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ScienceDirect
Journal of Sound and Vibration 300 (2007) 877–895
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Transverse vibration and stability of spinning circular plates of constant thickness and different boundary conditions

Helmut F. Bauer, Werner Eidel*

Universität der Bundeswehr München, Institut für Raumfahrttechnik, Werner-Heisenberg-Weg 39, 85577 Neubiberg, Germany

Received 16 February 2006; received in revised form 11 July 2006; accepted 8 September 2006
Available online 3 November 2006

Abstract

The lower approximate natural frequencies of a spinning circular plate are determined for various boundary conditions. The plate is rotating with constant speed of spin about the axis of symmetry perpendicular to its plane. The boundary conditions treated here are: clamped, simply supported, free and guided. It is assumed that the normal displacement of the plate is small and that it does not affect during the oscillation the angular and radial variable stress distribution due to the rotating of the plate. This divides the problem at hand into two parts, of which the first part determines the radial and angular variable planar stresses in the plate, while the second part requires the solution of the partial differential equation with variable coefficients. The approximate lower natural frequencies are determined as functions of the speed of spin and exhibit in the plates with clamped, guided and simply supported boundary conditions instabilities.

1. Introduction

In many engineering systems spinning circular plates appear as components. The fact that flexural vibrations in rotating discs may cause fractures and may therefore lead to failures of spinning mechanical systems, their treatment has become of quite some interest. It was found that at certain speeds of rotation such vibrations become very pronounced and produce repeated cycles of fatigue, resulting in gradual development of cracks at places where stress concentrations are present. In many engineering systems such transverse vibrations are induced by oscillatory exterior forces, introducing resonance phenomena, which may lead to large amplitude transverse oscillations and therefore high stresses. The knowledge of their behaviour must be understood to enhance a proper design. The following investigation treats transverse vibrations of a circular plate of uniform thickness, which is rotating about its axis perpendicular to the plane of the disc with constant angular speed. The spinning plate may exhibit various boundary conditions, of which those of clamped, guided, simply supported and free boundary shall be investigated. The major difficulty in the treatment arises from the fact, that due to the spinning of the plate, the centrifugal force is present as a more or less strong stress of changing magnitude in radial and angular direction, depending on the magnitude of the rotational spin. Depending on the boundary condition of the spinning plate the plane state of stress exhibits tension and

*Corresponding author.
E-mail address: Werner.Eidel@unibw.muenchen.de (W. Eidel).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2006.09.001

Nomenclature

a	radius of circular plate
D	stiffness of plate ($D = (Eh^3/12(1-\overline{v}^2))$)
E	modulus of elasticity
$F_{mn}(\lambda\xi)$	eigenfunction of non-spinning plate
F_r, F_φ	specific volume force components in r- and φ-direction, respectively
G	shear modulus
h	thickness of plate
I_m	modified Bessel function
J_m	Bessel function
k	distributed stiffness of translational springs (force/unit length)
K	distributed stiffness of spiral springs (moment/unit length)
M_r^*	bending moment
$\overline{N}$	in-plane force
r, φ	polar coordinates
t	time
T_0	tension of membrane
$\overline{u}$	radial displacement
$\overline{v}$	angular displacement
V_r	Kelvin–Kirchhoff edge reaction
$\overline{w}(r, \varphi, t)$	displacement of plate
λ_{mn}	eigenvalues of plate according to the treated boundary condition, $m = 0, 1, 2, \ldots, n = 1, 2, \ldots$
$\overline{v}$	Poisson's ratio
$\xi = r/a$	$0 \leqslant \xi \leqslant 1$
ϱ	mass density of plate
σ_r, σ_φ	radial and azimuthal stress respectively
$\tau_{r\varphi}$	shear stress
$\Phi = \overline{N}a^2/D$	tension–compression parameter ($\Phi < 0$: compression, $\Phi > 0$: tension)
ω	natural circular frequency, $\omega^* = \omega/\sqrt{D/\varrho h a^4}$
Ω_0	constant speed of rotation about axis perpendicular to the plate (centre), $\Omega_0^* = \Omega_0/\sqrt{D/\varrho h a^4}$
c, ss, g, f	represent as superscripts the values for the boundary conditions clamped, simply supported, guided or free, respectively

compression ranges in radial and circumferential direction and may lead to buckling as it is observed for non-spinning plates with in-plane forces. The problem exhibits partial differential equations with variable coefficients, for which a closed-form solution is not possible. We therefore have to resort to the Ritz–Galerkin method, which uses the modal expressions satisfying the boundary conditions at hand of the non-rotating plate. For small transverse amplitudes we may separate the problem into two parts by assuming the stresses due to the deflection of the plate sufficiently small (linearised theory) in comparison to the magnitude of tension or compression in the spinning plate so that the elastic contribution of the stresses do not significantly change the elasticity behaviour of the plate in oscillation. Therefore two parts of investigations have to be performed. The first part determines the planar relations of a constantly spinning circular plate with respect to the applied boundary conditions of the plate and yields the radial and circumferential stresses. Both of them depend only on the radial coordinate and the square of the rotational speed. The differential equations of the problem may be combined to one equation in the transverse vibration amplitude with variable coefficients, which is solved with the Ritz–Galerkin method, yielding after applying the orthogonality relation and the appropriate eigenfunctions and eigenvalues an infinite system of algebraic equations for the determination of the natural frequencies as functions of the rotational speed. By truncating the homogeneous algebraic system to a finite system, the vanishing determinant represents the approximate lower natural frequencies. The complexity of the procedure is somewhat reduced and simplified by using an auxiliary differential equation with constant coefficients, thus reducing the order of the partial differential equation, yielding a second-order partial differential equation. We are able to obtain the lower natural frequency with satisfactory engineering accuracy.

A few investigations on spinning structural members have been performed previously. For a beam Bauer [1] has determined the vibration behaviour of a beam spinning about its longitudinal axis for all possible combinations of free, clamped, simply supported and guided boundaries. The natural frequencies for all these cases exhibit either (linear with the speed of spin) a decrease or increase. For a beam oriented perpendicular to the axis of rotation Bauer and Eidel [2] present the fundamental natural frequencies for all possible combinations of free, clamped, simply supported and guided boundary conditions. They investigated the effect of the speed of spin and the hub radius of the beam, which play a pronounced influence upon the natural frequencies and buckling. Increasing speed of rotation may—depending upon the boundary conditions—increase or decrease the natural frequency. Increase of the speed of spin leads in some boundary cases, i.e. clamped–hinged, free–clamped, clamped–guided, guided–clamped, hinged–hinged, hinged–guided, guided–hinged, free–hinged and free–guided to lateral buckling. Here the first boundary condition (left) is that of the hub, while the second (right) notation is that at the outer end of the beam. It also was noticed that the magnitude of the hub-ratio b/l (hub radius to the length of the beam) has a pronounced effect on the natural frequencies, and on instability as well.

The problem at hand, i.e. the vibration of a spinning plate, which became classical, has been treated by Lamb and Southwell [3] and Southwell [4]. Lamb and Southwell treated the problem of the completely free circular spinning plate, neglecting the flexural rigidity, and found by assuming a series expansion in r/a an approximate relation for the natural frequencies as a function of the speed of spin. In Ref. [4], the basic equation for the plate clamped at the inner boundary and free at the outside has been derived. Lacking these investigations is the thorough numerical evaluation of those equations, and in particular the investigation of the behaviour of a spinning plate subject to other boundary conditions. Other investigation such as the formulation of an approximate method has been given by Prescott [5]. Various other studies on rotating plates and space boom structures have been performed in Refs. [6–9].

In the last quarter of the past century the investigation of transverse vibrations of spinning plates has drawn some attention. This interest is mainly based on the relevance of vibrations as they appear in turbine rotors, circular saws and floppy discs. Barash and Chen [10] solved the problem of a spinning plate, clamped at the inner radius and free at the outer radius by reducing the fourth-order differential equation of motion to a set of four first-order equations and solving them by a numerical method. Adams [11] treats the same problem of a clamped inner and a free outer boundary, assuming also that the in-plane forces of rotation are unaffected by the transverse motion of the plate. He includes an elastic foundation parameter, which is based on the viscosity of the air, a case, which is of importance for the operation of floppy discs. Ratios of critical speeds are given exhibiting the effect of the foundation stiffness and a clamping radius ratio. Since flexible spinning plates are very sensitive to transverse loadings Cole and Benson [12] present an effective technique for the determination of the forced response to space fixed point loads by using an eigenfunction expansion. They are able to identify the modes, which are most important to the response. This was achieved by introducing Green's functions, which were computed by numerical integration. In another paper, Shen and Sony [13] investigate the rotating plate under stationary in-plane and concentrated radial and tangential edge loads, as they may occur in rotating saw plates under cutting conditions. They detected that edge loads will yield parametric resonance in transversely excited spinning plates. A perturbation technique was employed by Mignolet et al. [14] to estimate the free vibration characteristics of an annular spinning plate with the inner boundary being clamped and a free outer boundary. They mainly present mode shapes. Tutuncu and Durdu [15] determined the buckling for spinning plates exhibiting polar orthotropy. It was performed for a full plate and for a plate, which was fixed to a rigid shaft. A recent paper by Nayfeh et al. [16] treated transverse oscillations of a centrally clamped rotating plate of uniform thickness and spinning with constant angular speed. The periphery is considered free. The authors use the nonlinear equations, as developed by von Kármán, which were derived for a spinning circular plate by Nowinski [17], solving the coupled nonlinear equations for the transverse deflection and the stress function. They also found that the system exhibits at the primary resonance of one of the asymmetric modes a Duffing-type nonlinear vibration equation, which exhibits for this case a hardening behaviour. The linear case is in good agreement with the results of other authors.

For eccentrically rotating circular plates Maretic [18] investigates transverse vibrations and stability. For the first mode the critical angular speed at which instability occurs is determined and shows for increasing eccentricity a decrease of the critical speed. All papers treat essentially annular spinning plate and represent the effect of certain system parameters.

2. Planar stress relation of a spinning circular plate

If we assume the oscillatory displacement $\overline{w}$ of a spinning circular plate of small magnitude, is unaffected by the stress in the plate, the state of planar tension and compression is predominantly due to the spinning of the plate about its axis perpendicular to its plane. For constant thickness h of the plate (Fig. 1) the two-dimensional equilibrium conditions are given by

$$\frac{\partial\sigma_r}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\varphi}}{\partial\varphi}+\frac{1}{r}(\sigma_r-\sigma_\varphi)+F_r=0 \quad \text{and} \quad \frac{\partial\tau_{r\varphi}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_\varphi}{\partial\varphi}+\frac{2}{r}\tau_{r\varphi}+F_\varphi=0. \tag{1}$$

In addition we have to observe the stress–strain relation, i.e. the Hook law

$$\varepsilon_r=\frac{\partial\overline{u}}{\partial r}=\frac{1}{E}(\sigma_r-\overline{v}\sigma_\varphi), \quad \varepsilon_\varphi=\frac{1}{r}\frac{\partial\overline{v}}{\partial\varphi}+\frac{1}{r}\overline{u}=\frac{1}{E}(\sigma_\varphi-\overline{v}\sigma_r), \tag{2}$$

$$\gamma_{r\varphi}=\frac{1}{r}\frac{\partial\overline{u}}{\partial\varphi}+\frac{\partial\overline{v}}{\partial r}-\frac{\overline{v}}{r}=\frac{1}{G}\tau_{r\varphi}, \tag{3}$$

where σ_r is the radial, σ_φ the angular stress, F_r and F_φ the specific volume force components and $\overline{u}$, $\overline{v}$ the radial and angular displacement, respectively. The strains are given by ε_r and ε_φ, and $\gamma_{r\varphi}$ is the angular displacement due to the shear stress $\tau_{r\varphi}$, E is Young's elasticity modulus and $G=E/2(1+\overline{v})$ is the shear modulus, $\overline{v}$ the Poisson ratio.

For a spinning plate with Ω_0 as the constant speed of rotation $\partial/\partial\varphi=0$ and $F_\varphi=0$, $F_r=\varrho\Omega_0^2 r$. This yields for the above strongly coupled equations (1)–(3) the expressions

$$r\frac{\mathrm{d}\sigma_r}{\mathrm{d}r}+(\sigma_r-\sigma_\varphi)+\varrho\Omega_0^2r^2=0, \quad \frac{\mathrm{d}\tau_{r\varphi}}{\mathrm{d}r}+\frac{2}{r}\tau_{r\varphi}=0 \tag{4}$$

and

$$E\frac{\mathrm{d}\overline{u}}{\mathrm{d}r}=\sigma_r-\overline{v}\sigma_\varphi, \quad E\frac{\overline{u}}{r}=\sigma_\varphi-\overline{v}\sigma_r, \quad Gr\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\overline{v}}{r}\right)=\tau_{r\varphi}. \tag{5}$$

For a uniformly spinning plate it is $\tau_{r\varphi}=\overline{v}=0$. From Eqs. (3) we obtain

$$r\left(\frac{\mathrm{d}\sigma_\varphi}{\mathrm{d}r}-\overline{v}\frac{\mathrm{d}\sigma_r}{\mathrm{d}r}\right)+(1+\overline{v})(\sigma_\varphi-\sigma_r)=0. \tag{6}$$

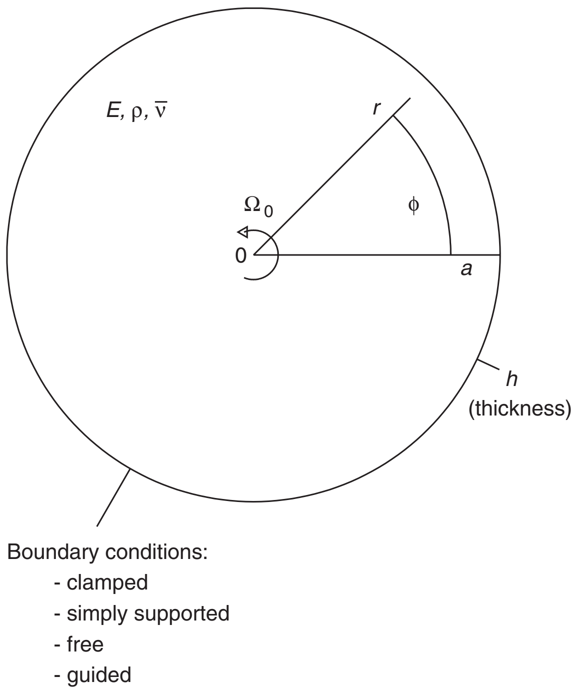

Fig. 1. Geometry of the circular plate.

The two coupled Euler differential equations (2) and (4) have a solution of the form

$$\sigma_r = A - \alpha\Omega_0^2 r^2, \quad \sigma_\varphi = A - \beta\Omega_0^2 r^2, \tag{7}$$

where $\alpha = \varrho(3+\bar{v})/8$ and $\beta = \varrho(1+3\bar{v})/8$. This solution may also be obtained by eliminating σ_φ from Eqs. (4) and (6), yielding for the radial stress σ_r the Euler differential equation

$$r^2\frac{\mathrm{d}^2\sigma_r}{\mathrm{d}r^2} + 3r\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} = -(3+\bar{v})\varrho\Omega_0^2 r^2, \tag{8}$$

which exhibits the solution (7). From Eq. (5) we may determine the radial displacement $\bar{u}$ as

$$\bar{u} = \frac{1-\bar{v}}{E} r\left[A - \frac{1}{8}\varrho\Omega_0^2(1+\bar{v})r^2\right]. \tag{9}$$

The magnitude of the integration constant A depends on the boundary conditions of the plate, of which we distinguish four cases at $r = a$:

$$\text{1. Free boundary at} \quad r = a: \quad \sigma_r = 0, \tag{10}$$

$$\text{2. Clamped boundary at} \quad r = a: \quad \bar{u} = 0, \tag{11}$$

$$\text{3. Hinged boundary at} \quad r = a: \quad \bar{u} = 0, \tag{12}$$

$$\text{4. Guided boundary at} \quad r = a: \quad \bar{u} = 0. \tag{13}$$

We notice that we have to treat for the planar stress only two cases. Case I (free boundary) renders the radial and angular stress

$$\sigma_r = \tfrac{1}{8}\varrho\Omega_0^2(3+\bar{v})(a^2 - r^2), \quad \sigma_\varphi = \tfrac{1}{8}\varrho\Omega_0^2\left[(3+\bar{v})a^2 - (1+3\bar{v})r^2\right], \tag{14}$$

while case II comprising the attachment cases 2–4 yields

$$\sigma_r = \tfrac{1}{8}\varrho\Omega_0^2\left[(1+\bar{v})a^2 - (3+\bar{v})r^2\right], \quad \sigma_\varphi = \tfrac{1}{8}\varrho\Omega_0^2\left[(1+\bar{v})a^2 - (1+3\bar{v})r^2\right]. \tag{15}$$

These results have to be included as variable in-plane forces into the equation of the plate.

3. Basic equation of a spinning plate

With the above in-plane stress forces the partial differential equation of a spinning circular plate is given by [3,4]

$$D\nabla^4\bar{w} - \frac{h}{r}\frac{\partial}{\partial r}\left(\sigma_r r\frac{\partial\bar{w}}{\partial r}\right) - \frac{h}{r^2}\sigma_\varphi\frac{\partial^2\bar{w}}{\partial\varphi^2} + \varrho h\frac{\partial^2\bar{w}}{\partial t^2} = 0, \tag{16}$$

where the second and third term are the in-plane forces due to the spinning of the plate, ϱ the mass density of the plate, $D = Eh^3/12(1-\bar{v}^2)$ the flexural rigidity, h the thickness of the plate, $\bar{w}(r,\varphi,t)$ the normal displacement and

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}.$$

If the stress is larger zero the plate is under tension, while for the stress smaller zero it is under compression. Introducing the above results (10), (11) and (12) or (13) into Eq. (16) we obtain with

$$\bar{w}(r,\varphi,t) = \mathrm{e}^{\mathrm{i}\omega t}\sum_{n=1}^{\infty(N)}\sum_{m=0}^{\infty(N)} W_{mn}(r)\cos m\varphi, \tag{17}$$

where ω is the yet unknown natural frequency of the plate and m (integer) the angular mode number, the ordinary differential equation with variable coefficients. It is with $\xi = r/a$

$$E_m[W_{mn};\xi] = \frac{d^4 W_{mn}}{d\xi^4} + \frac{2}{\xi}\frac{d^3 W_{mn}}{d\xi^3} - \frac{2m^2+1}{\xi^2}\frac{d^2 W_{mn}}{d\xi^2} + \frac{2m^2+1}{\xi^3}\frac{dW_{mn}}{d\xi} - \frac{m^2(4-m^2)}{\xi^4}W_{mn} - \frac{\varrho h a^4}{D}\omega^2 W_{mn} - \frac{h\varrho\Omega_0^2 a^4}{8\xi D}\frac{d}{d\xi}\left[\left(C^*\xi - D^*\xi^3\right)\frac{dW_{mn}}{d\xi}\right] + \frac{m^2 h\varrho\Omega_0^2 a^4}{8\xi^2 D}\left(C^* - E^*\xi^2\right)W_{mn} = 0, \tag{18}$$

where

$$C^* = \begin{cases} (3+\overline{v}) & \text{for free boundary condition,} \\ (1+\overline{v}) & \text{for clamped, hinged and guided boundary conditions,} \end{cases} \tag{19}$$

$$D^* = (3+\overline{v}), \quad E^* = (1+3\overline{v}) \quad \text{for all four boundary conditions.} \tag{20}$$

For a non-spinning plate, i.e. $\Omega_0 = 0$ the last four terms vanish, and we obtain the well-known differential equation for the vibrating circular plate. Eq. (18) has to be solved with the considered boundary conditions, which are either

$$\begin{array}{ll} \text{(a) clamped :} & W_{mn} = 0 \quad \text{and} \quad \frac{dW_{mn}}{dr} = 0 \quad \text{at} \quad \xi = 1, \\ \text{(b) hinged :} & W_{mn} = 0 \quad \text{and} \quad M_r^* = 0 \quad \text{at} \quad \xi = 1, \\ \text{(c) guided :} & \frac{dW_{mn}}{dr} = 0 \quad \text{and} \quad V_r^* = 0 \quad \text{at} \quad \xi = 1, \\ \text{(d) free :} & M_r^* = 0 \quad \text{and} \quad V_r^* = 0 \quad \text{at} \quad \xi = 1, \\ \text{(e) elastically supported :} & M_r^* - K\frac{dW_{mn}}{dr} = 0 \quad \text{and} \quad V_r^* + kW_{mn} = 0 \quad \text{at} \quad \xi = 1, \end{array} \tag{21}$$

where K is the distributed spiral spring stiffness, i.e. moment of unit length, opposing the edge rotation, and where k is the distributed spring stiffness, i.e. the force per unit length opposing the translational motion $\overline{w}$ in normal direction. Eq. (21e), not being used in the paper, should indicate that such an elastic supported boundary could also be treated by the followed method. The above indicated bending moment is

$$M_r^* = -D\left[\frac{d^2 W_{mn}}{d\xi^2} + \frac{\overline{v}}{\xi}\frac{dW_{mn}}{d\xi} - \frac{\overline{v}m^2}{\xi^2}W_{mn}\right], \tag{22}$$

while

$$V_r^* = -D\left[\frac{d^3 W_{mn}}{d\xi^3} + \frac{1}{\xi}\frac{d^2 W_{mn}}{d\xi^2} - \frac{1+(2-\overline{v})m^2}{\xi^2}\frac{dW_{mn}}{d\xi} + \frac{(3-\overline{v})m^2}{\xi^3}W_{mn}\right]. \tag{23}$$

4. Method of solution

The above ordinary differential equations (18) shall be solved by first solving a truncated auxiliary differential equation [19] that can be solved with the given boundary conditions. This means that the given original differential equation is truncated to one, rendering an exact solution with the boundary conditions of the problem. For the spinning plate equation (18) the truncated differential equation consists of the first five terms yielding a solution $\lambda_{mn}^4 W_{mn}$. The original differential equation reads then with $\omega^{*2} = (\varrho h a^4/D)\omega^2$ and $\Omega_0^{*2} = (\varrho h a^4/D)\Omega_0^2$

$$\left(\lambda_{mn}^4 - \omega^{*2}\right)W_{mn} - \frac{\Omega_0^{*2}}{8}[C^* - D^*\xi^2]\frac{d^2 W_{mn}}{d\xi^2} - \frac{\Omega_0^{*2}}{8\xi}[C^* - 3D^*\xi^2]\frac{dW_{mn}}{d\xi} + \frac{m^2\Omega_0^{*2}}{8\xi^2}(C^* - E^*\xi^2)W_{mn} = 0, \tag{24}$$

in which we have to introduce the appropriate values C^*, D^* and E^* according to the treated boundary conditions (see Eqs. (19) and (20)). λ_{mn} are the eigenvalues of the plate as obtained from the solution

$$W_{mn}(\xi) = A_{mn}J_m(\lambda_{mn}\xi) + B_{mn}I_m(\lambda_{mn}\xi) \tag{25}$$

and the boundary conditions. Assuming the solution of the differential equation as a series expansion in the eigenfunctions (satisfying the boundary conditions) and applying the Ritz–Galerkin method yields from the Ritz–Galerkin condition

$$\int_0^1 E_m[W_{mn};\xi]\xi F_{m\nu}(\lambda_{m\nu}\xi)\,\mathrm{d}\xi = 0, \quad \text{for} \quad \nu = 1, 2, \ldots \tag{26}$$

finally an infinite system of homogeneous algebraic equations, which vanishing coefficient determinant renders the eigenvalue equation. Truncating this determinant to a finite order yields the approximate values (and natural frequencies) of the lower eigenvalues.

4.1. Spinning clamped plate

A clamped spinning plate has to satisfy the boundary conditions $W_{mn} = 0$ and $\mathrm{d}W_{mn}/\mathrm{d}\xi = 0$ at $\xi = 1$. The solution

$$W_{mn}(\xi) = A_{mn}[J_m(\lambda_{mn}^{(c)}\xi)I_m(\lambda_{mn}^{(c)}) - J_m(\lambda_{mn}^{(c)})I_m(\lambda_{mn}^{(c)}\xi)] = A_{mn}F_{mn}(\lambda_{mn}^{(c)}\xi) \tag{27}$$

with the eigenvalues $\lambda_{mn}^{(c)}$ obtained from ($n = 1, 2, \ldots$)

$$J_m(\lambda)I_m'(\lambda) - J_m'(\lambda)I_m(\lambda) = 0 \tag{28}$$

satisfies the boundary conditions. The eigenvalues are shown in Table 1. With the Ritz–Galerkin condition (26) we obtain an infinite set of algebraic equations. It is (see upper scripts in "Nomenclature")

$$\int_0^1 \sum_{n=1}^{\infty} A_{mn}\left\{\left(\lambda_{mn}^4 - \frac{\varrho h a^4\omega^2}{D}\right)\xi F_{mn}(\lambda\xi)F_{m\nu}(\lambda\xi) - \frac{\varrho h a^4\lambda^2\Omega_0^2}{8D}[(1+\bar{\nu})\xi - (3+\bar{\nu})\xi^3]F_{mn}''(\lambda\xi)F_{m\nu}(\lambda\xi)\right.$$

Table 1
λ_{mn}^2 for clamped, guided, simply supported and free plate

$n\backslash m$	0	1	2	3	Boundary condition
1	10.2158	21.2604	34.8770	51.0300	$\lambda_{mn}^{(c)2}$ Clamped
2	39.7711	60.8287	84.5826	111.0214	
3	89.1041	120.0792	153.8151	190.3038	
4	158.1842	199.0534	242.7206	289.1799	
1	0	3.0825	8.7849	16.9020	$\lambda_{mn}^{(g)2}$ Guided ($\bar{\nu} = 0.3$)
2	14.6820	28.3988	44.9041	64.1304	
3	49.2185	72.8590	99.3610	128.6775	
4	103.4995	137.0254	173.4422	212.7161	
1	4.9351	13.8982	25.6133	39.9573	$\lambda_{mn}^{(ss)2}$ Simply supported ($\bar{\nu} = 0.3$)
2	29.7200	48.4789	70.1170	94.5490	
3	74.1561	102.7733	134.2978	168.6749	
4	138.3181	176.8012	218.2026	262.4847	
1	0	0	5.3583	12.4390	$\lambda_{mn}^{(f)2}$ Free ($\bar{\nu} = 0.3$)
2	9.0031	20.4746	35.2601	53.0078	
3	38.4432	59.8116	84.3662	111.9450	
4	87.7502	118.9573	153.3059	190.6918	

$$- \frac{\varrho h a^4 \lambda \Omega_0^2}{8D}[1+\bar{v}-3(3+\bar{v})\xi^2]F'_{mn}(\lambda\xi)F_{mv}(\lambda\xi)$$
$$+ \frac{m^2 \varrho h a^4 \Omega_0^2}{8D}\left[(1+\bar{v})\frac{1}{\xi}-(1+3\bar{v})\xi\right]F_{mn}(\lambda\xi)F_{mv}(\lambda\xi)\Bigg\}\,\mathrm{d}\xi = 0$$
$$\text{for} \quad v = 1,2,\ldots \,. \tag{29}$$

This system of equations requires the determination of the following integrals. The orthogonality relation is with the help of [20,21]

$$I^{(1,0)}_{mnv} = \int_0^1 \xi F_{mn}(\lambda^{(c)}_{mn}\xi)F_{mv}(\lambda^{(c)}_{mv}\xi)\,\mathrm{d}\xi = \begin{cases} J_m^2(\lambda^{(c)}_{mn})I_m^2(\lambda^{(c)}_{mn}) & \text{for} \quad v = n, \\ 0 & \text{for} \quad v \neq n. \end{cases} \tag{30}$$

while the remaining integrals are for fixed $m = 0,1,2,\ldots$ and $n, v = 1,2,3,\ldots$

$$\int_0^1 \xi^k F^{(l)}_{mn}(\lambda_{mn}\xi)F_{mv}(\lambda_{mv}\xi)\,\mathrm{d}\xi = I^{(k,l)}_{mnv} \tag{31}$$

with $(k,l) = (1,2),(3,2),(0,1),(2,1),(-1,0)$ are solved numerically. The truncated vanishing determinant (29) represents with $\omega a^2/\sqrt{D/\varrho h} = \lambda^2$ the approximate lower natural frequencies of the clamped spinning plate. Introducing the above results (30) and (31) into Eq. (29) yields the determinant for the determination of the natural frequencies of a spinning clamped plate.

4.2. Spinning guided plate

For a guided plate satisfying the boundary condition $\mathrm{d}W_{mn}/\mathrm{d}\xi = 0$ and $V_r^* = 0$ at $\xi = 1$, the eigenvalues in Eq. (27) are to be replaced by $\lambda^{(g)}_{mn}$ (Table 1) from

$$2\lambda^3 J'_m(\lambda)I'_m(\lambda) - m^2(1-\bar{v})\left[J_m(\lambda)I'_m(\lambda) - J'_m(\lambda)I_m(\lambda)\right] = 0 \tag{32}$$

and the solution

$$F^{(g)}_{mn}(\lambda^{(g)}_{mn}\xi) = [J_m(\lambda^{(g)}_{mn}\xi)I'_m(\lambda^{(g)}_{mn}) - J'_m(\lambda^{(g)}_{mn})I_m(\lambda^{(g)}_{mn}\xi)]. \tag{33}$$

The orthogonality relation has for $n = v$ the value $I^{(1,0)}_{mnn} = \frac{1}{2}[J'^2_m(\lambda)I^2_m(\lambda) + I'^2_m(\lambda)J^2_m(\lambda)] + (m^2/2\lambda^2)[J'^2_m(\lambda) I^2_m(\lambda) - I'^2_m(\lambda)J^2_m(\lambda)] + (1/\lambda)[J'^2_m(\lambda)I'_m(\lambda)I_m(\lambda) - I'^2_m(\lambda)J'_m(\lambda)J_m(\lambda)]$ with $\lambda = \lambda^{(g)}_{mn}$ and zero for $n \neq v$. The procedure for the determination of the approximate lower natural frequencies is similar to that of above.

4.3. Spinning simply supported plate

For a simply supported plate we have to satisfy the boundary conditions $W_{mn} = 0$ and $M_r^* = 0$ at $\xi = 1$. The eigenvalues in Eq. (27) are to be replace by $\lambda^{(ss)}_{mn}$ (Table 1) from

$$\left[J''_m(\lambda) + \frac{v}{\lambda}J'_m(\lambda)\right]I_m(\lambda) - \left[I''_m(\lambda) + \frac{v}{\lambda}I'_m(\lambda)\right]J_m(\lambda) = 0 \tag{34}$$

and the solution by

$$F^{(ss)}_{mn}(\lambda^{(ss)}_{mn}\xi) = [J_m(\lambda^{(ss)}_{mn}\xi)I_m(\lambda^{(ss)}_{mn}) - J_m(\lambda^{(ss)}_{mn})I_m(\lambda^{(ss)}_{mn}\xi)]. \tag{35}$$

The orthogonality relation is $I^{(1,0)}_{mnn} = (-1/1-\bar{v})J_m(\lambda)I_m(\lambda)\{(1+\bar{v})J_m(\lambda)I_m(\lambda) + \lambda[J'_m(\lambda)I_m(\lambda) + J_m(\lambda)I'_m(\lambda)]\}$ with $\lambda = \lambda^{(ss)}_{mn}$ for $n = v$ and zero for $n \neq v$.

4.4. Spinning free plate

For a free plate we have to satisfy the boundary conditions $M_r^* = 0$ and $V_r^* = 0$ at $\xi = 1$. The eigenfunction is given by

$$F^{(f)}_{mn}(\lambda^{(f)}_{mn}\xi) = [J_m(\lambda^{(f)}_{mn}\xi) - \chi_m(\lambda^{(f)}_{mn},\bar{v})I_m(\lambda^{(f)}_{mn}\xi)], \tag{36}$$

where

$$\chi_m(\lambda,\bar{\nu}) = \frac{(1-\bar{\nu})\lambda J_m'(\lambda) + \left[\lambda^2 - m^2(1-\bar{\nu})\right] J_m(\lambda)}{(1-\bar{\nu})\lambda I_m'(\lambda) - \left[\lambda^2 + m^2(1-\bar{\nu})\right] I_m(\lambda)}.$$

The eigenvalues in Eq. (27) are to be replaced by $\lambda_{mn}^{(f)}$ (Table 1) from

$$\begin{aligned}&\{(1-\bar{\nu})\lambda J_m'(\lambda) + [\lambda^2 - m^2(1-\bar{\nu})]J_m(\lambda)\}\{\lambda^3 I_m'(\lambda) - m^2(1-\bar{\nu})[\lambda I_m'(\lambda) - I_m(\lambda)]\}\\ &\quad + \{(1-\bar{\nu})\lambda I_m'(\lambda) - [\lambda^2 + m^2(1-\bar{\nu})]I_m(\lambda)\}\{\lambda^3 J_m'(\lambda) + m^2(1-\bar{\nu})[\lambda J_m'(\lambda) - J_m(\lambda)]\} = 0. \end{aligned} \tag{37}$$

In addition we have to observe the different stress values (14). This would mean that in the second term of Eq. (29) with F'' the expression $(1+\bar{\nu})$ has to be replaced by $(3+\bar{\nu})$ and in the third term with F' the value $(1+\bar{\nu})$ has also to be replaced by $(3+\bar{\nu})$, while in the last term $(1+\bar{\nu})$ must also be replaced by $(3+\bar{\nu})$. The orthogonality relation is

$$I_{mnn}^{(1,0)} = \tfrac{1}{2}\left(1 - \frac{m^2}{\lambda^2}\right) J_m^2(\lambda) + \tfrac{1}{2} J_m'^2(\lambda) + \chi_m^2\left[\tfrac{1}{2}\left(1+\tfrac{m^2}{\lambda^2}\right) I_m^2(\lambda) - \tfrac{1}{2} I_m'^2(\lambda)\right] - \tfrac{\chi_m}{\lambda}[J_m(\lambda) I_m'(\lambda) - I_m(\lambda) J_m'(\lambda)]$$

with $\lambda = \lambda_{mn}^{(f)}$ for $n = v$ and zero for $n \neq v$.

4.5. Special case

We consider here the special case of a fast spinning clamped plate for which the elastic restoring forces are small in comparison with the centrifugal force. This is represented by neglecting elastic terms ($D = 0$) of the plate equation (18), i.e. by omitting the first five terms. The governing differential equation may then be written as

$$\begin{aligned}&[(1+\bar{\nu}) - (3+\bar{\nu})\xi^2]\frac{d^2 W_{mn}}{d\xi^2} + [(1+\bar{\nu}) - 3(3+\bar{\nu})\xi^2]\frac{1}{\xi}\frac{dW_{mn}}{d\xi}\\ &\quad - \frac{m^2}{\xi^2}[(1+\bar{\nu}) - (1+3\bar{\nu})\xi^2]W_{mn} + 8\frac{\omega^2}{\Omega_0^2}W_{mn} = 0. \end{aligned} \tag{38}$$

for the clamped, guided and simply supported plate. For the free plate $(1+\bar{\nu})$ has to be replaced by $(3+\bar{\nu})$. We obtain with the Ritz–Galerkin condition the system of equations (m fixed integer)

$$\begin{aligned}\sum_{n=1}^{\infty} A_{mn}\Bigg\{&\lambda^2\int_0^1 \xi F_{mn}''(\lambda_{mn}\xi)F_{mv}(\lambda_{mv}\xi)\,d\xi - \frac{3+\bar{\nu}}{(1+\bar{\nu})}\lambda^2\int_0^1 \xi^3 F_{mn}''(\lambda_{mn}\xi)F_{mv}(\lambda_{mv}\xi)\,d\xi\\ &+\lambda\int_0^1 F_{mn}'(\lambda_{mn}\xi)F_{mv}(\lambda_{mv}\xi)\,d\xi - \frac{3(3+\bar{\nu})}{(1+\bar{\nu})}\lambda\int_0^1 \xi^2 F_{mn}'(\lambda_{mn}\xi)F_{mv}(\lambda_{mv}\xi)\,d\xi\\ &-m^2\int_0^1 \frac{1}{\xi}F_{mn}(\lambda_{mn}\xi)F_{mv}(\lambda_{mv}\xi)\,d\xi + m^2\frac{1+3\bar{\nu}}{(1+\bar{\nu})}\int_0^1 \xi F_{mn}(\lambda_{mn}\xi)F_{mv}(\lambda_{mv}\xi)\,d\xi\\ &+\frac{8\omega^2}{(1+\bar{\nu})\Omega_0^2}\int_0^1 \xi F_{mn}(\lambda_{mn}\xi)F_{mv}(\lambda_{mv}\xi)\,d\xi\Bigg\} = 0 \quad \text{for} \quad v = 1, 2, \ldots \end{aligned} \tag{39}$$

Introducing the appropriate eigenvalues λ, the eigenfunctions and the result of the above integrals yields after truncation to a finite system of homogeneous algebraic equations, of which the vanishing coefficient determinant shall produce the lower approximate natural frequencies of the spinning clamped, guided or simply supported plate. For a spinning free plate the values $(1+\bar{\nu})$ in the above equation has to be replaced by $(3+\bar{\nu})$. In addition the eigenvalues λ, eigenfunctions F_{mn} and the results of the integrals for that case have to be introduced. In the above equation the second integral has unity as a coefficient, the fourth integral 3 and the fifth integral $m^2(1+3\bar{\nu})/(3+\bar{\nu})$.

5. Numerical evaluations and conclusions

The analytically obtained results of above have been evaluated numerically. For the determination of the planar stress relation two cases appear and were considered. The first case I (10) is valid for the free boundary ($\sigma_r = 0$ at $r = a$) and yields the results (14). The stresses $\sigma_r/\varrho\Omega_0^2a^2$ (—) and $\sigma_\varphi/\varrho\Omega_0^2a^2$ (- - -) are presented in Fig. 2 and show with increasing radius from the centre a decrease for the radial stress to zero. The circumferential stress also decreases with the increase of the radius r. The freely spinning plate is totally in a state of tension. Fig. 3 exhibits the results of case II, which comprises the clamped, hinged and guided boundary ($u = 0$ at $r = a$) (see Eq. (15)). With increasing distance from the centre axis of the plate the stresses decrease in the lower r/a-region where we observe tension. At about $r/a \approx 0.63$ the radial stress vanishes and shows for $r/a > 0.63$ increasing compression in radial direction. The stress in angular direction is larger in $0 \leqslant r/a < 0.83$, exhibiting tension and has at $r \approx 0.83a$, after which it exhibits a smaller r-region of compression. All stresses are proportional to the square of the speed of spin. The state of stress is presented in the sketch in the left corner of the graph.

The lower natural frequencies ω_{0n}^* for the *clamped* plate as obtained from the solution of Eq. (29) with the orthogonality relation (30) are presented for the axisymmetric modes $m = 0$, $n = 1, 2, 3$ (Fig. 4a) and the ratio $\omega/\omega_{mn}^{(0)}$, $m = 0, 1, 2$ and $n = 1, 2, 3$ in Fig. 4b. The natural frequencies $\omega_{mn}/\sqrt{D/\varrho ha^4} = \omega_{mn}^*$ are exhibited as a function of the spin frequency $\Omega_0/\sqrt{D/\varrho ha^4} = \Omega_0^*$ ($0 \leqslant \Omega_0^* \leqslant 100$). For $n = 1$ the natural frequency increases slightly above the natural frequency without spin, then decreases until the frequency of spin reaches at $\Omega_0^* \approx 8.9$ the frequency of a non-spinning plate again (dashed line). For $\Omega_0^* > 8.9$ the natural frequency of the clamped plate decreases rapidly until it reaches for $\Omega_0^* \approx 20.6$ the value zero, i.e. buckling of the plate. For $m = 0$ and $n = 2$ the natural frequency ω_{02}^* exhibits from $\Omega_0^* = 0$ (the non-spinning value) increasing to $\omega_{02}^* \approx 45.6$ at a frequency of spin of about $\Omega_0^* \approx 25$, then decreases again until it reaches the natural frequency of the non-spinning plate at $\Omega_0^* \approx 36.1$. At the frequency of spin $\Omega_0^* \approx 46.8$ the clamped plate is buckling in the second mode. The third mode $m = 0$, $n = 3$ reaches instability at $\Omega_0^* \approx 73.3$. We conclude from these results that a spinning clamped plate buckles at $\Omega_0^* \approx 20.6$, i.e. at a speed of spin of $\Omega_0 \approx 20.6 \cdot \sqrt{D/\varrho a^4 h}$. By choosing in Ref. [24] the elastic parameters $C = K = \infty$ the results may be applied to the spinning plate with clamped boundary. This means that the translational and torsional spring stiffness assume both the magnitude

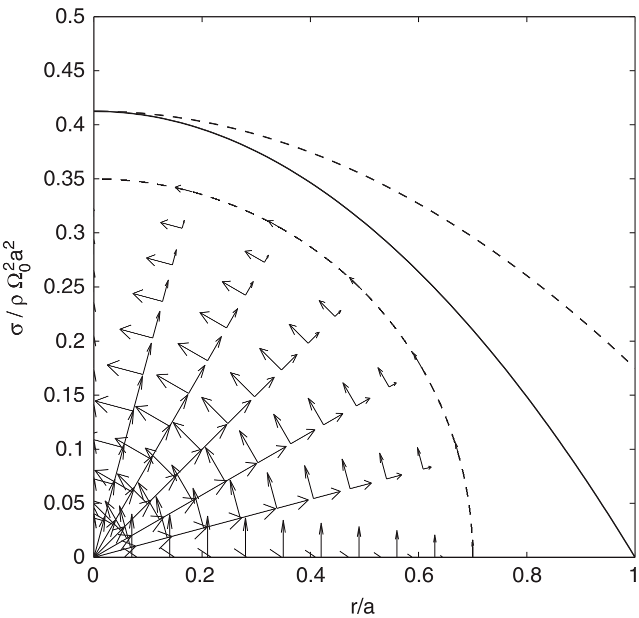

Fig. 2. Radial ($\sigma_r/\varrho\Omega_0^2a^2$) and angular ($\sigma_\varphi/\varrho\Omega_0^2a^2$, dashed line) stress for spinning plate with free boundary ($\bar{\nu} = 0.3$).

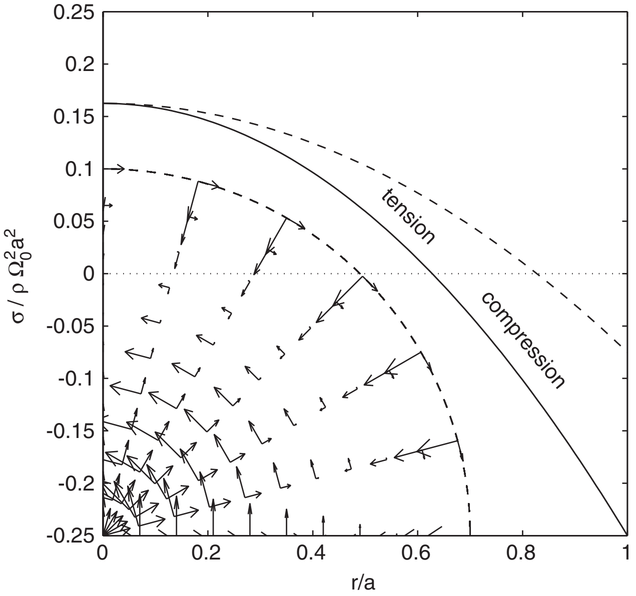

Fig. 3. Radial ($\sigma_r/\varrho\Omega_0^2a^2$) and angular ($\sigma_\varphi/\varrho\Omega_0^2a^2$, dashed line) stress for spinning plate with clamped, hinged or guided boundary ($\bar{\nu} = 0.3$).

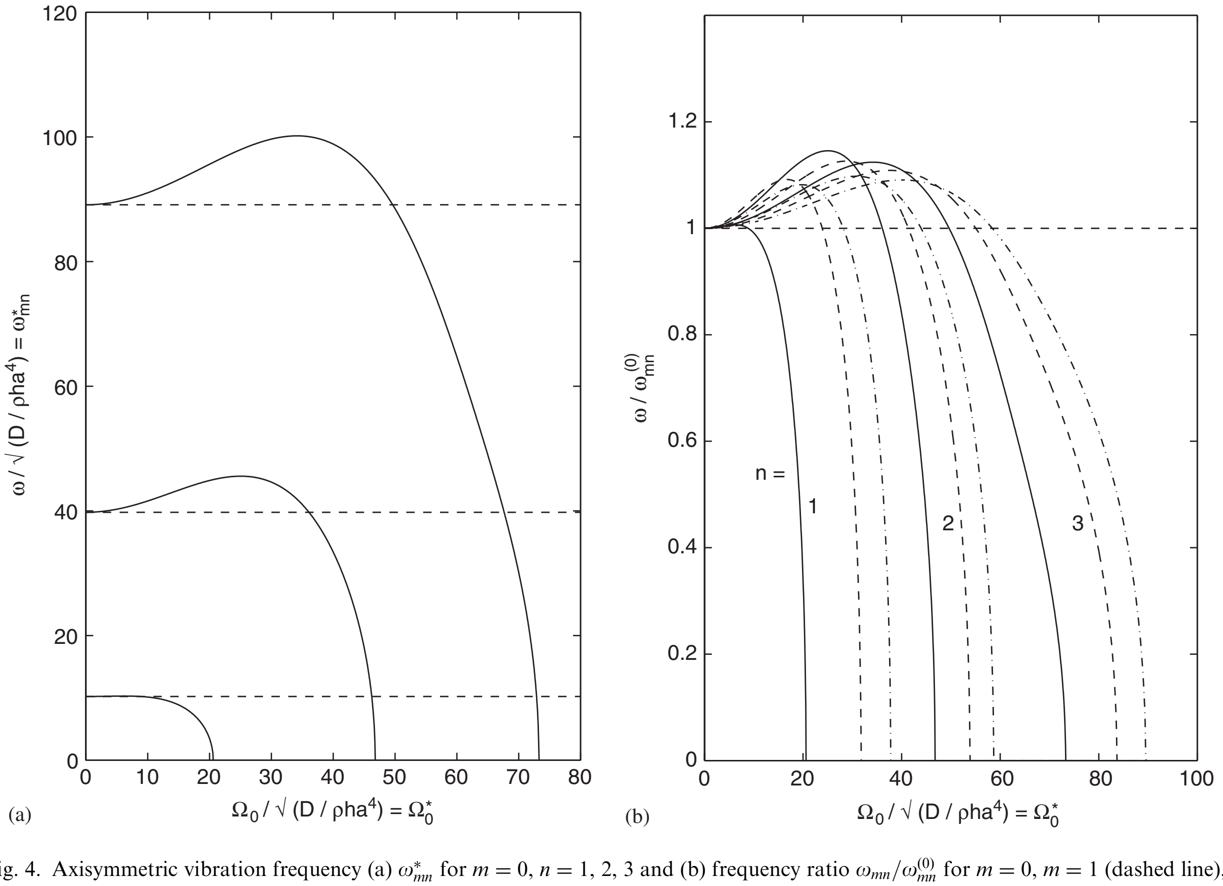

Fig. 4. Axisymmetric vibration frequency (a) ω^*_{mn} for $m = 0$, $n = 1, 2, 3$ and (b) frequency ratio $\omega_{mn}/\omega^{(0)}_{mn}$ for $m = 0$, $m = 1$ (dashed line), $m = 2$ (dash-dot line) and $n = 1, 2, 3$ as a function of the speed of spin Ω_0 for clamped plate ($\bar{\nu} = 0.3$, $N = 30$).

infinity. The critical speed parameter exhibits the magnitude of 38.853. In the present treatment it is $\Omega^*_{0,\text{crit}} = (\Omega_0/\sqrt{Eh^2})\sqrt{12(1-\bar{\nu}^2\varrho a^4} = 20.6$. To compare with the value $\lambda_{\text{crit}} = \Omega_0^2\varrho a^4/Eh^2$ of the paper by Maretic [24] we find $\Omega^{*2}_{0,\text{crit}} = 424.36 = 12(1-\bar{\nu}^2)\Omega_0^2\varrho a^4/Eh^2$, $\lambda_{\text{crit}} = 38.866$, which exhibits good agreement.

The spinning *simply supported* plate has been treated in a similar way, observing the eigenvalues $\lambda^{(ss)}_{mn}$ (Table 1), the eigenfunctions $F^{(ss)}_{mn}$ (Eq. (35)) and the presented orthogonality conditions. The results of the natural frequency ratios $\omega/\omega^{(0)}_{mn}$ are presented for the speed of spin Ω^*_0 in Fig. 5. We notice that the natural frequency $m = 0$, $n = 1$ is always below that of the non-spinning plate (dashed line) and that buckling occurs at a quite small value $\Omega^*_0 \approx 6.6$. The fundamental mode $m = n = 1$ buckles at $\Omega^*_0 \approx 14.1$.

For a spinning *free* circular plate the natural frequencies as a function of the speed of spin are presented for the various modes in Fig. 6. For the axisymmetric mode $m = 0$ the results for $\omega/\sqrt{D/\varrho a^4 h}$ are exhibited in Fig. 6a for $n = 2$, 3 and 4. The natural frequency of the plate shows an increase due to the increasing tension caused by the increasing speed of spin, $\omega^*_{02} = \omega/\sqrt{D/\varrho a^4 h} = 9.0$ to $\omega^*_{02} \approx 107$ (*note*: $n = 1$ represents the translational motion of the plate (see Table 1)).

The second mode frequency ω^*_{03} increases in the spin range $0 \leqslant \Omega^*_0 \leqslant 50$ from $\omega^*_{03} = 38.4$ to about $\omega^*_{03} = 170.2$, while the third axisymmetric frequency ω^*_{04} increased in that range from $\omega^*_{04} = 87.75$ to 256.6. If we neglect in the treatment of the spinning plate flexural rigidity $D = 0$, then we have to solve Eq. (39). This has been performed for a spinning plate of which the slope has been determined in the vicinity of $\Omega^*_0 = 50$, a

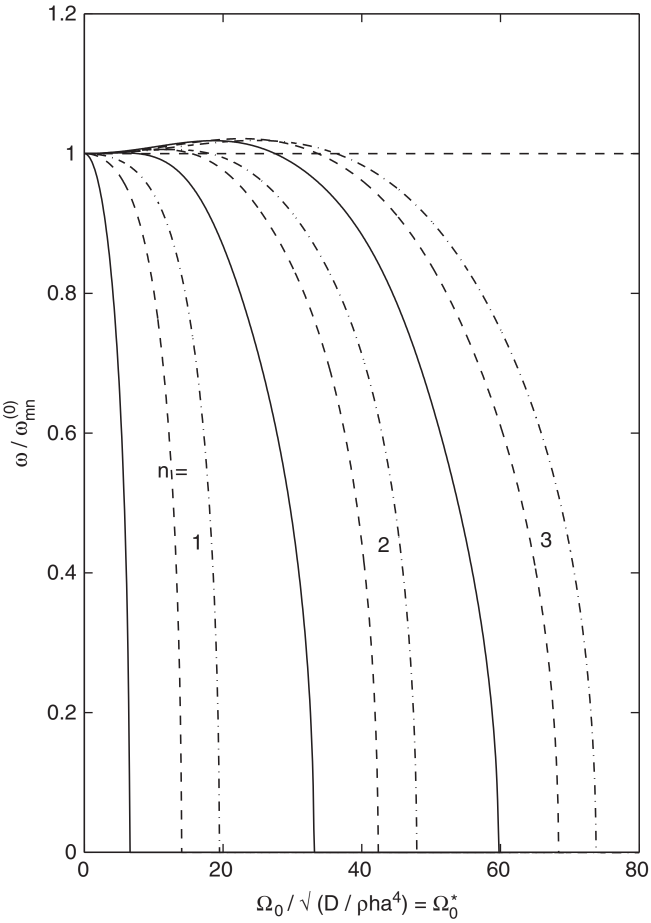

Fig. 5. Vibration frequency ratio $\omega_{mn}/\omega^{(0)}_{mn}$ as a function of the speed of spin Ω_0 for simply supported plate for $m = 0$, $m = 1$ (dashed line), $m = 2$ (dash-dot line) and $n = 1, 2, 3$, ($\bar{\nu} = 0.3$, $N = 30$).

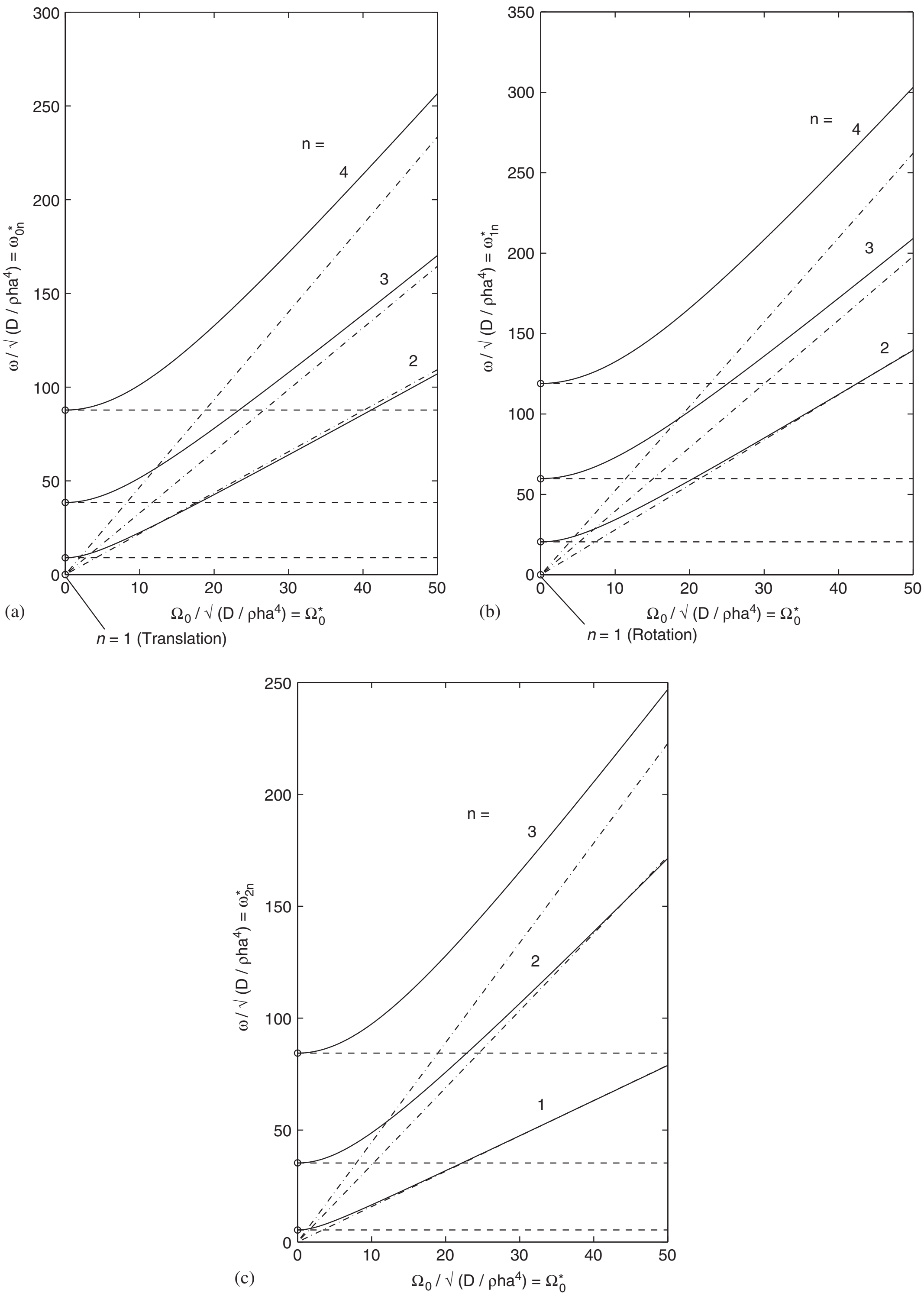

Fig. 6. Vibration frequency ω^*_{mn}: (a) axisymmetric ($m = 0$) for $n = 2, 3, 4$, (b) asymmetric for $m = 1$, $n = 2, 3, 4$ and (c) $m = 2$, $n = 1, 2, 3$ as a function of the speed of spin Ω_0 for free plate ($\bar{\nu} = 0.3$, $N = 30$).

relatively large speed of spin Ω_0. These results exhibiting at $\Omega_0 = 0$ the magnitude $\omega = 0$ and at $\Omega_0^* = 50$ the determined slope, is represented as the dash-dotted straight lines (-.-.). It shows that flexural rigidity $D \neq 0$ exhibits large deviations of the natural frequencies, depending on the modal number. With increasing radial mode number n the deviations increase. It also shows that deviations of the natural frequencies are pronounced for smaller speeds of spin Ω_0^*. For the asymmetric mode $m = 1$ the results are presented in Fig. 6b. The natural frequency of the spinning free plate is presented for the speed of spin in the range $0 \leqslant \Omega_0^* \leqslant 50$. We notice an increase of the natural frequency with increasing speed of spin. The fundamental mode $m = 1$, $n = 2$ starts for $\Omega_0^* = 0$ at $\omega_{12}^* = 20.47$ and reaches at $\Omega_0^* = 50$ the value $\omega_{12}^* \approx 139.6$, while the second mode

Table 2
Change of magnitude of modal frequency in the range $\Omega_0^* = 0$ to 50

$n\backslash m$	0	1	2
1	–	–	14.7
2	11.9	6.8	4.9
3	4.4	3.5	2.9
4	2.9	2.6	

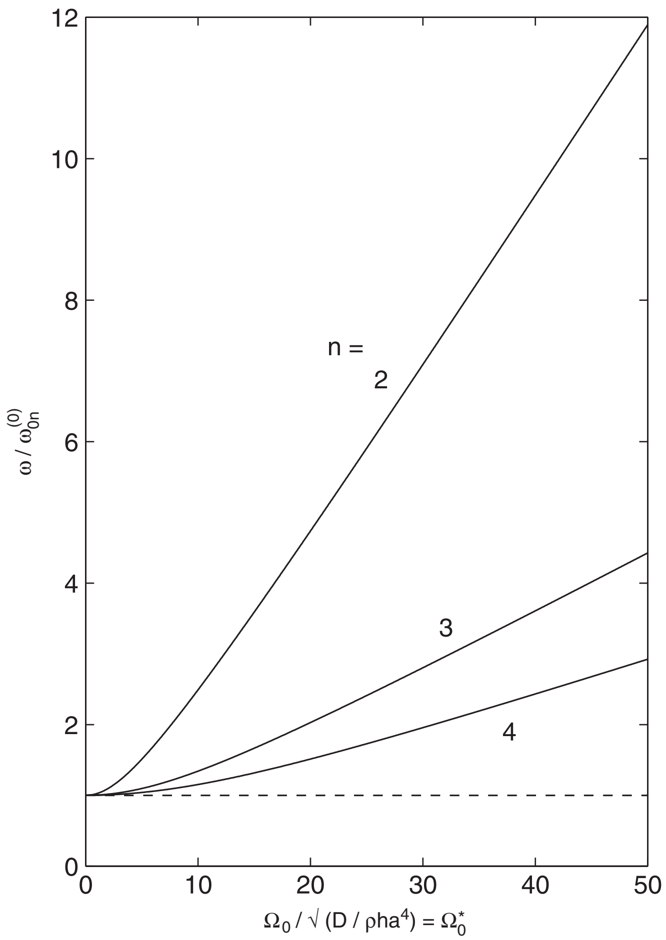

Fig. 7. Axisymmetric vibration frequency ratio $\omega_{mn}/\omega_{mn}^{(0)}$ as a function of the speed of spin Ω_0 for free plate for $m = 0$, $n = 2$, 3, 4, ($\bar{\nu} = 0.3$, $N = 30$).

frequency increases in this range from $\omega^*_{13} = 59.8$ to 209.1 and the third mode ($m = 1$, $n = 4$) frequency from $\omega^*_{14} = 118.96$ to 303.1. The difference between the natural frequencies at a large magnitude Ω^*_0 to that of a non-spinning plate is largest in the fundamental modes n and is presented in Table 2.

It may be noticed (see Fig. 7) that the first axisymmetric mode ($m = 0$, $n = 2$) exhibits in the considered range $0 \leqslant \Omega^*_0 \leqslant 50$ the largest increase. It increases its natural frequency nearly 11.9-fold. With the higher modes the increase diminishes as the radial mode number n increases. For the mode $m = 0$, $n = 3$ the natural frequency increases to a magnitude ($\omega/\omega^{(0)}_{03}$) 4.4-times of that of the non-spinning free plate. Finally, we notice that the natural frequency of the mode $m = 0$, $n = 4$ at $\Omega^*_0 = 50$ is only 2.9 times that of the non-spinning plate. In Fig. 8 we represent the results for natural frequencies $m = 0$, $n = 2$, 3, 4 and $m = 1$, 2, $n = 1$, 2, 3 of the spinning *guided* circular plate. For increasing speed of spin the natural frequency $\omega/\omega^{(0)}_{02}$ decreases and reaches at $\Omega^*_0 \approx 22.7$ the buckling point of the plate. For the asymmetric mode $m = 1$ and $n = 1$ the natural frequency ratio shows with increasing speed of spin a natural frequency which is always smaller than that of the non-spinning guided plate. Buckling occurs at about $\Omega^*_{0,\mathrm{buckl}} = 5.3$. More detailed representation may be found in Ref. [23].

The accuracy of the results could be improved by choosing in Eq. (29) the magnitude $n = N$, i.e. the amount of homogeneous equations to be evaluated. If we chose the number of equations $N = 10$ thus solving a vanishing determinant of degree 10, the lower natural frequencies are for any engineering purposed already quite adequate. An increase of N to 20 and 30 exhibits hardly any difference any more to these results for the

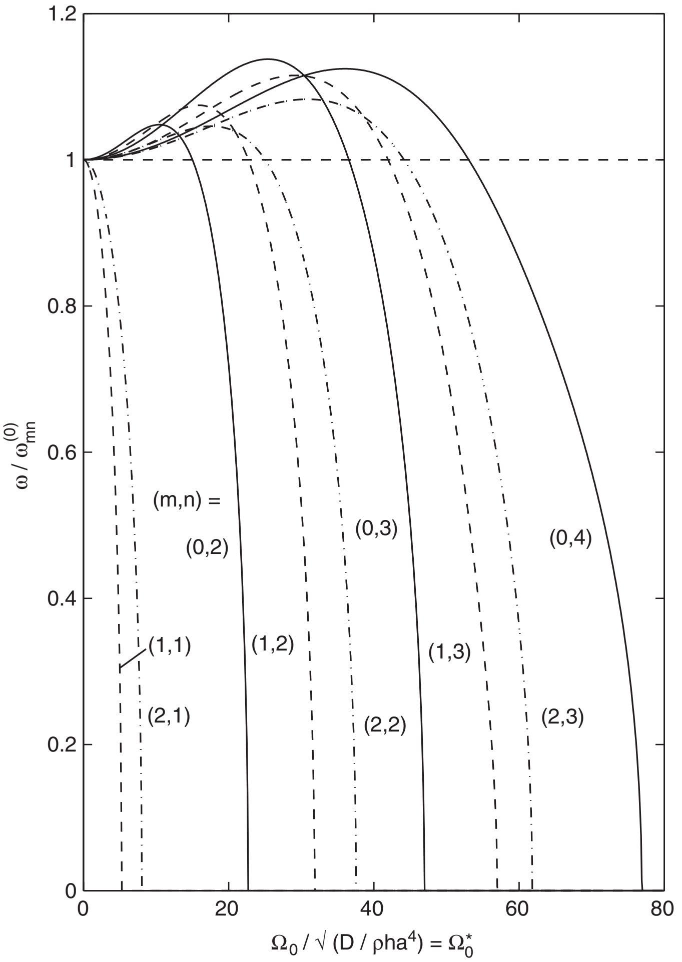

Fig. 8. Vibration frequency ratio $\omega_{mn}/\omega^{(0)}_{mn}$ as a function of the speed of spin Ω_0 for guided plate for $m = 0$, $n = 2$, 3, 4, $m = 1$ (dashed line), $m = 2$ (dash-dot line) and $n = 1$, 2, 3, ($\bar{v} = 0.3$, $N = 30$).

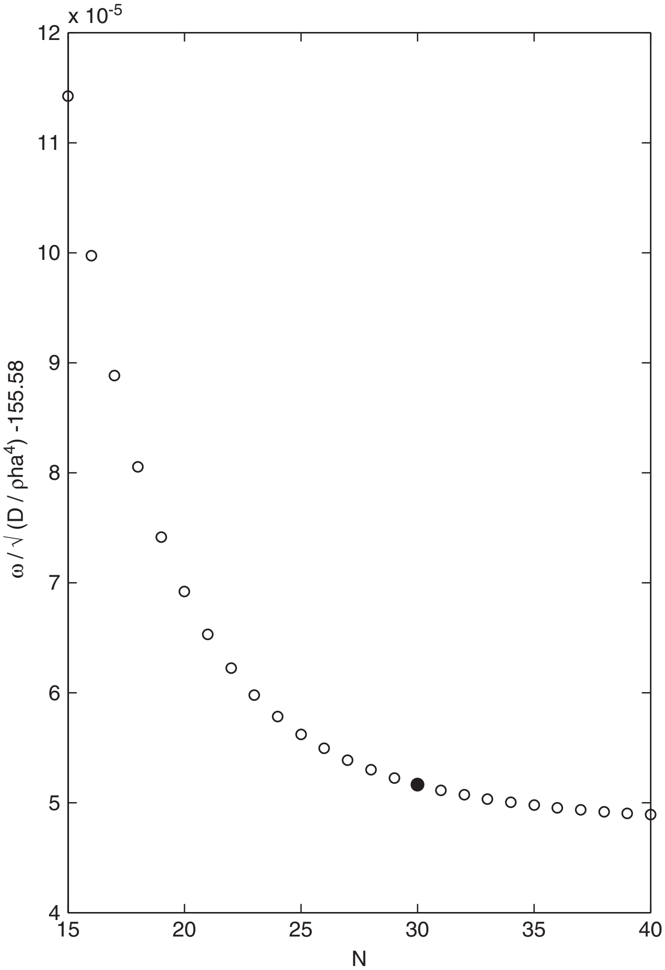

Fig. 9. Accuracy of results at various speeds of spin Ω_0 and orders N of the system of equations (• used for all previous numerical evaluations), ($\bar{\nu} = 0.3$, $\Omega_0^* = 10$, $m = 2$, $n = 3$).

modes $m = 0, 1, 2$ and $n = 1, 2$. Only for the higher mode $n = 3$ a small difference appears between $N = 10, 20$ and 30. For all numerical evaluations we restrict the truncation number N to $N = 30$. Fig. 9 exhibits the magnitude of the natural frequencies ω_{23}^* for various Ω_0^*-values, i.e. $\Omega_0^* = 10$ for the range of truncation numbers $15 \leqslant N \leqslant 40$. We notice that the magnitude of ω_{23}^* exhibits the same value 155.580 in the three-digit writing, expressing that there is up to three digits no change any more. The point • indicates the value $N = 30$, which has been used in the above computations. The results exhibit that for smaller speed of spin Ω_0^* the difference of the highest mode ($m = 2$, $n = 3$) considered in our investigation is very small, yielding a good acceptance for engineering purposes. This indicates that $N = 30$ is more than adequate.

Appendix

In the following we investigate the vibration and bucking behaviour of a non-rotating circular plate of constant thickness under the action of an in-plane force $\overline{N}$, to show the quality of the here presented method. The partial differential equation, describing this system is given by

$$D\nabla^4 w - \overline{N}\nabla^2 w + \varrho h w_{tt} = 0. \tag{40}$$

Using the same notations as in the above treatment

$$\xi = \frac{r}{a}, \quad w = e^{i\omega t} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} W_{mn}(\xi) \cos m\varphi, \tag{41}$$

we obtain ($W_{mn} = W$)

$$E_m[W;\xi] = W^{(IV)} + \frac{2}{\xi} W''' - \frac{2m^2+1}{\xi^2} W'' + \frac{2m^2+1}{\xi^3} W' - \frac{m^2(4-m^2)}{\xi^4} W - \frac{\varrho h a^4}{D} \omega^2 W - \frac{\overline{N} a^2}{D} \left[W'' + \frac{1}{\xi} W' + \frac{m^2}{\xi^2} W \right] = 0.$$

$\overline{N} > 0$ means tension in the plate, while $\overline{N} < 0$ describes the compression in the plate. We just investigate here only the case of a clamped plate. With the auxiliary differential equation we obtain

$$(\lambda^4 - \omega^{*2}) W - \frac{\overline{N} a^2}{D} \left[W'' + \frac{1}{\xi} W' - \frac{m^2}{\xi^2} W \right] = 0, \quad \omega^{*2} = \frac{\varrho a^4 h}{D} \omega^2 \tag{42}$$

and with the Ritz–Galerkin condition we obtain the system of equations

$$\int_0^1 \sum_{n=1}^{\infty} y A_{mn} \left\{ (\lambda^2 - \omega^{*2}) \xi F_n F_v - \Phi \left[\xi F_n'' F_v + F_n' F_v - \frac{m^2}{\xi} F_n F_v \right] \right\} d\xi = 0, \quad m = 0, 1, 2, \dots . \tag{43}$$

To exhibit the quality of the above-presented method, some numerical results are presented for a non-spinning clamped plate with the in-plane force $\overline{N}$. In Fig. 10a, we present the investigated results of the natural frequency ω_{0n}^{*} and in Fig. 10b the ratio $\omega/\omega_{mn}^{(0)}$ ($m = 0, 1, 2, n = 1, 2, 3$), of the non-spinning plate under the action of the in-plane force $\Phi = \overline{N} a^2/D$. We notice instability in the region of compression ($\Phi < 0$) and increase of the natural frequency for tension ($\Phi > 0$). The results also agree with those presented in Leissa [22] and show that the here presented method is quite adequate for the determination of the natural frequencies of spinning circular plates. The value $\omega/\omega_{mn}^{(0)}$ represents the frequency ratio, in which $\omega_{mn}^{(0)}$ is the natural frequency of the plate without in-plane forces (Fig. 10b) ($\overline{N} = 0$). It may be noticed that we used the same number of equations, as expressed by $N = 30$. The accuracy is very good, as we may find from the results given in Leissa [22]. More detailed representation may be found in Ref. [23].

The above method may also be applied to plates of annular geometry [25,26]. Assuming the same assumption that for small deflections of the plate the stress relation is unaffected during transverse vibrations, the planar stress relations may be obtained with the above Eqs. (1)–(3) yielding with Eqs. (7) and (8) the two coupled Euler equations (7) which exhibit for the radial and angular stress

$$\sigma_r = A + \frac{B}{r^2} - \frac{3+\overline{v}}{8} \varrho \Omega_0^2 r^2, \quad \sigma_\varphi = A - \frac{B}{r^2} - \frac{1+3\overline{v}}{8} \varrho \Omega_0^2 r^2 \tag{44}$$

and the radial displacement

$$\overline{u} = A - \frac{B}{r^2} - \frac{1+\overline{v}}{8} \varrho \Omega_0^2 r^2, \tag{45}$$

where A and B are integration constants to be determined from the boundary conditions at $r = a$ and b, the radius of inner boundary. There will be 16 different boundary combinations. Observing the abbreviations c (clamped), ss (simply supported), f (free), and g (guided), we distinguish the combinations: c–c, c–ss, c–f, c–g, ss–c, f–c, g–c, ss–ss, ss–g, g–ss, g–g, f–g, f–ss, ss–f, g–f and f–f. Depending on the conditions considered we are able to determine the integration constants A and B, where we have to observe for the cases the vanishing of two of the values σ_r, σ_φ or $\overline{u}$ from Section 2 at the boundaries $r = a, b$. Introducing these results for the appropriate boundaries of the annular plate into Eq. (16)—observing the inclusion of the Bessel and modified Bessel functions of the second kind Y_m and K_m, respectively—and applying the above procedure shall yield vibration frequencies and instabilities of any annular plate.

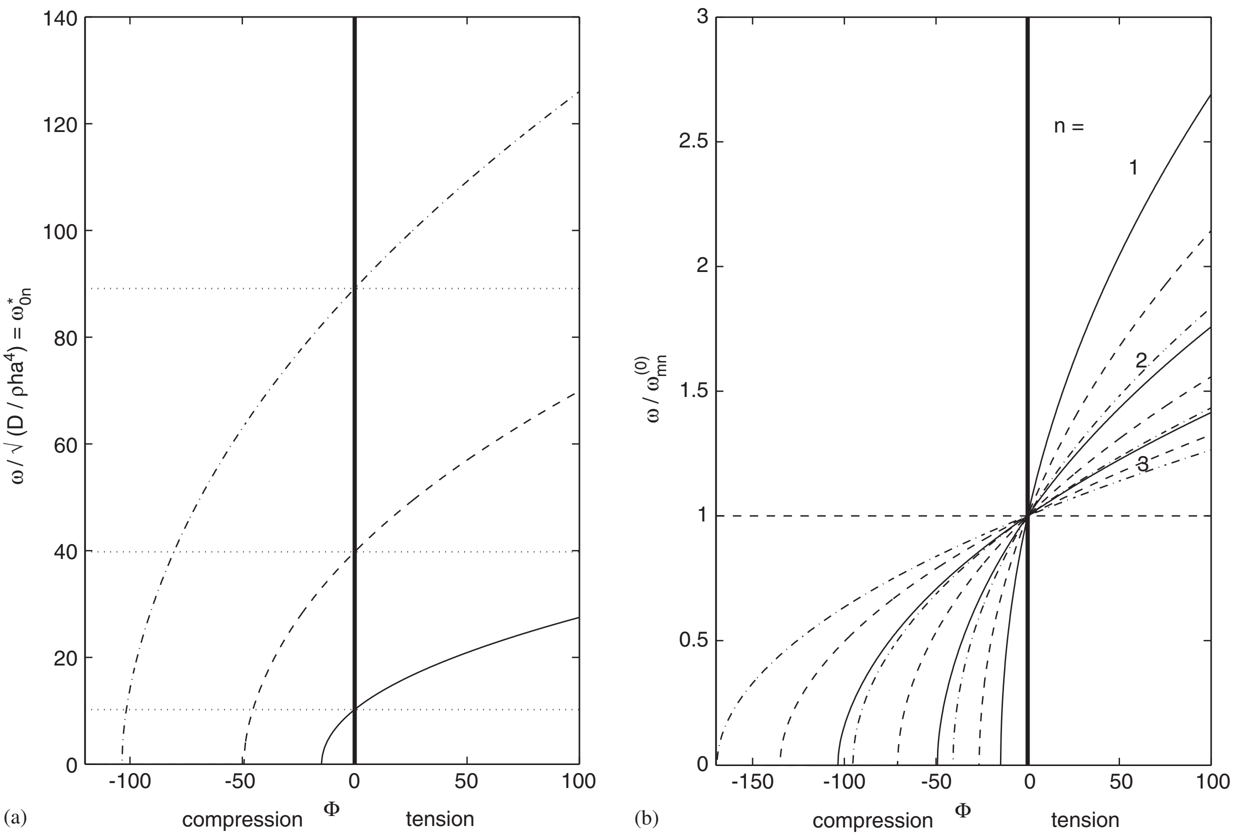

Fig. 10. Natural frequency (a) ω^*_{mn} for $m = 0$, $n = 1$, $n = 2$ (dashed line), $n = 3$ (dash-dot line) and (b) frequency ratio $\omega_{mn}/\omega^{(0)}_{mn}$, ($\omega^{(0)}_{mn}$ frequency without in-plane force) as function of the in-plane force $\Phi = \overline{N}a^2/D$ for $m = 0$, $m = 1$ (dashed line), $m = 2$ (dash-dot line) and $n = 1, 2, 3$ of the non-spinning clamped plate ($\overline{\nu} = 0.3$, $N = 30$).

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