Available online at www.sciencedirect.com
ScienceDirect
Journal of Sound and Vibration 300 (2007) 695–708
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Adiabatic invariants of oscillators with one degree of freedom

Ivana Kovacic*

Faculty of Technical Sciences, Department of Mechanics, 21121 Novi Sad, Serbia and Montenegro

Received 21 February 2006; received in revised form 11 August 2006; accepted 21 August 2006
Available online 20 October 2006

Abstract

Adiabatic invariants for dynamical systems with one degree of freedom are derived. The method developed for linear dynamical systems with constant parameters is extended to systems with slowly varying parameters. The method is based on the field method concept of obtaining a conservation law from an incomplete solution of a partial differential equation. The method results in a complete set of adiabatic invariants specifying the approximate solutions for motion. A few examples, including the classical time-dependent oscillator and the Duffing oscillator with slowly varying parameters, are given to illustrate the theory.

1. Introduction

It is well known that the knowledge of conserved quantities of a dynamical system deepens understanding of its essence. This must have been and still is one of the reasons for a lasting interest in the study of adiabatic invariants—quantities which remain approximately conserved during a slow variation of systems parameters over a time scale long compared to a natural period. Namely, for a function I which is an adiabatic invariant to $O(\varepsilon^N)$ with $0<\varepsilon\ll 1$, the expression "approximately conserved" reflects the property that its time derivative is of $N+1$ order. If that derivative is equal to zero, the function I represents an exact invariant (the first integral) [1].

Burgers [2] was a pioneer in adiabatic invariants research. Kruskal [3] and many others in after years [4–9] dealt with adiabatic invariants of the Hamiltonian systems, which are the systems completely described by Hamiltonian. In order to find adiabatic invariants of purely non-conservative systems, which cannot be described by Hamiltonian canonical equations, Djukic [10] developed the theory which enables one to obtain adiabatic invariants of dynamical systems with one degree of freedom and slowly varying parameters. This theory, which is the combination of the Noether's theory and Krylov–Bogolubov–Mitropolski method, was also extended to two weakly coupled oscillators with slowly varying parameters [11] as well as to the systems with one degree of freedom and large cubic nonlinearity [12].

In this paper, the procedure for obtaining adiabatic invariants of a system with one degree of freedom and slowly varying parameters is proposed. It is based on the field method [13] and its extension to vibrational

*Tel.: +38 121 6350122; fax: +38 121 458133.
E-mail address: ivanakov@uns.ns.ac.yu.

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2006.08.036

problems [13–15]. The field method is used since it does not have any limitation regarding conservative character of the system being considered, i.e. it is applicable to any system whose equations of motion can be written in the form of first-order ordinary differential equations. The basic field method concept is combined with the multiple variable expansion procedure. Similar to Djukic's results [10], the adiabatic invariant derived in this paper has the leading term quadratic with respect to the state variables. However, this quadratic invariant is derived by algebraic transformations from a complete set of new linear adiabatic invariants. These linear adiabatic invariants specify the solution for motion so that the proposed procedure enables the complete study of the system considered: qualitative (on the basis of the adiabatic invariants) and quantitative (on the basis of the approximate solution for motion).

Three examples are given to illustrate the procedure developed: the classical harmonic oscillator with slowly varying frequency, the Duffing oscillator with slowly varying parameters and the weakly and slowly pulsating undamped oscillator.

2. The field method approach

Consider an oscillator with slowly varying parameters whose differential equation of motion can be presented in the form:

$$\dot{x} = y,$$

$$\dot{y} = -K(\tau)x + \varepsilon f(x, p, \tau), \tag{1}$$

where x and y are state variables, $K(\tau)$ is an arbitrary function of slowly varying time $\tau = \varepsilon t$, ε is a small constant parameter, t is time, $\dot{()} \equiv \mathrm{d}()/\mathrm{d}t$ and f is an arbitrary function of the variables x, p and τ.

According to the basic supposition of the field method, one of the state variables of the system, say the coordinate x, can be expressed as a field depending on time t and the other variable, in this case the variable y, i.e.:

$$x = U(t, y). \tag{2}$$

Differentiating this expression totally with respect to time and using Eqs. (1), a partial differential equation, the so-called basic field equation is derived:

$$\frac{\partial U}{\partial t} + \frac{\partial U}{\partial y}[-K(\tau)U + \varepsilon f(U, p, \tau)] - y = 0. \tag{3}$$

It has been shown [13,16] that an incomplete (single) solution of the basic field equation can be used for deriving conservation laws of linear one degree of freedom dynamical systems with constant parameters. Namely, for the case when $K = \text{const.}$ and $f = 0$ the incomplete solution of the simplified basic field equation:

$$\frac{\partial U}{\partial t} - \frac{\partial U}{\partial y}KU - y = 0. \tag{4}$$

can be assumed as [16]

$$x \equiv U = A\,y + F(t), \tag{5}$$

where A is a constant, while $F(t)$ is an unknown function of time. Substituting Eq. (5) into Eq. (4), and equating the terms containing y and the free terms with zero, one has:

$$A_{1/2} = \pm\frac{1}{\sqrt{K}}\mathrm{i}, \tag{6}$$

$$F(t) = C\mathrm{e}^{A^2 K t}, \tag{7}$$

where i is an imaginary unit and C is a constant of integration.

Using Eqs. (6) and (7), the solution for the field (5) can be written down as:

$$\left[x - \frac{\mathrm{i}y}{\sqrt{K}}\right]\mathrm{e}^{-\mathrm{i}t} = C',$$

$$\left[x + \frac{\mathrm{i}y}{\sqrt{K}}\right]\mathrm{e}^{\mathrm{i}t} = C'', \tag{8}$$

where $C' = C(A_1)$ and $C'' = C(A_2)$. The values of these constants can be calculated from the initial values $x(0)$ and $y(0)$.

Eqs. (8) represent two independent conservation laws, which give the solution for motion $\bar{x}$ (the solution along trajectory):

$$\bar{x} = \frac{C'\mathrm{e}^{\mathrm{i}t} + C''\mathrm{e}^{-\mathrm{i}t}}{2}. \tag{9}$$

The product of C' and C'' yields the total energy conservation law:

$$Kx^2 + y^2 = 2E = \text{const.} \tag{10}$$

So, on the basis of the incomplete solution of the basic field equation, two independent conservation laws can be found. They specify the motion of the system with one degree of freedom, i.e. they represent a complete set of invariants, enabling the construction of the solution for motion.

In order to harness the power of this approach, the previously given concept is to be adapted for establishing a complete set of adiabatic invariants of the oscillator with slowly varying parameters (1). Firstly, fast time is introduced [1]:

$$T = \frac{1}{\varepsilon}\int_0^{\tau} \sqrt{K(s)}\,\mathrm{d}s. \tag{11}$$

Then, both the field U and the variable y are developed in series with respect to the small parameter according to the following expressions:

$$U(t, y, \varepsilon) = U_0(T, \tau, y_0) + \varepsilon\, U_1(T, \tau, y_0) + \cdots\ , \tag{12}$$

$$y(t, \varepsilon) = y_0(T, \tau) + \varepsilon y_1(T, \tau) + \cdots\ . \tag{13}$$

It is also assumed that the dependence of the field on the corresponding variable is not affected by the step of approximation [13], that is:

$$\frac{\partial U}{\partial y} = \frac{\partial U_0}{\partial y_0} = \frac{\partial U_1}{\partial y_1}. \tag{14}$$

Besides, the function f can be expanded with respect to the small parameter:

$$f(U, p, \tau) = f_0(\bar{x}_0, \bar{y}_0, \tau) + \varepsilon f_1(\bar{x}_0, \bar{y}_0, \bar{x}_1, \bar{y}_1, \tau) + \cdots\ . \tag{15}$$

Using Eqs. (11)–(15), the basic field equation (3) transforms to:

$$\frac{\partial U_0}{\partial T}\sqrt{K(\tau)} - \frac{\partial U_0}{\partial y_0} K(\tau) U_0 - y_0 = 0, \tag{16}$$

$$\frac{\partial U_1}{\partial T}\sqrt{K(\tau)} - \frac{\partial U_1}{\partial y_1} K(\tau) U_1 - y_1 = -\frac{\partial U_0}{\partial \tau}\Big|_{\bar{y}_0} - \frac{\partial U_1}{\partial y_1} f_0(\bar{x}_0, \bar{y}_0, \tau). \tag{17}$$

Since the right-hand side of Eq. (17) depends on the firstly found quantity U_0, the notation $\partial U_0/\partial\tau|_{\bar{y}_0}$ means that this derivative must be calculated for the solution along trajectory $\bar{y}_0$.

A trial incomplete solution of Eq. (16) will be assumed in the form analogous to the incomplete (5), but the existence of slow and fast time requires assuming its terms as varying in times:

$$U_0 = F_1(\tau) y_0 + F_2(T, \tau). \tag{18}$$

Substituting it into Eq. (16), equating the terms containing y_0 and the free terms with zero, and integrating them, one has:

$$F_1 = \pm \frac{1}{\sqrt{K(\tau)}} \mathrm{i}. \tag{19}$$

Two forms of this function will be denoted as

$$F_1' = \frac{1}{\sqrt{K(\tau)}} \mathrm{i}, \quad F_1'' = -\frac{1}{\sqrt{K(\tau)}} \mathrm{i}. \tag{20}$$

As a consequence, the function F_2 has two forms:

$$F_2' = C_0'(\tau) \mathrm{e}^{F_1' \sqrt{K(\tau)} T}, \quad F_2'' = C_0''(\tau) \mathrm{e}^{F_1'' \sqrt{K(\tau)} T}, \tag{21}$$

where C_0' and C_0'' are unknown functions of slow time τ. The solutions along trajectory for the first components $\bar{x}_0$ and $\bar{y}_0$ are:

$$\bar{x}_0 = \frac{C_0'(\tau)\mathrm{e}^{\mathrm{i}T} + C_0''(\tau)\mathrm{e}^{-\mathrm{i}T}}{2}, \tag{22}$$

$$\bar{y}_0 = \frac{-C_0'(\tau)\mathrm{e}^{\mathrm{i}T} + C_0''(\tau)\mathrm{e}^{-\mathrm{i}T}}{2\mathrm{i}} \sqrt{K(\tau)}. \tag{23}$$

The form of the functions C_0' and C_0'' will be found from the equation for the component U_1 (17), whose solution is assumed as

$$U_1 = F_1(\tau) y_1 + C_1(T, \tau) \mathrm{e}^{F_1 \sqrt{K(\tau)} T}, \tag{24}$$

with C_1 being unknown and having two values $C_1' = C_1(F_1')$ and $C_1'' = C_1(F_1'')$.

Now, Eq. (17) turns into:

$$\sqrt{K(\tau)} \frac{\mathrm{d}C_1}{\mathrm{d}T} = -\left[\frac{\mathrm{d}F_1}{\mathrm{d}\tau} \frac{-C_0'\mathrm{e}^{\mathrm{i}T} + C_0''\mathrm{e}^{-\mathrm{i}T}}{2\mathrm{i}} \sqrt{K(\tau)} + \frac{\mathrm{d}C_0}{\mathrm{d}\tau} \mathrm{e}^{F_1\sqrt{K(\tau)}T} \right] \mathrm{e}^{-F_1\sqrt{K(\tau)}T} - F_1 f_0(\bar{x}_0, \bar{y}_0, \tau) \mathrm{e}^{-F_1\sqrt{K(\tau)}T}. \tag{25}$$

Taking into consideration two possible values of F_1 (20), Eq. (25) yields two differential equations for C_1' and C_1'':

$$\sqrt{K(\tau)} \frac{\mathrm{d}C_1'}{\mathrm{d}T} = \frac{1}{4K} \frac{\mathrm{d}K}{\mathrm{d}\tau} [C_0''\mathrm{e}^{-2\mathrm{i}T} - C_0'] - \frac{\mathrm{d}C_0'}{\mathrm{d}\tau} - \frac{\mathrm{i}}{\sqrt{K(\tau)}} f_0(\bar{x}_0, \bar{y}_0, \tau) \mathrm{e}^{-\mathrm{i}T}, \tag{26}$$

$$\sqrt{K(\tau)} \frac{\mathrm{d}C_1''}{\mathrm{d}T} = \frac{1}{4K} \frac{\mathrm{d}K}{\mathrm{d}\tau} [-C_0'' + C_0'\mathrm{e}^{2\mathrm{i}T}] - \frac{\mathrm{d}C_0''}{\mathrm{d}\tau} + \frac{\mathrm{i}}{\sqrt{K(\tau)}} f_0(\bar{x}_0, \bar{y}_0, \tau) \mathrm{e}^{\mathrm{i}T}. \tag{27}$$

Firstly, in order to find C_0' and C_0'', the requirement of no appearance of the secular terms will be used. This will result in the first-order differential equations for C_0' and C_0''. The initial values $C_0'(0)$ and $C_0''(0)$ are defined with the initial values of the state variables. If $x(t=0) = a$ and $y(t=0) = b$, Eqs. (22) and (23) give:

$$C_0'(0) = a - \frac{\mathrm{i}b}{\sqrt{K(0)}}, \quad C_0''(0) = a + \frac{\mathrm{i}b}{\sqrt{K(0)}}. \tag{28}$$

Then, Eqs. (26) and (27) are to be integrated with respect to T. Since expression (24) implies:

$$\bar{x}_1 = \frac{C_1'(T,\tau)\mathrm{e}^{\mathrm{i}T} + C_1''(T,\tau)\mathrm{e}^{-\mathrm{i}T}}{2}, \tag{29}$$

$$\bar{y}_1 = \frac{-C_1'(T,\tau)e^{iT} + C_1''(T,\tau)e^{-iT}}{2i}\sqrt{K(\tau)}. \tag{30}$$

one derives:

$$C_1'(0,0) = C_1''(0,0) = 0. \tag{31}$$

2.1. Adiabatic invariants

According to the assumed forms of components (18) and (24), the field U (12) can be expressed as

$$x \equiv U = F_1(\tau)y_0 + C_0(\tau)e^{F_1\sqrt{K(\tau)}T} + \varepsilon[F_1(\tau)y_1 + C_1(T,\tau)e^{F_1\sqrt{K(\tau)}T}]. \tag{32}$$

Replacing two possible values of F_1, C_0 and C_1, and using Eq. (13), one has:

$$\left[x - \frac{iy}{\sqrt{K(\tau)}}\right]e^{-iT} = C_0' + \varepsilon C_1'(T,\tau), \tag{33}$$

$$\left[x + \frac{iy}{\sqrt{K(\tau)}}\right]e^{iT} = C_0'' + \varepsilon C_1''(T,\tau). \tag{34}$$

These relations represent two independent adiabatic invariants of system (1).

Multiplying them, the following adiabatic invariant is obtained:

$$\frac{K(\tau)x^2 + y^2}{K(\tau)} - \varepsilon[C_0'(\tau)C_1''(T,\tau) + C_0''(\tau)C_1'(T,\tau)] + O(\varepsilon^2) = C_0'(\tau)C_0''(\tau). \tag{35}$$

The leading term is obviously the energy of the linear harmonic oscillator (the double one) divided by frequency squared. The correction for the arbitrary oscillator (1) is defined by the terms in the brackets, that is, by C_0'-C_1''. Consequently, this correction will depend on slow and fast time only, which is the basic difference between this term and those derived by some other authors who obtained them in the mixed form of the state variables and time (for example, Refs. [6,10,17]). Since the product $C_0'C_0''$ may depend on slow time, it might be expected that, after transforming Eq. (35) in such a way that a pure constant appears on the right-hand side, the leading term will become the ratio of the energy and some function of slowly varying frequency.

2.2. Solution for motion

With a sufficient number of independent invariants in hand (33)–(34), one can effectively find the approximate solution for motion in the second approximation:

$$x = \frac{C_0'e^{iT} + C_0''e^{-iT}}{2} + \varepsilon\frac{C_1'e^{iT} + C_1''e^{-iT}}{2}. \tag{36}$$

Further consideration is based on the function $f(x,p,\tau)$. Hence, some forms will be assigned to it and the adiabatic invariants (33)–(35) as well as the solution for motion (36) will be found completely.

3. Examples

3.1. Classical time-dependent harmonic oscillator

Consider the classical time-dependent harmonic oscillator, whose differential equation of motion is given by Eq. (1), where $f = 0$.

The corresponding differential equation (25) for the second component of the field is

$$\sqrt{K(\tau)}\frac{\mathrm{d}C_1}{\mathrm{d}T} = -\left[\frac{\mathrm{d}F_1}{\mathrm{d}\tau}\bar{y}_0 + \frac{\mathrm{d}C_0}{\mathrm{d}\tau}\mathrm{e}^{F_1\sqrt{K(\tau)}T}\right]\mathrm{e}^{-F_1\sqrt{K(\tau)}T}, \tag{37}$$

i.e.:

$$\sqrt{K(\tau)}\frac{\mathrm{d}C_1'}{\mathrm{d}T} = \frac{1}{4K(\tau)}\frac{\mathrm{d}K}{\mathrm{d}\tau}[C_0''\mathrm{e}^{-2\mathrm{i}T} - C_0'] - \frac{\mathrm{d}C_0'}{\mathrm{d}\tau}, \tag{38}$$

$$\sqrt{K(\tau)}\frac{\mathrm{d}C_1''}{\mathrm{d}T} = \frac{1}{4K(\tau)}\frac{\mathrm{d}K}{\mathrm{d}\tau}[-C_0'' + C_0'\mathrm{e}^{2\mathrm{i}T}] - \frac{\mathrm{d}C_0''}{\mathrm{d}\tau}. \tag{39}$$

The elimination of secular terms from Eqs. (38) and (39) imposes:

$$\frac{\mathrm{d}C_0'}{\mathrm{d}\tau} + \frac{1}{4K(\tau)}\frac{\mathrm{d}K}{\mathrm{d}\tau}C_0' = 0, \tag{40}$$

$$\frac{\mathrm{d}C_0''}{\mathrm{d}\tau} + \frac{1}{4K(\tau)}\frac{\mathrm{d}K}{\mathrm{d}\tau}C_0'' = 0. \tag{41}$$

Their solutions are:

$$C_0' = C_0'(0)\sqrt[4]{\frac{K(0)}{K(\tau)}}, \quad C_0'' = C_0''(0)\sqrt[4]{\frac{K(0)}{K(\tau)}}, \tag{42}$$

where $C_0'(0)$ and $C_0''(0)$ are defined by Eq. (28).

Integrating what remained in Eqs. (38) and (39) with respect to T and considering slow time τ as a constant, one finds:

$$C_1' = -\frac{1}{8\mathrm{i}K^{3/2}(\tau)}\frac{\mathrm{d}K}{\mathrm{d}\tau}C_0''\mathrm{e}^{-2\mathrm{i}T} + P'(\tau), \tag{43}$$

$$C_1'' = \frac{1}{8\mathrm{i}K^{3/2}(\tau)}\frac{\mathrm{d}K}{\mathrm{d}\tau}C_0'\mathrm{e}^{2\mathrm{i}T} + P''(\tau), \tag{44}$$

where $P'(\tau)$ and $P''(\tau)$ are arbitrary functions of slow time. They must be of such forms that the initial values of C_1' and C_1'' (31) are satisfied. Underlining that these forms are not unique, we will suppose them as:

$$P'(\tau) = \frac{1}{8\mathrm{i}K^{3/2}(\tau)}\frac{\mathrm{d}K}{\mathrm{d}\tau}C_0'', \tag{45}$$

$$P''(\tau) = \frac{1}{-8\mathrm{i}K^{3/2}(\tau)}\frac{\mathrm{d}K}{\mathrm{d}\tau}C_0'. \tag{46}$$

They were derived from Eqs. (43) and (44) by substituting $T = 0$.

3.1.1. Adiabatic invariants

The complex adiabatic invariants (33), (34) of the classical time-dependent harmonic oscillator are:

$$\begin{aligned}\left[x - \frac{\mathrm{i}y}{\sqrt{K(\tau)}}\right]\mathrm{e}^{-\mathrm{i}T} &= \left(a - \frac{\mathrm{i}b}{\sqrt{K(0)}}\right)\sqrt[4]{\frac{K(0)}{K(\tau)}} \\ &+ \varepsilon\frac{\mathrm{d}K}{\mathrm{d}\tau}\left(a + \frac{\mathrm{i}b}{\sqrt{K(0)}}\right)\sqrt[4]{\frac{K(0)}{K(\tau)}}\left[-\frac{1}{8\mathrm{i}K^{3/2}(\tau)}\mathrm{e}^{-2\mathrm{i}T} + \frac{1}{8\mathrm{i}K^{3/2}(\tau)}\right],\end{aligned} \tag{47}$$

$$\left[x+\frac{\mathrm{i}y}{\sqrt{K(\tau)}}\right]\mathrm{e}^{\mathrm{i}T}=\left(a+\frac{\mathrm{i}b}{\sqrt{K(0)}}\right)\sqrt[4]{\frac{K(0)}{K(\tau)}} + \varepsilon\frac{\mathrm{d}K}{\mathrm{d}\tau}\left(a-\frac{\mathrm{i}b}{\sqrt{K(0)}}\right)\sqrt[4]{\frac{K(0)}{K(\tau)}}\left[-\frac{1}{8\mathrm{i}K^{3/2}(\tau)}\mathrm{e}^{2\mathrm{i}T}+\frac{1}{8\mathrm{i}K^{3/2}(\tau)}\right]. \tag{48}$$

The corresponding first-order adiabatic invariant (35) is

$$\frac{K(\tau)x^2+y^2}{\sqrt{K(\tau)}}-\varepsilon\frac{\mathrm{d}K}{\mathrm{d}\tau}\left[\frac{K(0)a^2-b^2}{\sqrt{K(0)}}\frac{1}{4K^{3/2}(\tau)}\sin 2T-\frac{ab}{2K^{3/2}(\tau)}\cos 2T+\frac{ab}{2K^{3/2}(\tau)}\right]=\frac{K(0)a^2+b^2}{\sqrt{K(0)}}. \tag{49}$$

To the best of author's knowledge, this adiabatic invariant in which the terms next to the small parameter depend only on time is unknown in the literature.

3.1.2. Solution for motion

From Eqs. (22), (28) and (42), the solution in the first approximation is obtained:

$$x_0=a\sqrt[4]{\frac{K(0)}{K(\tau)}}\cos T+\frac{b}{\sqrt[4]{K(0)K(\tau)}}\sin T. \tag{50}$$

This is the same solution as one usually given in the literature (see, for example, Ref. [1] or [19]). Here, following the proposed field method technique and using Eqs. (43)–(46) the solution for motion in the second approximation (36) is also found:

$$x_F=a\sqrt[4]{\frac{K(0)}{K(\tau)}}\cos T+\frac{b}{\sqrt[4]{K(0)K(\tau)}}\sin T+\varepsilon a\frac{\sqrt[4]{K(0)}}{4K^{7/4}(\tau)}\frac{\mathrm{d}K}{\mathrm{d}\tau}\sin T. \tag{51}$$

In Fig. 1 the numerical solution x_N of the Eq. (1) for the case $f=0$ is compared with the approximate solution x_F (51) and x_A according to Ref. [10] (Eqs. (9), (45), (30) therein). The values of the parameters and the initial conditions are: $K(\tau)=1+\tau$, $\varepsilon=0.01$, $a=1$ and $b=0$. For smaller values of time, both approximate solutions agree well with the numerical result x_N. As time passes, the solution obtained by using the field method x_F agrees excellently with the numerical one, while the solution x_A shows some deviation.

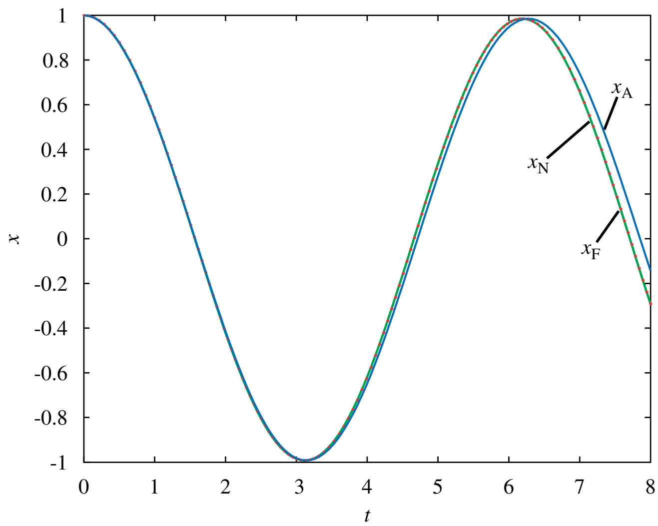

Fig. 1. Approximate analytical x_F, x_A and numerical solutions x_N of the classical time-dependent harmonic oscillator for $K(\tau)=1+\tau$, $\varepsilon=0.01$ and the initial values $x(0)=1$ and $y(0)=0$.

It is interesting to note that by using Eq. (50) for the case when $b=0$ and deriving $y_0 = a\sqrt[4]{K(0)K(\tau)}\sin T$, the terms in the square brackets of Eq. (49) can be transformed into the form making the adiabatic invariant Eq. (49) equal to that given in Ref. [10] (Eq. (32) therein).

Zeroth-order invariants: For the case when $\varepsilon = 0$, the adiabatic invariants (47)–(48) become the invariants of the zeroth order:

$$\left[x - \frac{\mathrm{i}y}{\sqrt{K(\tau)}}\right]\mathrm{e}^{-\mathrm{i}T} = \left(a - \frac{\mathrm{i}b}{\sqrt{K(0)}}\right)\sqrt[4]{\frac{K(0)}{K(\tau)}}, \tag{52}$$

$$\left[x + \frac{\mathrm{i}y}{\sqrt{K(\tau)}}\right]\mathrm{e}^{\mathrm{i}T} = \left(a + \frac{\mathrm{i}b}{\sqrt{K(0)}}\right)\sqrt[4]{\frac{K(0)}{K(\tau)}}. \tag{53}$$

If one multiplies Eqs. (52) and (53) mutually, the following exact invariant is obtained:

$$\frac{K(\tau)x^2 + y^2}{\sqrt{K(\tau)}} = \frac{K(0)a^2 + b^2}{\sqrt{K(0)}}. \tag{54}$$

This is a classical invariant for a linear harmonic oscillator with slowly varying parameters [1,18]—the energy of the oscillator (in this case the double one) divided by frequency.

The complex invariants (52), (53) can be written down as:

$$\sqrt[4]{K(\tau)}x\cos T - \frac{y}{\sqrt[4]{K(\tau)}}\sin T = a\sqrt[4]{K(0)}, \tag{55}$$

$$\sqrt[4]{K(\tau)}x\sin T + \frac{y}{\sqrt[4]{K(\tau)}}\cos T = \frac{b}{\sqrt[4]{K(0)}}. \tag{56}$$

When $K = \text{const.}$, these invariants become equal to a general form of the exact invariants of a simple harmonic oscillator [7].

3.2. Duffing oscillator with slowly varying parameters

In this case the characteristic function f is nonlinear with respect to the coordinate:

$$f(U) = -\alpha(\tau)U^3, \tag{57}$$

where α is an arbitrary function of τ. The initial conditions are $x(t=0)=a$, $y(t=0)=0$.

Since $f_0 = -\alpha\bar{x}_0^3$, where $\bar{x}_0$ is given by Eq. (22), Eq. (25) becomes:

$$\sqrt{K(\tau)}\frac{\mathrm{d}C_1}{\mathrm{d}T} = -\left[\frac{\mathrm{d}F_1}{\mathrm{d}\tau}\frac{-C_0'\mathrm{e}^{\mathrm{i}T} + C_0''\mathrm{e}^{-\mathrm{i}T}}{2\mathrm{i}}\sqrt{K(\tau)} + \frac{\mathrm{d}C_0}{\mathrm{d}\tau}\mathrm{e}^{F_1\sqrt{K(\tau)}T}\right]\mathrm{e}^{-F_1\sqrt{K(\tau)}T} + F_1\alpha(\tau)\left[\frac{C_0'\mathrm{e}^{\mathrm{i}T} + C_0''\mathrm{e}^{-\mathrm{i}T}}{2}\right]^3\mathrm{e}^{-F_1\sqrt{K(\tau)}T}. \tag{58}$$

In accordance with Eqs. (26) and (27), it can be separated into:

$$\sqrt{K(\tau)}\frac{\mathrm{d}C_1'}{\mathrm{d}T} = \frac{1}{4K(\tau)}\frac{\mathrm{d}K}{\mathrm{d}\tau}[C_0''\mathrm{e}^{-2\mathrm{i}T} - C_0'] - \frac{\mathrm{d}C_0'}{\mathrm{d}\tau} + \frac{\mathrm{i}\alpha(\tau)}{8\sqrt{K(\tau)}}[C_0'^3\mathrm{e}^{2\mathrm{i}T} + 3C_0'^2C_0'' + 3C_0'C_0''^2\mathrm{e}^{-2\mathrm{i}T} + C_0''^3\mathrm{e}^{-4\mathrm{i}T}], \tag{59}$$

$$\sqrt{K(\tau)}\frac{\mathrm{d}C_1''}{\mathrm{d}T} = \frac{1}{4K}\frac{\mathrm{d}K}{\mathrm{d}\tau}[-C_0'' + C_0'\mathrm{e}^{2\mathrm{i}T}] - \frac{\mathrm{d}C_0''}{\mathrm{d}\tau} - \frac{\mathrm{i}\alpha(\tau)}{8\sqrt{K(\tau)}}[C_0'^3\mathrm{e}^{4\mathrm{i}T} + 3C_0'^2C_0''\mathrm{e}^{2\mathrm{i}T} + 3C_0'C_0''^2 + C_0''^3\mathrm{e}^{-2\mathrm{i}T}]. \tag{60}$$

The elimination of secular terms leads to:

$$\frac{\mathrm{d}C_0'}{\mathrm{d}\tau}+\frac{1}{4K(\tau)}\frac{\mathrm{d}K}{\mathrm{d}\tau}C_0'-\frac{3\mathrm{i}\alpha(\tau)}{8\sqrt{K(\tau)}}C_0'^2C_0''=0, \tag{61}$$

$$\frac{\mathrm{d}C_0''}{\mathrm{d}\tau}+\frac{1}{4K(\tau)}\frac{\mathrm{d}K}{\mathrm{d}\tau}C_0''+\frac{3\mathrm{i}\alpha(\tau)}{8\sqrt{K(\tau)}}C_0'C_0''^2=0. \tag{62}$$

Integrating them, and using Eq. (28) for $b=0$, one has:

$$C_0'=a\sqrt[4]{\frac{K(0)}{K(\tau)}}\mathrm{e}^{\mathrm{i}\Lambda(\tau)},\quad C_0''=a\sqrt[4]{\frac{K(0)}{K(\tau)}}\mathrm{e}^{-\mathrm{i}\Lambda(\tau)}, \tag{63}$$

where

$$\Lambda(\tau)=\frac{3a^2\sqrt{K(0)}}{8}\int_0^\tau\frac{\alpha(s)}{K(s)}\,\mathrm{d}s. \tag{64}$$

Now, the integration of Eqs. (59) and (60) with respect to T is to be done. The assumed form of the function C_1 (24) as the one depending on fast and slow time, implies that after this integration, the unknown functions of τ are to be added to C_1' and C_1''. Each of these functions will be separated into two terms: the first ones existing because of slowly varying frequency, i.e. $\mathrm{d}K/\mathrm{d}t$, and the second ones existing because of the characteristic function f, i.e. $\alpha(\tau)$. The first terms will be assumed analogously as $P'(\tau)$ and $P''(\tau)$ in the previous example. At this level of approximation, the dependence of the second terms on slow time is assumed in the same form as the dependence of C_0' and C_0'' on slow time, that is, proportional to $\mathrm{e}^{\mathrm{i}\Lambda(\tau)}$ and $\mathrm{e}^{-\mathrm{i}\Lambda(\tau)}$. Further, these functions must have such initial values that $C_1'(0,0)=C_1''(0,0)=0$. So, the solution for the functions C_1' and C_1'' are as follows:

$$\begin{aligned}C_1'=&-\frac{1}{8\mathrm{i}K^{3/2}(\tau)}\frac{\mathrm{d}K}{\mathrm{d}\tau}C_0''\mathrm{e}^{-2\mathrm{i}T}+\frac{\alpha(\tau)}{16K(\tau)}[C_0'^3\mathrm{e}^{2\mathrm{i}T}-3C_0'C_0''^2\mathrm{e}^{-2\mathrm{i}T}-\frac{C_0''^3}{2}\mathrm{e}^{-4\mathrm{i}T}]\\&+\frac{1}{8\mathrm{i}K^{3/2}(\tau)}\frac{\mathrm{d}K}{\mathrm{d}\tau}C_0''+\frac{5a^3\alpha(\tau)K^{3/4}(0)}{32K^{7/4}(\tau)}\mathrm{e}^{\mathrm{i}\Lambda(\tau)},\end{aligned} \tag{65}$$

$$\begin{aligned}C_1''=&\frac{1}{8\mathrm{i}K^{3/2}(\tau)}\frac{\mathrm{d}K}{\mathrm{d}\tau}C_0'\mathrm{e}^{-2\mathrm{i}T}-\frac{\alpha(\tau)}{16K(\tau)}\left[\frac{C_0'^3}{2}\mathrm{e}^{4\mathrm{i}T}+3C_0'^2C_0''\mathrm{e}^{2\mathrm{i}T}-C_0''^3\mathrm{e}^{-2\mathrm{i}T}\right]\\&-\frac{1}{8\mathrm{i}K^{3/2}(\tau)}\frac{\mathrm{d}K}{\mathrm{d}\tau}C_0'+\frac{5a^3\alpha(\tau)K^{3/4}(0)}{32K^{7/4}(\tau)}\mathrm{e}^{-\mathrm{i}\Lambda(\tau)}.\end{aligned} \tag{66}$$

3.2.1. *Adiabatic invariants*

The complex adiabatic invariants (33) and (34) of the Duffing oscillator with slowly varying parameters are:

$$\begin{aligned}\left[x-\frac{\mathrm{i}y}{\sqrt{K(\tau)}}\right]\mathrm{e}^{-\mathrm{i}T}=a\sqrt[4]{\frac{K(0)}{K(\tau)}}\mathrm{e}^{\mathrm{i}\Lambda(\tau)}+\varepsilon\Bigg[&-\frac{a\sqrt[4]{K(0)}}{8\mathrm{i}K^{7/4}(\tau)}\frac{\mathrm{d}K}{\mathrm{d}\tau}\mathrm{e}^{-\mathrm{i}(2T+\Lambda(\tau))}\\&+\frac{a^3\alpha(\tau)K^{3/4}(0)}{16K^{7/4}(\tau)}\left[\mathrm{e}^{\mathrm{i}(2T+3\Lambda(\tau))}-3\mathrm{e}^{-\mathrm{i}(2T+\Lambda(\tau))}-\frac{1}{2}\mathrm{e}^{-\mathrm{i}(4T+3\Lambda(\tau))}\right]\\&+\frac{a\sqrt[4]{K(0)}}{8\mathrm{i}K^{7/4}(\tau)}\frac{\mathrm{d}K}{\mathrm{d}\tau}\mathrm{e}^{-\mathrm{i}\Lambda(\tau)}+\frac{5a^3\alpha(\tau)K^{3/4}(0)}{32K^{7/4}(\tau)}\mathrm{e}^{\mathrm{i}\Lambda(\tau)}\Bigg],\end{aligned} \tag{67}$$

$$\left[x+\frac{\mathrm{i}y}{\sqrt{K(\tau)}}\right]\mathrm{e}^{\mathrm{i}T}=a\sqrt[4]{\frac{K(0)}{K(\tau)}}\mathrm{e}^{-\mathrm{i}\Lambda(\tau)}+\varepsilon\left[\frac{a\sqrt[4]{K(0)}}{8\mathrm{i}K^{7/4}(\tau)}\frac{\mathrm{d}K}{\mathrm{d}\tau}\mathrm{e}^{\mathrm{i}(2T+\Lambda(\tau))}\right.$$
$$-\frac{a^3\alpha(\tau)K^{3/4}(0)}{16K^{7/4}(\tau)}\left[\frac{1}{2}\mathrm{e}^{\mathrm{i}(4T+3\Lambda(\tau))}+3\mathrm{e}^{\mathrm{i}(2T+\Lambda(\tau))}-\mathrm{e}^{-\mathrm{i}(2T+3\Lambda(\tau))}\right]$$
$$\left.-\frac{a\sqrt[4]{K(0)}}{8\mathrm{i}K^{7/4}(\tau)}\frac{\mathrm{d}K}{\mathrm{d}\tau}+\frac{5a^3\alpha(\tau)K^{3/4}(0)}{32K^{7/4}(\tau)}\mathrm{e}^{-\mathrm{i}\Lambda(\tau)}\right]. \tag{68}$$

Their product, after some transformations, yields the adiabatic invariant:

$$I\equiv\frac{K(\tau)x^2+y^2}{\sqrt{K(\tau)}}-\varepsilon\left[\frac{a^2\sqrt{K(0)}}{4K^{3/2}(\tau)}\frac{\mathrm{d}K}{\mathrm{d}\tau}\sin(2T+2\Lambda(\tau))-\frac{a^2\sqrt{K(0)}}{4K^{3/2}(\tau)}\frac{\mathrm{d}K}{\mathrm{d}\tau}\sin(2\Lambda(\tau))\right.$$
$$-\frac{a^4\alpha(\tau)K(0)}{16K^{3/2}(\tau)}\cos(4T+4\Lambda(\tau))-\frac{a^4\alpha(\tau)K(0)}{4K^{3/2}(\tau)}\cos(2T+2\Lambda(\tau))$$
$$\left.+\frac{5a^4\,\alpha(\tau)K(0)}{16K^{3/2}(\tau)}\right]=a^2\sqrt{K(0)}. \tag{69}$$

The leading term of this adiabatic invariant is the energy divided by frequency. The terms next to the small parameter depend only on slow and fast time explicitly, unlike those given for example, in Refs. [6,10], which are of the mixed form of time, coordinate and momentum.

In Fig. 2 the adiabatic invariant I (69) is plotted for the case in which $K(\tau)=\alpha(\tau)=1+\tau$, $a=1$, $b=0$ and different values of the small parameter ε. It is shown that each corresponding adiabatic invariant has a tendency of slow increase.

3.2.2. Solution for motion

Solution for motion (36) is

$$x_A=a\sqrt[4]{\frac{K(0)}{K(\tau)}}\cos(T+\Lambda(\tau))+\varepsilon\left[\frac{a\sqrt[4]{K(0)}}{8K^{7/4}(\tau)}\frac{\mathrm{d}K}{\mathrm{d}\tau}\sin(T+\Lambda(\tau))\right.$$

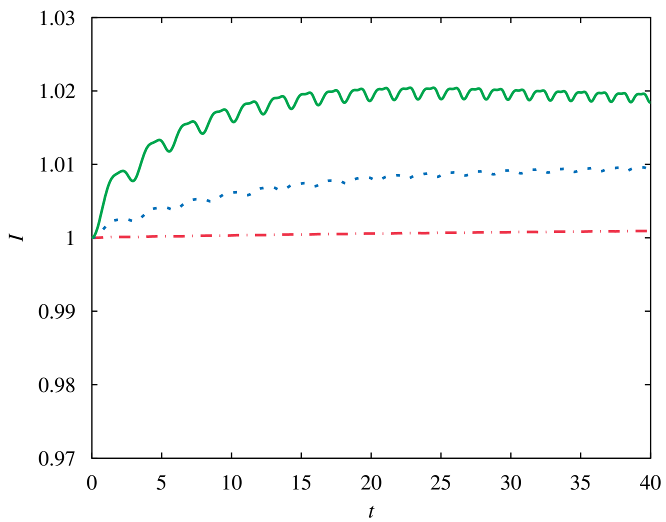

Fig. 2. Adiabatic invariants for the Duffing oscillator with slowly varying parameters for $K(\tau)=\alpha(\tau)=1+\tau$ and the initial values $x(0)=1$, $y(0)=0$: -.-.-. $\varepsilon=0.01$, ... $\varepsilon=0.05$, — $\varepsilon=0.1$.

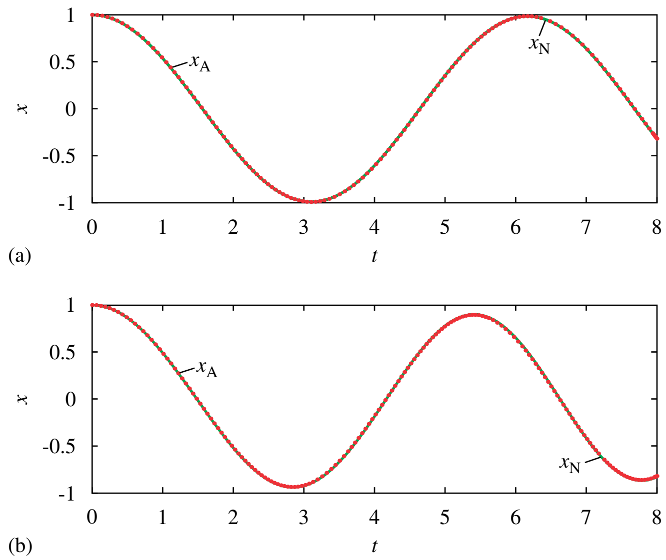

Fig. 3. Approximate analytical x_A (. . .) and numerical solutions x_N (—) of the Duffing oscillator with slowly varying parameters for $K(\tau) = \alpha(\tau) = 1 + \tau$, the initial values $x(0) = 1$, $y(0) = 0$: (a) the small parameter is $\varepsilon = 0.01$, (b) the small parameter is $\varepsilon = 0.1$.

$$+\frac{a\sqrt[4]{K(0)}}{8K^{7/4}(\tau)}\frac{\mathrm{d}K}{\mathrm{d}\tau}\sin(T - \Lambda(\tau)) + \frac{a^3\alpha(\tau)K^{3/4}(0)}{32K^{7/4}(\tau)}\cos(3T + 3\Lambda(\tau))$$
$$\left. - \frac{a^3\alpha(\tau)K^{3/4}(0)}{32K^{7/4}(\tau)}\cos(T + \Lambda(\tau))\right]. \tag{70}$$

In Fig. 3 the approximate analytical solution x_A defined by Eq. (70) and the numerical solution x_N are plotted for the case when $K(\tau) = \alpha(\tau) = 1 + \tau$, $a = 1$, $b = 0$ and two different values of the small parameter: $\varepsilon = 0.01$ and $\varepsilon = 0.1$. In both cases the difference between the solutions is negligible.

3.3. *Weakly and slowly pulsating undamped oscillator*

In Ref. [7], invariants of the damped weakly pulsating oscillator are derived. Here, we will consider the undamped weakly and slowly pulsating oscillator with mass m:

$$m = m_0 \mathrm{e}^{2\sin(\nu\tau)}, \tag{71}$$

where m_0 and ν are constant parameters.

Its equation of motion:

$$\ddot{x} + \frac{\dot{m}}{m}\dot{x} + Kx = 0, \tag{72}$$

can be presented in the form:

$$\dot{x} = y,$$
$$\dot{y} = -Kx - 2\varepsilon\nu y\cos(\nu\tau). \tag{73}$$

The mathematical model (73) is a special case of Eq. (1) with $K = \text{const.}$ and $f = -2\nu y\cos(\nu\tau)$.

For such oscillator the function F_1 in Eq. (19) is constant, as a result of which Eq. (25) reads as

$$\sqrt{K}\frac{\mathrm{d}C_1}{\mathrm{d}T} = -\frac{\mathrm{d}C_0}{\mathrm{d}\tau} + 2\nu F_1\cos(\nu\tau)\frac{-C_0'(\tau)\mathrm{e}^{\mathrm{i}T} + C_0''(\tau)\mathrm{e}^{-\mathrm{i}T}}{2\mathrm{i}}\sqrt{K}\mathrm{e}^{-F_1\sqrt{K}T}, \tag{74}$$

i.e.:

$$\sqrt{K}\frac{\mathrm{d}C_1'}{\mathrm{d}T} = -\frac{\mathrm{d}C_0'}{\mathrm{d}\tau} + v\cos(v\tau)[-C_0'(\tau) + C_0''(\tau)\mathrm{e}^{-2\mathrm{i}T}], \tag{75}$$

$$\sqrt{K}\frac{\mathrm{d}C_1''}{\mathrm{d}T} = -\frac{\mathrm{d}C_0''}{\mathrm{d}\tau} - v\cos(v\tau)[-C_0'(\tau)\mathrm{e}^{2\mathrm{i}T} + C_0''(\tau)]. \tag{76}$$

To eliminate the secular terms one requires that:

$$\frac{\mathrm{d}C_0'}{\mathrm{d}\tau} + C_0'(\tau)v\cos(v\tau) = 0, \tag{77}$$

$$\frac{\mathrm{d}C_0''}{\mathrm{d}\tau} + C_0''(\tau)v\cos(v\tau) = 0. \tag{78}$$

Integrating these equations and using Eq. (28) one has:

$$C_0' = \left(a - \frac{\mathrm{i}b}{\sqrt{K}}\right)\mathrm{e}^{-\sin(v\tau)}, \quad C_0'' = \left(a + \frac{\mathrm{i}b}{\sqrt{K}}\right)\mathrm{e}^{-\sin(v\tau)}. \tag{79}$$

On the basis of Eqs. (75), (76) and (31), the solutions for the functions C_1' and C_1'' can be taken in the form:

$$C_1' = \frac{v}{2\mathrm{i}\sqrt{K}}\left(a + \frac{\mathrm{i}b}{\sqrt{K}}\right)\mathrm{e}^{-\sin(v\tau)}\cos(v\tau)[-\mathrm{e}^{-2\mathrm{i}T} + 1], \tag{80}$$

$$C_1'' = \frac{v}{2\mathrm{i}\sqrt{K}}\left(a - \frac{\mathrm{i}b}{\sqrt{K}}\right)\mathrm{e}^{-\sin(v\tau)}\cos(v\tau)[\mathrm{e}^{2\mathrm{i}T} - 1]. \tag{81}$$

Invariants (33) and (34) yield:

$$I_1 \equiv \left(x\cos T - \frac{y}{\sqrt{K}}\sin T\right)\mathrm{e}^{\sin(v\tau)} - \varepsilon\left[\left(\frac{va}{2\sqrt{K}}\sin 2T - \frac{vb}{2K}\cos 2T\right)\cos(v\tau) + \frac{vb}{2K}\cos(v\tau)\right] = a, \tag{82}$$

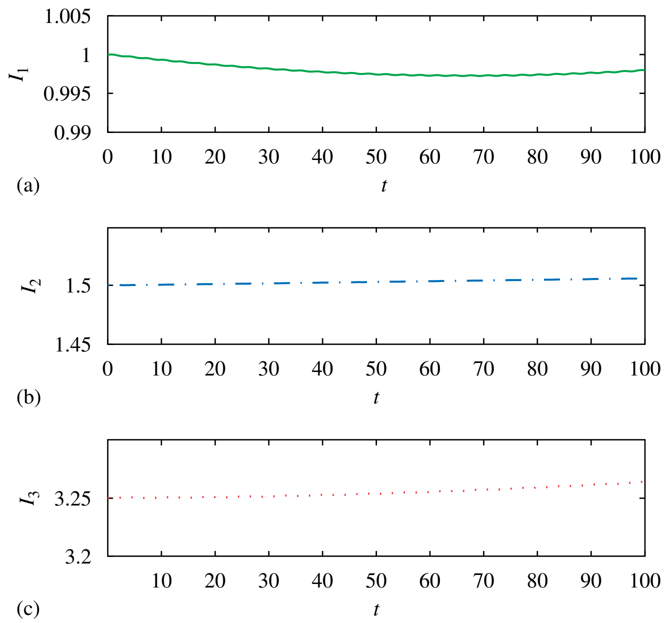

Fig. 4. Adiabatic invariants for the weakly and slowly pulsating undamped oscillator for $\varepsilon = 0.01$, $v = 1$, $K = 1$ and the initial conditions $x(0) = 1$, $y(0) = 1.5$: (a) — the adiabatic invariant I_1, (b) -.-.-. the adiabatic invariant I_2, (c) . . . . the adiabatic invariant I_3.

$$I_2 \equiv \left(x\sin T + \frac{y}{\sqrt{K}}\cos T\right)\mathrm{e}^{\sin(v\tau)} + \varepsilon\left[\left(\frac{va}{2\sqrt{K}}\cos 2T + \frac{vb}{2K}\sin 2T\right)\cos(v\tau) - \frac{va}{2K}\cos(v\tau)\right] = \frac{b}{\sqrt{K}}, \quad (83)$$

while Eq. (35) becomes:

$$I_3 \equiv \frac{Kx^2+y^2}{K}\mathrm{e}^{2\sin(v\tau)} - \varepsilon\frac{v}{\sqrt{K}}\cos(v\tau)\left[\left(a^2-\frac{b^2}{K}\right)\sin 2T - \frac{2ab}{\sqrt{K}}\cos 2T + \frac{2ab}{\sqrt{K}}\right] = \frac{Ka^2+b^2}{K}\cdot \quad (84)$$

In Fig. 4 the adiabatic invariants I_1–I_3 are shown for the case when $\varepsilon = 0.01$, $v = 1$, $K = 1$, $x(0) = 1$, $y(0) = 1.5$. It can be seen that all adiabatic invariants are almost constant.

4. Conclusion

In this paper a procedure for obtaining adiabatic invariants of oscillators with one degree of freedom and slowly varying parameters has been proposed. The procedure is based on the field method approach to deriving exact invariants for the systems with constant parameters. This approach is combined with the multiple variable expansion technique. As a result, two linear time-dependent adiabatic invariants are derived. Combining them, the solution for motion in the second approximation can be found. Besides, multiplying them, the adiabatic invariant whose leading term is quadratic with respect to the state variables can be obtained. The term next to the small parameter in the expression for this invariant is the function of time, unlike those derived previously by some other authors who obtained them in the mixed form of the state variables and time.

Acknowledgement

This investigation is supported by the Ministry of Science and Environmental Protection, Republic of Serbia (Project No.144008).

References

[1] J. Kevorkian, J.D. Cole, *Perturbation Methods in Applied Mathematics*, Springer, Berlin, 1981.
[2] J.M. Burgers, Die adiabatischen Invarianten bedingt periodischer Systeme, *Annalen der Physik* 52 (1917) 195–202.
[3] M. Kruskal, Asymptotic theory of Hamiltonian and other systems with all solutions nearly periodic, *Journal of Mathematical Physics* 3 (1962) 806–828.
[4] H.R. Lewis, Class of exact invariants for classical and quantum time-dependent harmonic oscillators, *Journal of Mathematical Physics* 9 (1968) 1976–1986.
[5] H.R. Lewis, W.B. Riesenfeld, An exact quantum theory of the time-dependent harmonic oscillators and of a charged particle in a time-dependent electomagnetic field, *Journal of Mathematical Physics* 10 (1969) 1458–1473.
[6] K.R. Symon, The adiabatic invariants of the linear or nonlinear oscillator, *Journal of Mathematical Physics* 11 (1970) 1320–1330.
[7] R.K. Colegrave, M.A. Mannan, Invariants of the time-dependent harmonic oscillator, *Journal of Mathematical Physics* 29 (1988) 1580–1587.
[8] P. Helander, M. Lisak, V.E. Semenov, Generalized adiabatic invariants in one-dimensional Hamiltonian systems, *Physical Review Letters* 68 (1992) 3659–3662.
[9] X.-W. Chen, Y.-M. Li, Y.-H. Zhao, Lie symmetries, perturbation to symmetries and adiabatic invariants of Lagrange system, *Physics Letters A* 337 (2005) 274–278.
[10] Dj.S. Djukic, Adiabatic invariants for dynamical systems with one degree of freedom, *International Journal of Non-linear Mechanics* 16 (1981) 489–498.
[11] L. Cveticanin, Adiabatic invariants of dynamic systems with two degrees of freedom, *International Journal of Non-linear Mechanics* 29 (1994) 799–808.
[12] L. Cveticanin, Adiabatic invariants of quasi-pure cubic oscillator, *Journal of Sound and Vibration* 183 (1995) 881–888.
[13] B.D. Vujanovic, S.E. Jones, *Variational Methods in Nonconservative Phenomena*, Academic Press, Boston, 1989.
[14] B. Vujanovic, A.M. Strauss, Applications of a field method to the theory of vibrations, *Journal of Sound and Vibration* 114 (1987) 375–387.
[15] I. Kovacic, Applications of the field method to the non-linear theory of vibrations, *Journal of Sound and Vibration* 264 (2003) 1073–1090.

[16] B.D. Vujanovic, A.M. Strauss, Linear and quadratic first integrals of a forced linearly damped oscillator with a single degree of freedom, *Journal of Acoustical Society of America* 69 (1981) 1213–1214.
[17] J. Kevorkian, Perturbation techniques for oscillatory systems with slowly varying coefficients, *SIAM Review* 29 (1987) 391–461.
[18] N.N. Bogolubov, J.A. Mitropolskij, *Asimptoticheskie Metodi v Teorii Nelinejnih Kolebanij*, Nauka, Moscow, 1974.
[19] A.H. Nayhef, *Perturbation Methods*, Wiley, New York, 1973.