicted by these two methods are always less than 1%, and the frequencies predicted by FL and M are more accurate than those obtained with a direct harmonic balance technique based on the first Fourier coefficient and a modified harmonic balance method that treats independently even and odd values of $1/q$ [23].

Table 3
Exact (EX) angular frequencies and those determined with FL and M with $k = 0.0001$ for Example 10

q	$\omega_{\mathsf{EX}}(p)$	$\omega_{\mathsf{FL}}(p)$	$\omega_{\mathsf{M}}(p)$
1	1.000000	1.0082490	1.0082460
3/4	1.024957	1.0334099	1.0334098
5/7	1.028660	1.0366503	1.0371397
2/3	1.033652	1.0418939	1.0421772
3/5	1.040749	1.0490220	1.0493302
1/2	1.051637	1.0603005	1.0603088
3/7	1.059596	1.0688881	1.0683346
1/3	1.070451	1.0792801	1.0792801
1/4	1.080181	1.0890382	1.0890852
1/5	1.086126	1.0949623	1.0950869
1/6	1.090133	1.0990256	1.0991208
1/7	1.093018	1.1016784	1.1020499
1/8	1.095194	1.1037508	1.1042191
1/9	1.096894	1.1053072	1.1059405
1/10	1.098258	1.1067789	1.1073096
1/11	1.099377	1.1076425	1.1084489
0	1.110721	1.1171964	1.1199019

Table 4
Angular frequencies determined with FL as functions of k for Example 10

q	$k = 0.01$	$k = 0.001$	$k = 0.0001$
1	0.9997868	1.0005486	1.0082490
3/4	1.0248130	1.0255226	1.0334099
5/7	1.0284455	1.0292321	1.0366503
2/3	1.0332914	1.0342154	1.0418939
3/5	1.0406101	1.0413201	1.0490220
1/2	1.0500473	1.0522296	1.0603005
3/7	1.0560917	1.0600264	1.0688881
1/3	1.0656362	1.0708772	1.0792801
1/4	1.1675404	1.0795840	1.0890382
1/5	1.0591701	1.0856153	1.0949623
1/6	1.0354817	1.1039122	1.0990256
1/7	1.0538553	1.0917708	1.1016784
1/8	1.0484813	1.0892577	1.1037508
1/9	1.0671917	1.0921813	1.1053072
1/10	1.0488810	1.0914827	1.1067789
1/11	1.0206510	1.1141843	1.1076425
0	0.9100739	1.0852358	1.1171964

Table 5
Maximum (max) and minimum (min) values of $e = E(t)/E(0)$ determined with FL and M for $0 \leqslant t \leqslant 100$ and $k = 0.0001$ for Example 10

q	e_{FL}^{max}	e_{FL}^{min}	e_{M}^{max}	e_{M}^{min}
1	1.0000	1.0000	1.0001	0.9999
3/4	1.0000	1.0000	1.0000	1.0000
5/7	1.0058	1.0000	1.0000	1.0000
2/3	1.0028	1.0000	1.0000	1.0000
3/5	1.0025	1.0000	1.0000	1.0000
1/2	1.0001	1.0000	1.0001	1.0000
3/7	1.0000	0.9973	1.0001	0.9999
1/3	1.0000	1.0000	1.0001	0.9999
1/4	1.0002	1.0000	1.0001	0.9999
1/5	1.0004	1.0000	1.0001	0.9999
1/6	1.0003	1.0000	1.0001	0.9999
1/7	1.0009	1.0000	1.0001	0.9999
1/8	1.0012	1.0000	1.0001	0.9999
1/9	1.0014	1.0000	1.0001	0.9999
1/10	1.0012	1.0000	1.0001	0.9999
1/11	1.0018	1.0000	1.0001	0.9999
0	1.0052	1.0000	1.0001	0.9999

The effects of k on the angular frequencies predicted by FL are illustrated in Table 4 which shows that the accuracy of this method increases as k is decreased as one expects because the method is based on local linearization of the nonlinear terms.

Table 5 shows that energy is conserved for $k = 0.0001$ and that M preserves the energy slightly better than FL. Although not shown here, the violation of energy conservation increases as k is increased.

Discussion: As indicated in Section 2, the Picard–Lindelof's theorem ensures the convergence and uniqueness of the solution of Eq. (1) if $f(t, x, \dot{x})$ is continuous with respect to its three arguments and satisfies a uniform Lipschitz condition with respect to the second and third arguments. Moreover, if f is sufficiently differentiable, one can apply a variety of numerical methods for the numerical solution of Eq. (1), and prove their convergence by means of Taylor's series expansions. However, if $f(t, x, \dot{x})$ is non-smooth, rigorous convergence results for numerical methods for Eq. (1) are difficult to obtain because of lack of differentiability [1], and, as a consequence, most of the numerical methods that have been used to-date in non-smooth mechanics problems have ignored differentiability, e.g., they employ the F method presented in this paper. It should be noted that Examples 1–6, 8, 9 (with $n \neq 0$) and 10 correspond to Group C, i.e., non-smooth problems characterized by vectors fields which are continuous, of the classification of non-smooth mechanical systems introduced by Brogliato [1].

The results of the 10 examples presented in this paper and others not shown here indicate that the piecewise linearization method, FL, can accurately predict the displacement, velocity, energy, and amplitude and frequency of nonlinear oscillators. In addition, the method provides results in accord with those of non-standard finite difference methods, although it may predict that energy is not strictly preserved in conservative systems such as those of Examples 8 and 9. However, depending on the nonlinearity, both FL and M may require small time steps to obtain accurate results. Furthermore, we showed in Section 2 that FL is second-order accurate for smooth $f(t, x, \dot{x})$, whereas the constant acceleration method is first-order accurate. The piecewise linearization technique presented in this paper applies full linearization when $f(t, x, \dot{x})$ is smooth, and partial piecewise linearization, otherwise. Therefore, for non-smooth $f(t, x, \dot{x})$, the piecewise linearization technique is expected to have an order of accuracy between 1 and 2. This has been verified by determining the numerical errors as

$$e_{\mathsf{N}}(t_n; k) = |y_e(t_n) - y_{\mathsf{N}}(t_n; k)| = Ak^p, \tag{48}$$

when the exact solution, y_e, is available, or

$$e_{\mathsf{N}}(t_n; k) = |y_{\mathsf{NE}}(t_n; k_m) - y_{\mathsf{N}}(t_n; k)| = Bk^p, \tag{49}$$

where $y_{\mathsf{N}}(t_n; k)$ and y_{NE} denote the numerical solution obtained with the N method using a time step equal to k and a very small time step equal to k_m, respectively, A and B are assumed to be (positive) constants, and p is the order of the N method.

From Eqs. (48) and (49), it is an easy matter to obtain

$$p = \frac{\log(e_{\mathsf{N}}(t_n; k_1)/e_{\mathsf{N}}(t_n; k_2))}{\log(k_1/k_2)}. \tag{50}$$

Application of Eq. (50) at t_n with $k_m = 10^{-4}$, $k_1 = 10^{-3}$ and $k_2 = 10^{-2}$ shows that the order of accuracy of FL is $p = 1.37$, 1.35, 1.34, 1.42, 1.35, 1.53, 2.01 and 1.35 for Examples 1–8, respectively, whereas that of F is 0.99, 0.98, 0.99, 0.98, 0.98, 0.99, 0.98 and 1.02 for the same examples, where t_n denotes the time just after the first occurrence of $y = 0$ for Examples 2, 3, 4, 6 and 8, the time just after the first occurrence of $\dot{y} = 0$ for Examples 1 and 5, and $t_n = 100$ for Example 7.

Just before the first occurrence of $y = 0$ in Examples 2, 3, 4, 6 and 8, and the first occurrence of $\dot{y} = 0$ in Examples 1 and 5, the order of accuracy of FL was found to be 2.02, 1.99, 1.99, 2.02, 2.01, 1.98 and 1.99 for Examples 1–6 and 8, respectively, in accord with the analysis presented in Section 2 which indicates that this method is second-order accurate when f is sufficiently smooth.

The order of accuracy of FL was found to decrease as t increased and tended towards 1 as t was increased when f was non-smooth as in Examples 1–6 and 8–10, and was two for Example 7. F is first-order accurate as shown in Section 2.

4. Conclusions

A piecewise linearization method based on the Taylor series expansion of nonlinear ordinary differential equations with respect to time, the displacement and velocity, has been developed for the study of one degree-of-freedom nonlinear oscillators with smooth and fractional-power nonlinearities. The method provides the exact solution (in the absence of round-off errors) of linear ordinary differential equations with constant coefficients and right-hand sides that depend linearly on time, displacement and velocity, and yields explicit nonstandard finite difference formulae for the discrete displacement and velocity. It also provides smooth solutions.

For smooth nonlinearities, it has been shown that the piecewise linearization method is absolute stable and second-order accurate provided that the linearization is performed with respect to time, displacement and velocity. If the nonlinearities are approximated by a piecewise constant term, the piecewise linearization method presented here is conditionally stable and first-order accurate. Piecewise partial linearization techniques are also conditionally stable and first-order accurate.

It has been shown that the piecewise linearization method presented in this paper preserves very well the amplitude and phase, energy and angular frequency of oscillators with fractional-power nonlinearities provided that the time step is properly chosen, and its accuracy degrades as the time step is increased. The method does not require explicit linear damping and/or linear stiffness terms in the governing equation, and it is more accurate than linearization techniques that freeze the nonlinearities at the previous time level or linearize the nonlinear terms with respect to only the displacement or the velocity.

The piecewise linearization method was found to predict slightly more (less) accurate results, i.e., displacement, velocity, energy and angular frequency, than Mickens' nonstandard finite difference schemes for fractional powers greater (less) than one. The reason for the discrepancies between the piecewise linearization method and Mickens's non-standard finite difference approach has been attributed to the non-differentiability of the nonlinear terms when the positive fractional powers are smaller than unity. However, the piecewise linearization methods presented in this paper are entirely general, whereas those proposed by Mickens have to be designed in a case-by-case manner.

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