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Second-order eigensensitivity analysis of asymmetric damped systems using Nelson's method

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Abstract

First-order eigensensitivity analysis using Nelson's method is first reviewed. Then, Nelson's approach is extended to the computation of the second-order derivatives of the eigenvalues and eigenvectors of symmetric and asymmetric damped systems. The computation of second-order derivatives may be required for large variation of design parameters and for some optimization algorithms. Nelson's method has the advantage of requiring only the knowledge of the eigenvector to be differentiated. Only systems with distinct eigenvalues are considered in this paper. Two numerical examples have been selected to illustrate the application and the utility of the derived expressions of the first- and second-order eigensensitivities. A four degree of freedom symmetric damped system is chosen for the first example whereas a finite element asymmetric 20 dof damped rotor model is considered in the second example.

1. Introduction

The study of vibration of structural and mechanical systems usually requires the computation of the eigensolutions. Owing to the environmental influences and real working conditions, variations in system parameters may occur, leading to small or sometimes large changes in the eigenvalues and eigenvectors. The derivatives of the eigensolutions can be used to gain insight into the extent of variation of the eigensolutions due to the change of design parameters. Furthermore, eigensensitivities are used in several other applications such as design modification, system optimization, damage detection and model updating.

One of the earliest methods for the computation of the first-order derivatives of the eigenvalues and eigenvectors is derived by Fox and Kapoor [1]. In this method, the derivative of any eigenvector with respect to a chosen design parameter is expressed as a linear combination of all eigenvectors. In 1976, Nelson introduced another approach for the exact computation of the eigenvector derivatives [2]. This method constitutes a significant contribution for the sensitivity computation since the calculation of the derivative of

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Nomenclature

c_i, d_i	scalars used to calculate the first-order eigenvectors derivatives
e_i, f_i	scalars used to calculate the second-order eigenvectors derivatives
$\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$	mass, damping, and stiffness matrices
N	number of generalized coordinates
p, q	system parameters
$\mathbf{q}(t)$	vector of generalized coordinates
t	time
$\mathbf{u}_i$, $\mathbf{v}_i$	right and left system eigenvectors
$\mathbf{x}_i$, $\mathbf{y}_i$	vectors used to calculate the first-order eigenvectors
$\mathbf{z}_i$, $\mathbf{w}_i$	vectors used to calculate the second-order eigenvectors
$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers
λ_i	system eigenvalues
ω_i	undamped natural frequencies
ζ_i	damping ratios
$\{o\}_{n_i}$	n_ith component of the ith vector
$(.)^{\mathrm{T}}$	transpose of (.)
$(.)^*$	complex conjugate of (.)
$(o)_{,p}$	partial derivatives of (.) with respect to p
$(o)_{,pq}$	partial derivatives of (.) with respect to p and q

an eigenvector requires only the knowledge of the eigenvector to be differentiated. Other approaches for sensitivity computation have also been derived later [3,4] by which eigenpair sensitivities are computed simultaneously for each mode. All these methods are limited to symmetric undamped systems. Sensitivity computation methods are developed later for damped and asymmetric systems. Lee et al. [5] have proposed a procedure for determining simultaneously the sensitivities of the eigenvalues and the eigenvectors of symmetric damped vibratory systems. Adhikari [6,7] derived a calculation method of the derivatives of complex modes based on the modal approach. Subsequent researches have also considered asymmetric systems. Friswell and Adhikari extended Nelson's method to computation of the first eigenderivatives of damped symmetric and asymmetric systems [8]. Recently, Choi et al. have also proposed an algebraic method for non-conservative asymmetric systems [9]. However, this method does not consider left eigenvectors and is limited to first-order derivatives.

Whereas a large body of literature has dealt with first-order sensitivity analysis, little attention has been devoted to second-order sensitivity. Nevertheless, the need for second-order sensitivities often arises. This occurs in optimization and reliability analysis algorithms, such as Newton's method and second-order reliability method (SORM) that require second-order information. In addition, second-order derivatives are needed for approximations where large variations in design parameters are considered or when natural frequencies are closely spaced [10,11]. In Ref. [12], a computation method for the second-order derivatives of eigensolutions of asymmetric damped systems has been derived using the modal method. This method, however, is only applicable when the system matrices are linear functions of the design variables and requires the knowledge of all eigenvectors for accurate eigenvector sensitivity computation. An algebraic approach has also been proposed in Ref. [13] for computing second and higher order eigensensitivities of damped systems with repeated eigenvalues; however, it is applicable to symmetric systems only.

In the present paper, Nelson's method for computing first-order eigensensitivities is reviewed and then extended to the calculation of second-order derivatives of both left and right eigenvectors for damped, symmetric as well as asymmetric systems with distinct eigenvalues.

Section 2 gives a summary of the calculation of eigenvalues and eigenvectors for asymmetric damped systems. Section 3 presents a brief review of Nelson's method for first-order eigenvector derivatives of symmetric and asymmetric damped systems. The derivation of second-order sensitivities of eigenvalues of asymmetric damped systems is detailed in Section 4. Next, in Sections 5 and 6, Nelson's method is extended to the second-order derivatives of eigenvectors for both symmetric and asymmetric systems. Finally, two numerical examples are considered in Section 7. The first example is a four degree of freedom symmetric damped system. The second is a 20 degree of freedom finite element model of a rotor which corresponds to a damped asymmetric system.

2. Computation of eigensolution of asymmetric damped systems

The equations of motion for the free vibration of a linear damped discrete system with N degrees of freedom can be expressed as

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = 0, \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K} \in \mathbb{R}^{N \times N}$ are mass, damping and stiffness matrices, respectively, $\mathbf{q}(t) \in \mathbb{R}^N$ is the vector of generalized coordinates and $t \in \mathbb{R}^+$ denotes time.

In this paper, the matrices $\mathbf{C}$ and $\mathbf{K}$ may be asymmetric and damping is assumed to be non-proportional. For this class of vibratory systems, the eigenvalues and their associated right and left eigenvectors are in general complex.

The eigenvalues of system (1) are the roots of the characteristic polynomial

$$\det(\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K}) = 0. \tag{2}$$

The order of the polynomial is $2N$ and the distinct eigenvalues λ_i for $i = 1, 2, \ldots, N$, assumed to be distinct, appear in complex conjugate pairs for under damped systems. The eigenvalues are usually sorted in increasing order of their imaginary parts, which correspond to the damped natural frequencies: $\lambda_1, \ldots, \lambda_N,\ \lambda_1^*, \ldots, \lambda_N^*$, where $(o)^*$ denotes complex conjugate of (o).

The right and left eigenvalue problems can be expressed, respectively, as follows:

$$\mathbf{F}_i\,\mathbf{u}_i = 0, \quad i = 1, \ldots, N, \tag{3}$$

$$\mathbf{v}_i^{\mathrm{T}}\mathbf{F}_i = 0^{\mathrm{T}}, \quad i = 1, \ldots, N, \tag{4}$$

where $\mathbf{F}_i = \left[\lambda_i^2\mathbf{M} + \lambda_i\mathbf{C} + \mathbf{K}\right]$, $\mathbf{u}_i \in \mathbb{C}^N$ is the ith right eigenvector, $\mathbf{v}_i \in \mathbb{C}^N$ is the left eigenvector and $(o)^{\mathrm{T}}$ denotes vector transpose.

If the mass matrix $\mathbf{M}$ is non-singular, then the eigenvectors are usually normalized so that

$$\mathbf{v}_i^{\mathrm{T}}\mathbf{G}_i\mathbf{u}_i = 1, \quad i = 1, \ldots, N, \tag{5}$$

where $\mathbf{G}_i = [2\lambda_i\mathbf{M} + \mathbf{C}]$.

However, the above normalization condition is not sufficient to ensure uniqueness of eigenvectors which is essential for existence of derivatives. For example, multiple solutions can be obtained by multiplying the left eigenvector by any scalar value and dividing the right eigenvector by the same scalar. To guarantee uniqueness of eigenvectors, an additional constraint, adopted by many authors [2,8,12], should be imposed which consists in setting one component to be equal in both left and right eigenvectors so that

$$\{\mathbf{u}_i\}_{n_i} = \{\mathbf{v}_i\}_{n_i}, \tag{6}$$

where $\{o\}_{n_i}$ denotes the n_ith component of the ith eigenvector. The index n_i is selected such that the product of the magnitudes of the corresponding components in the eigenvectors is the largest. Thus,

$$\left|\{\mathbf{u}_i\}_{n_i}\right|\left|\{\mathbf{v}_i\}_{n_i}\right| = \max_{k_i}\left(\left|\{\mathbf{u}_i\}_{k_i}\right|\left|\{\mathbf{v}_i\}_{k_i}\right|\right). \tag{7}$$

Using the above procedure, sorted complex eigenvalues and their associated normalized left and right eigenvector are computed.

In this paper, we are interested in finding the first- and second-order derivatives of the eigenvalues and eigenvectors. As a preliminary, Nelson's method for the computation of the first derivatives of the eigenvalues and eigenvectors [8] is reviewed, then it is extended to the calculation of the second derivatives of the eigensolutions. The expressions of the sensitivities are given essentially for asymmetric damped systems. The same expressions are also valid for the case of symmetric damped systems. The system matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are assumed to be dependent on one or several independent design parameters. For convenience, the following notation is used:

$$(o)_{,p} \equiv \frac{\partial(o)}{\partial p}, \ \ (o)_{,pq} \equiv \frac{\partial^2(o)}{\partial p \partial q},$$

where p and q denote, respectively, two design parameters.

3. First-order eigensensitivity computation

In this section, the expression of the first order eigenvalue derivative and the procedure for the computation of first-order derivatives of the left and right eigenvectors using Nelson's approach are presented.

Differentiation of Eq. (3) with respect to the design parameter p yields

$$\mathbf{F}_i\,\mathbf{u}_{i,p} = -\left(\lambda_{i,p}\mathbf{G}_i + \tilde{\mathbf{F}}_{i,p}\right)\mathbf{u}_i, \tag{8}$$

where $\tilde{\mathbf{F}}_{i,p} = [\lambda_i^2\mathbf{M}_{,p} + \lambda_i\mathbf{C}_{,p} + \mathbf{K}_{,p}]$ and $\mathbf{G}_i = [2\lambda_i\mathbf{M} + \mathbf{C}]$.

Premultiplying each side of Eq. (8) by $\mathbf{v}_i^{\mathrm{T}}$ and using the normalization Eq. (5), the ith eigenvalue first-order derivatives are given by

$$\lambda_{i,p} = -\mathbf{v}_i^{\mathrm{T}}\,\tilde{\mathbf{F}}_{i,p}\,\mathbf{u}_i. \tag{9}$$

The computation of $\lambda_{i,p}$ requires the knowledge of the eigenvalue λ_i, its associated left and right eigenvectors, $\mathbf{u}_i$ and $\mathbf{v}_i$, and the derivatives of the system matrices $\mathbf{M}_{,p}$, $\mathbf{C}_{,p}$ and $\mathbf{K}_{,p}$.

The right eigenvector derivative $\mathbf{u}_{i,p}$, however, cannot be computed directly using Eq. (8) since the matrix $\mathbf{F}_i$ is singular, of rank $(N-1)$ in the case of distinct eigenvalues.

For the computation of the left eigenvector derivative $\mathbf{v}_{i,p}$, an equation similar to Eq. (8) is required. Differentiating Eq. (4) with respect to p gives

$$\mathbf{v}_{i,p}^{\mathrm{T}}\,\mathbf{F}_i = -\mathbf{v}_i^{\mathrm{T}}\left(\lambda_{i,p}\mathbf{G}_i + \tilde{\mathbf{F}}_{i,p}\right). \tag{10}$$

Due to the singularity of $\mathbf{F}_i$, the left eigenvector derivative $\mathbf{v}_{i,p}$ also cannot be computed directly using the above equation.

An efficient way to overcome the singularity of $\mathbf{F}_i$ is to use the Nelson's approach. Nelson's method stabilizes the coefficient matrix $\mathbf{F}_i$ and solves the resultant well conditioned system in order to obtain a particular solution. It then adds a vector, as a homogenous solution, in the direction of the eigenvector, which defines the null space of the matrix, scaled to achieve the desired normalization. Applying this approach in the case of asymmetric damped systems, the right, respectively left, eigenvector derivative is expressed as a sum of a particular solution and a homogeneous solution:

$$\mathbf{u}_{i,p} = \mathbf{x}_i + c_i\mathbf{u}_i, \tag{11}$$

$$\mathbf{v}_{i,p} = \mathbf{y}_i + d_i\mathbf{v}_i, \tag{12}$$

where $\mathbf{x}_i$, $\mathbf{y}_i$ are the particular solutions and c_i, d_i are the coefficients of the homogenous solutions. However, these four quantities which need to be determined are not unique since any multiple of the eigenvector can be added to $\mathbf{x}_i$ (respectively, $\mathbf{y}_i$). In order to obtain unique solutions, the component number n_i of the eigenvector is first identified from Eq. (7), and then the n_ith elements of $\mathbf{x}_i$ and $\mathbf{y}_i$ are set to zero. Other elements of $\mathbf{x}_i$ and $\mathbf{y}_i$ could also be set to zero, however, this choice is most likely to produce a numerically well-conditioned problem according to Ref. [8].

Substituting Eqs. (11) and (12), respectively, into Eqs. (8) and (10), and using Eqs. (3) and (4) yields

$$\mathbf{F}_i\,\mathbf{x}_i = \mathbf{h}_i, \tag{13}$$

$$\mathbf{y}_i^{\mathrm{T}}\mathbf{F}_i = \mathbf{g}_i, \tag{14}$$

where $\mathbf{h}_i = -(\lambda_{i,p}\mathbf{G}_i + \tilde{\mathbf{F}}_{i,p})\mathbf{u}_i$ and $\mathbf{g}_i = -\mathbf{v}_i^{\mathrm{T}}(\lambda_{i,p}\mathbf{G}_i + \tilde{\mathbf{F}}_{i,p})$.

Setting the n_ith elements of $\mathbf{x}_i$ and $\mathbf{y}_i$ to zero and substituting the n_ith row and n_ith column of matrix $\mathbf{F}_i$ with the corresponding row and column of the identity matrix, the particular solutions $\mathbf{x}_i$ and $\mathbf{y}_i$ are obtained from the following two equivalent problems:

$$\bar{\mathbf{F}}_i \mathbf{x}_i = \bar{\mathbf{h}}_i, \tag{15}$$

$$\mathbf{y}_i^{\mathrm{T}} \bar{\mathbf{F}}_i = \bar{\mathbf{g}}_i, \tag{16}$$

where

$$\bar{\mathbf{F}}_i = \begin{array}{c} \left[\begin{array}{ccccccc} & & & 0 & & & \\ & \mathbf{F}_{i11} & & \vdots & & \mathbf{F}_{i13} & \\ & & & 0 & & & \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ & & & 0 & & & \\ & \mathbf{F}_{i31} & & \vdots & & \mathbf{F}_{i33} & \\ & & & 0 & & & \end{array}\right] n_i\text{th row}, \\ n_i\text{th column} \end{array} \quad \mathbf{x}_i = \begin{Bmatrix} x_{i1} \\ \vdots \\ x_{i(n_i-1)} \\ 0 \\ x_{i(n_i+1)} \\ \vdots \\ x_{iN} \end{Bmatrix} \text{ and } \bar{\mathbf{h}}_i = \begin{Bmatrix} h_{i1} \\ \vdots \\ h_{i(n_i-1)} \\ 0 \\ h_{i(n_i+1)} \\ \vdots \\ h_{iN} \end{Bmatrix}, \tag{17}$$

$\mathbf{y}_i$ and $\bar{\mathbf{g}}_i$ have, respectively, the same form as $\mathbf{x}_i$ and $\bar{\mathbf{h}}_i$.

Once the particular solutions are found, it remains to compute the scalar constants c_i and d_i of the homogenous solutions so that the derivatives of the eigenvectors are completely determined. Differentiating Eq. (5) and substituting expressions (11) and (12) yields

$$c_i + d_i = -\mathbf{v}_i^{\mathrm{T}} \mathbf{G}_i \mathbf{x}_i - \mathbf{y}_i^{\mathrm{T}} \mathbf{G}_i \mathbf{u}_i - \mathbf{v}_i^{\mathrm{T}} (2\mathbf{M}\lambda_{i,p} + \tilde{\mathbf{G}}_{i,p}) \mathbf{u}_i, \tag{18}$$

where $\tilde{\mathbf{G}}_{i,p} = \left[2\lambda_i \mathbf{M}_{,p} + \mathbf{C}_{,p}\right]$.

From Eq. (6), it can be deduced that

$$\left\{\mathbf{u}_{i,p}\right\}_{n_i} = \left\{\mathbf{v}_{i,p}\right\}_{n_i}. \tag{19}$$

Since the same component number n_i is chosen for the null component in both particular solutions $\mathbf{x}_i$ and $\mathbf{y}_i$, Eq. (19) yields

$$c_i = d_i. \tag{20}$$

Hence c_i and d_i can be calculated from Eq. (18), and the derivatives of the right and left eigenvectors $\mathbf{u}_{i,p}$ and $\mathbf{v}_{i,p}$ deduced from Eqs. (11) and (12).

4. Second-order derivatives of the eigenvalues

In this section, an expression is derived for the second-order derivative of the complex eigenvalues of asymmetric damped systems with respect to two design parameters p and q. Differentiating Eq. (8) with respect to q, premultiplying the result by $\mathbf{v}_i^{\mathrm{T}}$ and using Eqs. (3) and (4) yields

$$\begin{aligned} \lambda_{i,pq} = &-\frac{1}{\mathbf{v}_i^{\mathrm{T}} \mathbf{G}_i \mathbf{u}_i} \Big[\mathbf{v}_i^{\mathrm{T}} \Big(\tilde{\tilde{\mathbf{F}}}_{i,pq} + \lambda_{i,p} \tilde{\mathbf{G}}_{i,q} + \lambda_{i,q} \tilde{\mathbf{G}}_{i,p}\Big) \mathbf{u}_i + \mathbf{v}_i^{\mathrm{T}} \big(\tilde{\mathbf{F}}_{i,p} + \lambda_{i,p} \mathbf{G}_i\big) \mathbf{u}_{i,q} \\ &+ \mathbf{v}_i^{\mathrm{T}} \big(\tilde{\mathbf{F}}_{i,q} + \lambda_{i,q} \mathbf{G}_i\big) \mathbf{u}_{i,p} + 2\lambda_{i,p} \lambda_{i,q} \mathbf{v}_i^{\mathrm{T}} \mathbf{M} \, \mathbf{u}_i\Big], \end{aligned} \tag{21}$$

where

$$\begin{aligned} \mathbf{G}_i &= [2\lambda_i \mathbf{M} + \mathbf{C}], \\ \tilde{\mathbf{G}}_{i,\alpha} &= \left[2\lambda_i \mathbf{M}_{,\alpha} + \mathbf{C}_{,\alpha}\right], \quad \alpha \equiv p \text{ or } q, \\ \tilde{\mathbf{F}}_{i,\alpha} &= \left[\lambda_i^2 \mathbf{M}_{,\alpha} + \lambda_i \mathbf{C}_{,\alpha} + \mathbf{K}_{,\alpha}\right], \text{ and} \\ \tilde{\tilde{\mathbf{F}}}_{i,pq} &= \left[\lambda_i^2 \mathbf{M}_{,pq} + \lambda_i \mathbf{C}_{,pq} + \mathbf{K}_{,pq}\right]. \end{aligned} \tag{22}$$

The right-hand side of Eq. (21) is a function of the ith eigenvalue and eigenvectors and their first-order derivatives as well as the first and second derivatives of the system matrices. For symmetric damped systems, the left and right eigenvectors are equal, thus Eq. (21) is written as

$$\lambda_{i,pq} = -\frac{1}{\mathbf{u}_i^{\mathrm{T}}\mathbf{G}_i\mathbf{u}_i}\Big[\mathbf{u}_i^{\mathrm{T}}\Big(\tilde{\tilde{\mathbf{F}}}_{i,pq} + \lambda_{i,p}\tilde{\mathbf{G}}_{i,q} + \lambda_{i,q}\tilde{\mathbf{G}}_{i,p}\Big)\mathbf{u}_i + \mathbf{u}_i^{\mathrm{T}}\big(\mathbf{F}_{i,p} + \lambda_{i,p}\mathbf{G}_i\big)\mathbf{u}_{i,q}$$
$$+\mathbf{u}_i^{\mathrm{T}}\big(\mathbf{F}_{i,q} + \lambda_{i,q}\mathbf{G}_i\big)\mathbf{u}_{i,p} + 2\lambda_{i,p}\lambda_{i,q}\mathbf{u}_i^{\mathrm{T}}\mathbf{M}\,\mathbf{u}_i\Big]. \tag{23}$$

For the particular case of a double derivative, $\lambda_{i,pp}$ is expressed, respectively, in the case of asymmetric and symmetric damped systems as

$$\lambda_{i,pp} = -\frac{2}{\mathbf{v}_i^{\mathrm{T}}\mathbf{G}_i\mathbf{u}_i}\Big[0.5\mathbf{v}_i^{\mathrm{T}}\tilde{\tilde{\mathbf{F}}}_{i,pp}\mathbf{u}_i + \lambda_{i,p}\mathbf{v}_i^{\mathrm{T}}\tilde{\mathbf{G}}_{i,p}\mathbf{u}_i + \mathbf{v}_i^{\mathrm{T}}\big(\mathbf{F}_{i,p} + \lambda_{i,p}\mathbf{G}_i\big)\mathbf{u}_{i,p} + \lambda_{i,p}^2\mathbf{v}_i^{\mathrm{T}}\mathbf{M}\,\mathbf{u}_i\Big], \tag{24}$$

$$\lambda_{i,pp} = -\frac{2}{\mathbf{u}_i^{\mathrm{T}}\mathbf{G}_i\mathbf{u}_i}\Big[0.5\mathbf{u}_i^{\mathrm{T}}\tilde{\tilde{\mathbf{F}}}_{i,pp}\mathbf{u}_i + \lambda_{i,p}\mathbf{u}_i^{\mathrm{T}}\tilde{\mathbf{G}}_{i,p}\mathbf{u}_i + \mathbf{u}_i^{\mathrm{T}}\big(\mathbf{F}_{i,p} + \lambda_{i,p}\mathbf{G}_i\big)\mathbf{u}_{i,p} + \lambda_{i,p}^2\mathbf{u}_i^{\mathrm{T}}\mathbf{M}\,\mathbf{u}_i\Big]. \tag{25}$$

It should be noted that if the system matrices are linear functions of the design parameters, $\tilde{\tilde{\mathbf{F}}}_{i,pq}$ and $\tilde{\tilde{\mathbf{F}}}_{i,pp}$ are then equal to null matrices in the above equations.

The second-order derivatives of the eigenvalues of undamped asymmetric and symmetric systems can be deduced respectively from Eqs. (21) and (23). For an undamped asymmetric or symmetric system, $\mathbf{C} = 0$ and the eigenvalues are purely imaginary. The ith eigenvalue is expressed as $\lambda_i = j\omega_i$ where $j = \sqrt{-1}$ and $\omega_i \in \mathbb{R}$ is the ith undamped natural frequency. The right and left eigenvectors are real, i.e, $\mathbf{u}_i \in \mathbb{R}^N$, $\mathbf{v}_i \in \mathbb{R}^N$. The expressions of $\mathbf{G}_i$, $\tilde{\mathbf{G}}_{i,\alpha}, \tilde{\mathbf{F}}_{i,\alpha}$ and $\tilde{\tilde{\mathbf{F}}}_{i,pq}$ in Eq. (22) become

$$\begin{aligned}\mathbf{G}_i &= 2j\omega_i\mathbf{M},\\ \tilde{\mathbf{G}}_{i,\alpha} &= 2j\omega_i\mathbf{M}_{,\alpha},\ \ \alpha \equiv p \ \text{ or } \ q,\\ \tilde{\mathbf{F}}_{i,\alpha} &= \big[-\omega_i^2\mathbf{M}_{,\alpha} + \mathbf{K}_{,\alpha}\big], \ \text{ and}\\ \tilde{\tilde{\mathbf{F}}}_{i,pq} &= \big[-\omega_i^2\mathbf{M}_{,pq} + \mathbf{K}_{,pq}\big].\end{aligned} \tag{26}$$

Substituting the expressions given in Eq. (26) into Eq. (21) yields the second order derivative of the eigenvalues of asymmetric undamped systems expressed in the following frequently used form

$$\begin{aligned}\big(\omega_i^2\big)_{,pq} = \frac{1}{\mathbf{v}_i^{\mathrm{T}}\mathbf{M}\,\mathbf{u}_i}\big[&\mathbf{v}_i^{\mathrm{T}}\big(-\omega_i^2\mathbf{M}_{,pq} + \mathbf{K}_{,pq} - 2\omega_i\omega_{i,p}\mathbf{M}_{,q} - 2\omega_i\omega_{i,q}\mathbf{M}_{,p}\big)\mathbf{u}_i\\ &+ \mathbf{v}_i^{\mathrm{T}}\big(-\omega_i^2\mathbf{M}_{,p} + \mathbf{K}_{,p} - 2\omega_i\omega_{i,p}\mathbf{M}\big)\mathbf{u}_{i,q}\\ &+\mathbf{v}_i^{\mathrm{T}}\big(-\omega_i^2\mathbf{M}_{,q} + \mathbf{K}_{,q} - 2\omega_i\omega_{i,q}\mathbf{M}\big)\mathbf{u}_{i,p}\big].\end{aligned} \tag{27}$$

For symmetric undamped systems, $\mathbf{v}_i = \mathbf{u}_i$, thus, Eq. (27) can be simplified as follows:

$$\begin{aligned}\big(\omega_i^2\big)_{,pq} = \frac{1}{\mathbf{u}_i^{\mathrm{T}}\mathbf{M}\,\mathbf{u}_i}\big[&\mathbf{u}_i^{\mathrm{T}}\big(-\omega_i^2\mathbf{M}_{,pq} + \mathbf{K}_{,pq} - 2\omega_i\omega_{i,p}\mathbf{M}_{,q} - 2\omega_i\omega_{i,q}\mathbf{M}_{,p}\big)\mathbf{u}_i\\ &+ \mathbf{u}_i^{\mathrm{T}}\big(-\omega_i^2\mathbf{M}_{,p} + \mathbf{K}_{,p} - 2\omega_i\omega_{i,p}\mathbf{M}\big)\mathbf{u}_{i,q}\\ &+\mathbf{u}_i^{\mathrm{T}}\big(-\omega_i^2\mathbf{M}_{,q} + \mathbf{K}_{,q} - 2\omega_i\omega_{i,q}\mathbf{M}\big)\mathbf{u}_{i,p}\big].\end{aligned} \tag{28}$$

For the double derivative with respect to the same parameter, Eqs. (27) and (28) become

$$\begin{aligned}\big(\omega_i^2\big)_{,pp} = \frac{1}{\mathbf{v}_i^{\mathrm{T}}\mathbf{M}\,\mathbf{u}_i}\big[&\mathbf{v}_i^{\mathrm{T}}\big(-\omega_i^2\mathbf{M}_{,pp} + \mathbf{K}_{,pp} - 4\omega_i\omega_{i,p}\mathbf{M}_{,p}\big)\mathbf{u}_i\\ &+2\mathbf{v}_i^{\mathrm{T}}\big(-\omega_i^2\mathbf{M}_{,p} + \mathbf{K}_{,p} - 2\omega_i\omega_{i,p}\mathbf{M}\big)\mathbf{u}_{i,p}\big],\end{aligned} \tag{29}$$

$$\begin{aligned}\big(\omega_i^2\big)_{,pp} = \frac{1}{\mathbf{u}_i^{\mathrm{T}}\mathbf{M}\,\mathbf{u}_i}\big[&\mathbf{u}_i^{\mathrm{T}}\big(-\omega_i^2\mathbf{M}_{,pp} + \mathbf{K}_{,pp} - 4\omega_i\omega_{i,p}\mathbf{M}_{,p}\big)\mathbf{u}_i\\ &+2\mathbf{u}_i^{\mathrm{T}}\big(-\omega_i^2\mathbf{M}_{,p} + \mathbf{K}_{,p} - 2\omega_i\omega_{i,p}\mathbf{M}\big)\mathbf{u}_{i,p}\big].\end{aligned} \tag{30}$$

It should be noted that when the standard normalization condition for undamped systems is used, both products $\mathbf{v}_i^{\mathrm{T}}\mathbf{M}\,\mathbf{u}_i$, in Eqs. (27) and (29), and $\mathbf{u}_i^{\mathrm{T}}\mathbf{M}\,\mathbf{u}_i$, in Eqs. (28) and (30), are equal to unity. The second-order derivative of the ith undamped frequency $\omega_{i,pq}$ and $\omega_{i,pp}$ can be computed using, respectively, the following equations:

$$\omega_{i,pq} = \frac{\left(\omega_i^2\right)_{,pq} - 2\omega_{i,p}\omega_{i,q}}{2\omega_i}, \tag{31}$$

$$\omega_{i,pp} = \frac{\left(\omega_i^2\right)_{,pp} - 2\left(\omega_{i,p}\right)^2}{2\omega_i}. \tag{32}$$

5. Second-order derivatives of the eigenvectors of symmetric damped systems

In the following, the idea underlying Nelson's approach is applied to derive expressions for the second-order derivatives of the eigenvectors of symmetric damped systems with respect to two independent design parameters p and q.

For symmetric damped systems, the matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are symmetric and the left and right eigenvectors are equal. Differentiating Eq. (8) with respect to the design parameter q yields

$$\mathbf{F}_i\,\mathbf{u}_{i,pq} = \mathbf{k}_i, \tag{33}$$

where $\mathbf{k}_i$ is given by the following expression:

$$\begin{aligned}\mathbf{k}_i = -\Big[&\left(\tilde{\mathbf{F}}_{i,q} + \lambda_{i,q}\mathbf{G}_i\right)\mathbf{u}_{i,p} + \left(\tilde{\mathbf{F}}_{i,p} + \lambda_{i,p}\mathbf{G}_i\right)\mathbf{u}_{i,q} + \left(\tilde{\tilde{\mathbf{F}}}_{i,pq} + \lambda_{i,pq}\mathbf{G}_i\right)\mathbf{u}_i \\ &+\left(\tilde{\mathbf{G}}_{i,p}\lambda_{i,q} + \tilde{\mathbf{G}}_{i,q}\lambda_{i,p} + 2\lambda_{i,p}\lambda_{i,q}\mathbf{M}\right)\mathbf{u}_i\Big].\end{aligned} \tag{34}$$

As for the case of first-order derivatives, the second-order derivatives of the eigenvectors are written as a sum of a particular and a homogeneous solutions

$$\mathbf{u}_{i,pq} = \mathbf{z}_i + e_i\mathbf{u}_i. \tag{35}$$

The particular solution $\mathbf{z}_i$ can be obtained in a similar manner to that of the first derivatives $\mathbf{x}_i$, by solving the equivalent problem

$$\bar{\mathbf{F}}_i\mathbf{z}_i = \bar{\mathbf{k}}_i. \tag{36}$$

The matrix $\bar{\mathbf{F}}_i$ is given by Eq. (17) and the vector $\bar{\mathbf{k}}_i$ is obtained by zeroing the n_ith components of the vectors $\mathbf{k}_i$. The number n_i corresponds to the index of the element of the maximum magnitude in $\mathbf{u}_i$. The particular solution $\mathbf{z}_i$ can thus be computed from the non-singular system given by Eq. (36).

To compute the coefficient of the homogeneous solution e_i, Eq. (5) is differentiated twice, respectively, with respect to p and q after substituting $\mathbf{v}_i$ by $\mathbf{u}_i$. Furthermore, using the expression of the second order derivatives of the eigenvector, Eq. (35) and the normalization condition, Eq. (5), yields

$$\begin{aligned}e_i = -0.5\Big[&2\left(\mathbf{z}_i^{\mathrm{T}}\mathbf{G}_i\mathbf{u}_i\right) + \mathbf{u}_i^{\mathrm{T}}\left(\tilde{\tilde{\mathbf{G}}}_{i,pq} + \left(\tilde{\mathbf{G}}_{i,q} + 2\lambda_{i,q}\mathbf{M}\right) + 2\left(\lambda_{i,pq}\mathbf{M} + \lambda_{i,p}\mathbf{M}_{,q} + \lambda_{i,q}\mathbf{M}_{,p}\right)\right)\mathbf{u}_i \\ &+ \mathbf{u}_{i,p}^{\mathrm{T}}\left(\tilde{\mathbf{G}}_{i,q} + 2\lambda_{i,q}\mathbf{M}\right)\mathbf{u}_i + \mathbf{u}_{i,q}^{\mathrm{T}}\left(\tilde{\mathbf{G}}_{i,p} + 2\lambda_{i,p}\mathbf{M}\right)\mathbf{u}_i + \mathbf{u}_i^{\mathrm{T}}\left(\mathbf{G}_i + \tilde{\mathbf{G}}_{i,p} + 2\lambda_{i,p}\mathbf{M}\right)\mathbf{u}_{i,q} \\ &+\mathbf{u}_{i,q}^{\mathrm{T}}\mathbf{G}_i\,\mathbf{u}_{i,p}\Big],\end{aligned} \tag{37}$$

where $\tilde{\tilde{\mathbf{G}}}_{i,pq} = \left[2\lambda_i\mathbf{M}_{,pq} + \mathbf{C}_{,pq}\right]$.

Thus, using the particular solution $\mathbf{z}_i$ obtained from the solution of Eq. (36) and the scalar e_i, the second-order derivatives $\mathbf{u}_{i,pq}$ is computed using Eq. (35).

A particular case to consider is the second order derivatives of the eigenvectors with respect to the same design parameter ($p = q$). In this case, Eq. (33) becomes

$$\mathbf{F}_i\,\mathbf{u}_{i,pp} = \mathbf{k}_i \tag{38}$$

with the following expression of $\mathbf{k}_i$:

$$\mathbf{k}_i = -2\Big[\big(\tilde{\mathbf{F}}_{i,p} + \lambda_{i,p}\mathbf{G}_i\big)\mathbf{u}_{i,p} + 0.5\Big(\tilde{\tilde{\mathbf{F}}}_{i,pp} + \lambda_{i,pp}\mathbf{G}_i\Big)\mathbf{u}_i + \lambda_{i,p}\big(\tilde{\mathbf{G}}_{i,p} + \lambda_{i,p}\mathbf{M}\big)\mathbf{u}_i\Big] \tag{39}$$

and the coefficient of the homogenous solution is expressed as

$$\begin{aligned} e_i = &-0.5\Big[2\big(\mathbf{z}_i^{\mathrm{T}}\mathbf{G}_i\mathbf{u}_i\big) + \mathbf{u}_i^{\mathrm{T}}\Big(\tilde{\tilde{\mathbf{G}}}_{i,pp} + \big(\tilde{\mathbf{G}}_{i,p} + 2\lambda_{i,p}\mathbf{M}\big) + 2\big(\lambda_{i,pp}\mathbf{M} + 2\lambda_{i,p}\mathbf{M}_{,p}\big)\Big)\mathbf{u}_i \\ &+2\mathbf{u}_{i,p}^{\mathrm{T}}\big(\tilde{\mathbf{G}}_{i,p} + 2\lambda_{i,p}\mathbf{M}\big)\mathbf{u}_i + \mathbf{u}_i^{\mathrm{T}}\big(\mathbf{G}_i + \tilde{\mathbf{G}}_{i,p} + 2\lambda_{i,p}\mathbf{M}\big)\mathbf{u}_{i,p} + \mathbf{u}_{i,p}^{\mathrm{T}}\mathbf{G}_i\,\mathbf{u}_{i,p}\Big]. \end{aligned} \tag{40}$$

For symmetric undamped systems, $\mathbf{M} = \mathbf{M}^{\mathrm{T}}$, $\mathbf{K} = \mathbf{K}^{\mathrm{T}}$, $\mathbf{C} = 0$ the eigenvalues are purely imaginary so that $\lambda_i = j\omega_i$ and the eigenvectors are real. The expression of $\mathbf{k}_i$ becomes

$$\begin{aligned} \mathbf{k}_i = &-\big[\big[-\omega_i^2\mathbf{M}_{,q} + \mathbf{K}_{,q}\big]\mathbf{u}_{i,p} + \big[-\omega_i^2\mathbf{M}_{,p} + \mathbf{K}_{,p}\big]\mathbf{u}_{i,q} + \big[-\omega_i^2\mathbf{M}_{,pq} + \mathbf{K}_{,pq}\big]\mathbf{u}_i \\ &-2\omega_i\big[\omega_{i,q}\mathbf{M}\,\mathbf{u}_{i,p} + \omega_{i,p}\mathbf{M}\,\mathbf{u}_{i,q} + \big(\omega_{i,pq}\mathbf{M} + \omega_{i,q}\mathbf{M}_{,p} + \omega_{i,p}\mathbf{M}_{,q}\big)\mathbf{u}_i\big] \\ &-2\omega_{i,p}\omega_{i,q}\mathbf{M}\,\mathbf{u}_i\big]. \end{aligned} \tag{41}$$

The scalar e_i is computed by differentiating twice the usual mass orthogonality of the real undamped modes, which yields the following expression:

$$\begin{aligned} e_i = &-0.5\Big[\Big(2\mathbf{z}_i^{\mathrm{T}}\mathbf{M} + \mathbf{u}_{i,p}^{\mathrm{T}}\mathbf{M}_{,q} + \mathbf{u}_{i,q}^{\mathrm{T}}\mathbf{M}_{,p} + \mathbf{u}_i^{\mathrm{T}}\mathbf{M}_{,pq}\Big)\mathbf{u}_i \\ &+\Big(\mathbf{u}_{i,p}^{\mathrm{T}}\mathbf{M} + \mathbf{u}_i^{\mathrm{T}}\mathbf{M}_{,p}\Big)\mathbf{u}_{i,q} + \Big(\mathbf{u}_{i,q}^{\mathrm{T}}\mathbf{M} + \mathbf{u}_i^{\mathrm{T}}\mathbf{M}_{,q}\Big)\mathbf{u}_{i,p}\Big]. \end{aligned} \tag{42}$$

6. Second-order derivatives of the eigenvectors of asymmetric damped systems

Similarly to the case of first-order eigenvector derivatives, the second-order derivative of the left and right eigenvectors with respect to two independent design parameters p and q must be computed simultaneously. Differentiating Eqs. (8) and (10) with respect to the design parameter q yields

$$\mathbf{F}_i\,\mathbf{u}_{i,pq} = \mathbf{k}_i, \tag{43}$$

$$\mathbf{v}_{i,pq}^{\mathrm{T}}\,\mathbf{F}_i = \mathbf{l}_i, \tag{44}$$

where $\mathbf{k}_i$ and $\mathbf{l}_i$ are given by the following expressions:

$$\begin{aligned} \mathbf{k}_i = &-\Big[\big(\tilde{\mathbf{F}}_{i,q} + \lambda_{i,q}\mathbf{G}_i\big)\mathbf{u}_{i,p} + \big(\tilde{\mathbf{F}}_{i,p} + \lambda_{i,p}\mathbf{G}_i\big)\mathbf{u}_{i,q} + \Big(\tilde{\tilde{\mathbf{F}}}_{i,pq} + \lambda_{i,pq}\mathbf{G}_i\Big)\mathbf{u}_i \\ &+\big(\tilde{\mathbf{G}}_{i,p}\lambda_{i,q} + \tilde{\mathbf{G}}_{i,q}\lambda_{i,p} + 2\lambda_{i,p}\lambda_{i,q}\mathbf{M}\big)\mathbf{u}_i\Big], \end{aligned} \tag{45}$$

$$\begin{aligned} \mathbf{l}_i = &-\Big[\mathbf{v}_{i,p}^{\mathrm{T}}\big(\tilde{\mathbf{F}}_{i,q} + \lambda_{i,q}\mathbf{G}_i\big) + \mathbf{v}_{i,q}^{\mathrm{T}}\big(\tilde{\mathbf{F}}_{i,p} + \lambda_{i,p}\mathbf{G}_i\big) + \mathbf{v}_i^{\mathrm{T}}\Big(\tilde{\tilde{\mathbf{F}}}_{i,pq} + \lambda_{i,pq}\mathbf{G}_i\Big) \\ &+\mathbf{v}_i^{\mathrm{T}}\big(\tilde{\mathbf{G}}_{i,p}\lambda_{i,q} + \tilde{\mathbf{G}}_{i,q}\lambda_{i,p} + 2\lambda_{i,p}\lambda_{i,q}\mathbf{M}\big)\Big]. \end{aligned} \tag{46}$$

As for the case of first-order derivatives, the second-order derivatives of the right and left eigenvectors are written as a sum of a particular and a homogeneous solutions

$$\mathbf{u}_{i,pq} = \mathbf{z}_i + e_i\mathbf{u}_i, \tag{47}$$

$$\mathbf{v}_{i,pq} = \mathbf{w}_i + f_i\mathbf{v}_i. \tag{48}$$

The particular solutions $\mathbf{z}_i$ and $\mathbf{w}_i$ can be obtained in a similar manner to those of the first derivatives $\mathbf{x}_i$ and $\mathbf{y}_i$, by solving, respectively, the two equivalent problems

$$\bar{\mathbf{F}}_i\,\mathbf{z}_i = \bar{\mathbf{k}}_i, \tag{49}$$

$$\mathbf{w}_i^{\mathrm{T}}\bar{\mathbf{F}}_i = \bar{\mathbf{l}}_i. \tag{50}$$

The matrix $\bar{\mathbf{F}}_i$ is given by Eq. (17). The vectors $\bar{\mathbf{k}}_i$ and $\bar{\mathbf{l}}_i$ are obtained, respectively, by zeroing the n_ith components of the vectors $\bar{\mathbf{k}}_i$ and $\bar{\mathbf{l}}_i$. The number n_i is determined from Eq. (7). The particular solutions $\mathbf{z}_i$ and $\mathbf{w}_i$ can thus be computed from the non-singular systems given by Eqs. (49) and (50).

The coefficients of the homogeneous solutions e_i and f_i are computed by differentiating twice Eq. (5) with respect to p and q. Moreover, using the expressions of the second-order derivatives of the eigenvectors, Eqs. (47) and (48) and the normalization condition (5) yields

$$\begin{aligned} e_i + f_i = & -\Big[\mathbf{w}_i^{\mathrm{T}}\mathbf{G}_i\mathbf{u}_i + \mathbf{v}_i^{\mathrm{T}}\mathbf{G}_i\mathbf{z}_i + \mathbf{v}_i^{\mathrm{T}}\Big(\tilde{\tilde{\mathbf{G}}}_{i,pq} + 2\big(\lambda_{i,pq}\mathbf{M} + \lambda_{i,p}\mathbf{M}_{,q} + \lambda_{i,q}\mathbf{M}_{,p}\big)\Big)\mathbf{u}_i \\ & + \mathbf{v}_{i,p}^{\mathrm{T}}\big(\tilde{\mathbf{G}}_{i,q} + 2\lambda_{i,q}\mathbf{M}\big)\mathbf{u}_i + \mathbf{v}_{i,q}^{\mathrm{T}}\big(\tilde{\mathbf{G}}_{i,p} + 2\lambda_{i,p}\mathbf{M}\big)\mathbf{u}_i + \mathbf{v}_i^{\mathrm{T}}\big(\tilde{\mathbf{G}}_{i,q} + 2\lambda_{i,q}\mathbf{M}\big)\mathbf{u}_{i,p} \\ & + \mathbf{v}_i^{\mathrm{T}}\big(\mathbf{G}_i + \tilde{\mathbf{G}}_{i,p} + 2\lambda_{i,p}\mathbf{M}\big)\mathbf{u}_{i,q} + \mathbf{v}_{i,q}^{\mathrm{T}}\mathbf{G}_i\mathbf{u}_{i,p}\Big]. \end{aligned} \tag{51}$$

Since the n_ith elements of the right and left eigenvectors are equal, Eq. (6), so are the corresponding components of the second derivatives

$$\{\mathbf{u}_{i,pq}\}_{n_i} = \{\mathbf{v}_{i,pq}\}_{n_i}, \tag{52}$$

which leads to

$$e_i = f_i. \tag{53}$$

Eqs. (51) and (53) are sufficient to determine the constants e_i and f_i. Thus, using the particular solutions $\mathbf{z}_i$ and $\mathbf{w}_i$ obtained from the solution of Eqs. (49) and (50) and the constants e_i and f_i, the second-order derivatives $\mathbf{u}_{i,pq}$ and $\mathbf{v}_{i,pq}$ are computed using Eqs. (47) and (48).

One particular case of interest is the second-order derivatives of the eigenvectors with respect to the same design parameter ($p = q$). In this case, Eqs. (43) and (44) become

$$\mathbf{F}_i\,\mathbf{u}_{i,pp} = \mathbf{k}_i, \tag{54}$$

$$\mathbf{v}_{i,pp}^{\mathrm{T}}\,\mathbf{F}_i = \mathbf{l}_i, \tag{55}$$

where

$$\mathbf{k}_i = -2\Big[\big(\tilde{\mathbf{F}}_{i,p} + \lambda_{i,p}\mathbf{G}_i\big)\mathbf{u}_{i,p} + 0.5\Big(\tilde{\tilde{\mathbf{F}}}_{i,pp} + \lambda_{i,pp}\mathbf{G}_i\Big)\mathbf{u}_i + \Big(\tilde{\mathbf{G}}_{i,p}\lambda_{i,p} + \lambda_{i,p}^2\mathbf{M}\Big)\mathbf{u}_i\Big], \tag{56}$$

$$\mathbf{l}_i = -2\Big[\mathbf{v}_{i,p}^{\mathrm{T}}\big(\tilde{\mathbf{F}}_{i,p} + \lambda_{i,p}\mathbf{G}_i\big) + 0.5\mathbf{v}_i^{\mathrm{T}}\Big(\tilde{\tilde{\mathbf{F}}}_{i,pp} + \lambda_{i,pp}\mathbf{G}_i\Big) + \mathbf{v}_i^{\mathrm{T}}\Big(\tilde{\mathbf{G}}_{i,p}\lambda_{i,p} + \lambda_{i,p}^2\mathbf{M}\Big)\Big]. \tag{57}$$

The coefficients e_i and f_i of the homogenous solutions are identical and equal

$$\begin{aligned} e_i = f_i = & -0.5\Big[\mathbf{w}_i^{\mathrm{T}}\mathbf{G}_i\mathbf{u}_i + \mathbf{v}_i^{\mathrm{T}}\mathbf{G}_i\mathbf{z}_i + \mathbf{v}_i^{\mathrm{T}}\Big(\tilde{\tilde{\mathbf{G}}}_{i,pp} + 2\big(\lambda_{i,pp}\mathbf{M} + 2\lambda_{i,p}\mathbf{M}_{,p}\big)\Big)\mathbf{u}_i \\ & + 2\mathbf{v}_{i,p}^{\mathrm{T}}\big(\tilde{\mathbf{G}}_{i,p} + 2\lambda_{i,p}\mathbf{M}\big)\mathbf{u}_i + 2\mathbf{v}_i^{\mathrm{T}}\big(\tilde{\mathbf{G}}_{i,p} + 2\lambda_{i,p}\mathbf{M}\big)\mathbf{u}_{i,p} \\ & + \mathbf{v}_i^{\mathrm{T}}\mathbf{G}_i\mathbf{u}_{i,p} + \mathbf{v}_{i,p}^{\mathrm{T}}\mathbf{G}_i\mathbf{u}_{i,p}\Big]. \end{aligned} \tag{58}$$

For asymmetric undamped systems, $\mathbf{C} = 0$, the eigenvalues are purely imaginary. The right and left eigenvectors are real, i.e., $\mathbf{u}_i \in \mathbb{R}^N$, $\mathbf{v}_i \in \mathbb{R}^N$. To compute $\mathbf{u}_{i,pq}$ and $\mathbf{v}_{i,pq}$, Eqs. (43) and (44) are used, where $\mathbf{k}_i$ and $\mathbf{l}_i$ have the same expressions given by Eqs. (45) and (46). Using Eq. (26) yields

$$\begin{aligned} \mathbf{k}_i = & -\big[\big[-\omega_i^2\mathbf{M}_{,q} + \mathbf{K}_{,q}\big]\mathbf{u}_{i,p} + \big[-\omega_i^2\mathbf{M}_{,p} + \mathbf{K}_{,p}\big]\mathbf{u}_{i,q} + \big[-\omega_i^2\mathbf{M}_{,pq} + \mathbf{K}_{,pq}\big]\mathbf{u}_i \\ & - 2\omega_i\big[\omega_{i,q}\mathbf{M}\,\mathbf{u}_{i,p} + \omega_{i,p}\mathbf{M}\,\mathbf{u}_{i,q} + \big(\omega_{i,pq}\mathbf{M} + \omega_{i,q}\mathbf{M}_{,p} + \omega_{i,p}\mathbf{M}_{,q}\big)\mathbf{u}_i\big] \\ & - 2\omega_{i,p}\omega_{i,q}\mathbf{M}\,\mathbf{u}_i\big], \end{aligned} \tag{59}$$

Table 1
The algorithm for Nelson's method in the case of asymmetric damped systems

First order eigenderivatives
1. Compute $\lambda_{i,p} = -\mathbf{v}_i^T \tilde{\mathbf{F}}_{i,p} \mathbf{u}_i$.
2. Compute respectively $\mathbf{h}_i = -(\lambda_{i,p}\mathbf{G}_i + \tilde{\mathbf{F}}_{i,p})\mathbf{u}_i$ and $\mathbf{g}_i = -\mathbf{v}_i^T(\lambda_{i,p}\mathbf{G}_i + \tilde{\mathbf{F}}_{i,p})$.
3. Construct $\bar{\mathbf{F}}_i$ by zeroing out the n_i row and column of $\mathbf{F}_i$ and set the n_ith diagonal element to 1.
4. Construct $\bar{\mathbf{h}}_i$ and $\bar{\mathbf{g}}_i$ by zeroing the n_ith element respectively of $\mathbf{h}_i$ and $\mathbf{g}_i$.
5. Solve the two linear systems: $\bar{\mathbf{F}}_i \mathbf{x}_i = \bar{\mathbf{h}}_i$ and $\mathbf{y}_i^T \bar{\mathbf{F}}_i = \bar{\mathbf{g}}_i$.
6. Compute c_i and d_i using Eqs. (18) and (20).
7. Compute $\mathbf{u}_{i,p} = \mathbf{x}_i + c_i\mathbf{u}_i$ and $\mathbf{v}_{i,p} = \mathbf{y}_i + d_i\mathbf{v}_i$.
8. Repeat steps 1–7 for the design parameter q.

Second order eigenderivatives
9. Compute $\lambda_{i,pq}$ using Eq. (21).
10. Compute $\mathbf{k}_i$ and $\mathbf{l}_i$ using respectively Eqs. (45) and (46).
11. Construct $\bar{\mathbf{k}}_i$ and $\bar{\mathbf{l}}_i$ by zeroing the n_ith element respectively of $\mathbf{k}_i$ and $\mathbf{l}_i$.
12. Solve the two linear systems $\bar{\mathbf{F}}_i \mathbf{z}_i = \bar{\mathbf{k}}_i$ and $\mathbf{w}_i^T \bar{\mathbf{F}}_i = \bar{\mathbf{l}}_i$.
13. Compute e_i and f_i using Eqs. (51) and (53).
14. Compute $\mathbf{u}_{i,pq} = \mathbf{z}_i + e_i\mathbf{u}_i$ and $\mathbf{v}_{i,pq} = \mathbf{w}_i + f_i\mathbf{v}_i$.

$$\begin{aligned}\mathbf{l}_i = -\Big[&\mathbf{v}_{i,p}^{\mathrm{T}}\left[-\omega_i^2\mathbf{M}_{,q} + \mathbf{K}_{,q}\right] + \mathbf{v}_{i,q}^{\mathrm{T}}\left[-\omega_i^2\mathbf{M}_{,p} + \mathbf{K}_{,p}\right] + \mathbf{v}_i^{\mathrm{T}}\left[-\omega_i^2\mathbf{M}_{,pq} + \mathbf{K}_{,pq}\right]\\ &- 2\omega_i\Big[\omega_{i,q}\mathbf{v}_{i,p}^{\mathrm{T}}\mathbf{M} + \omega_{i,p}\mathbf{v}_{i,q}^{\mathrm{T}}\mathbf{M} + \mathbf{v}_i^{\mathrm{T}}\left(\omega_{i,pq}\mathbf{M} + \omega_{i,q}\mathbf{M}_{,p} + \omega_{i,p}\mathbf{M}_{,q}\right)\Big]\\ &-2\omega_{i,p}\omega_{i,q}\mathbf{v}_i^{\mathrm{T}}\mathbf{M}\Big]. \end{aligned} \tag{60}$$

To obtain the coefficients e_i and f_i of the homogenous solutions, the normalization condition $\mathbf{v}_i^{\mathrm{T}}\mathbf{M}\,\mathbf{u}_i = 1$, is differentiated twice and Eqs. (47) and (48) are used. In addition, the constraint, Eq. (52) is also used, yielding

$$\begin{aligned}e_i = f_i = -0.5\Big[&\left(\mathbf{w}_i^{\mathrm{T}}\mathbf{M} + \mathbf{v}_{i,p}^{\mathrm{T}}\mathbf{M}_{,q} + \mathbf{v}_{i,q}^{\mathrm{T}}\mathbf{M}_{,p} + \mathbf{v}_i^{\mathrm{T}}\mathbf{M}_{,pq}\right)\mathbf{u}_i\\ &+\left(\mathbf{v}_{i,p}^{\mathrm{T}}\mathbf{M} + \mathbf{v}_i^{\mathrm{T}}\mathbf{M}_{,p}\right)\mathbf{u}_{i,q} + \left(\mathbf{v}_{i,q}^{\mathrm{T}}\mathbf{M} + \mathbf{v}_i^{\mathrm{T}}\mathbf{M}_{,q}\right)\mathbf{u}_{i,p} + \mathbf{v}_i^{\mathrm{T}}\mathbf{M}\,\mathbf{z}_i\Big].\end{aligned} \tag{61}$$

Therefore, the expressions developed above constitute an extension of Nelson's method to the computation of the second-order derivatives of the eigenvectors for asymmetric damped systems with respect to two distinct or identical design parameters. It has also been shown that second-order derivatives of eigenvectors of undamped systems can be obtained as a particular case from the general derived expressions. The complete procedure, derived from Nelson's approach, for computing first- and second eigenderivatives in the case of asymmetric damped systems is summarized in Table 1.

7. Numerical examples

In order to apply the expressions derived in this paper and to demonstrate the importance of second order eigensolution sensitivities, two numerical examples are treated. In the first example, a 4-dof symmetric damped system is considered, while the second example consists of a rotor system described by a 20-dof finite element model.

7.1. A symmetric damped system example

The four degree of freedom symmetric damped system considered is shown in Fig. 1. The damping coefficient c_1 is chosen as a design parameter. The system matrices and their derivatives with respect to the

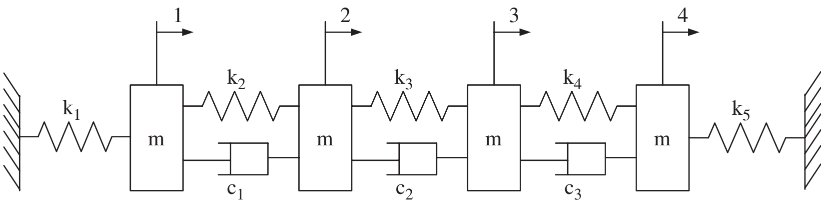

Fig. 1. The 4 dof discrete system, $m = 1\,\text{kg}$, $k_1 = k_3 = 1000\,\text{N/m}$, $k_2 = k_4 = 200\,\text{N/m}$, $c_1 = c_3 = 8\,\text{N s/m}$, $c_2 = 2\,\text{N s/m}$.

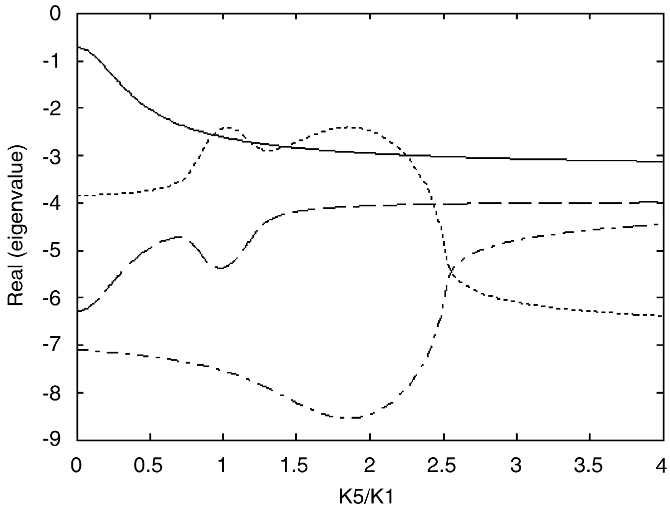

Fig. 2. The real parts of the eigenvalues for the 4-dof system. —— Eigenvalue 1, – – – eigenvalue 2, ----- eigenvalue 3, —-— eigenvalue 4.

design parameter c_1 are expressed as follows:

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2 + k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3 + k_4 & -k_4 \\ 0 & 0 & -k_4 & k_4 + k_5 \end{bmatrix},$$
$$\mathbf{C} = \begin{bmatrix} c_1 & -c_1 & 0 & 0 \\ -c_1 & c_1 + c_2 & -c_2 & 0 \\ 0 & -c_2 & c_2 + c_3 & -c_3 \\ 0 & 0 & -c_3 & c_3 \end{bmatrix} \quad \text{and} \quad \frac{\mathrm{d}\mathbf{C}}{\mathrm{d}c_1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{62}$$

The derivatives of the mass and the stiffness matrices with respect to the design parameter c_1 are equal to the null matrix. Figs. 2 and 3 show, respectively, the real and the imaginary parts of the eigenvalues as functions of the ratio k_5/k_1. It should be noted that small values of k_5 simulate flexible connection between the fourth mass and the ground, whereas large values correspond to a nearly rigid connection. From examination of Figs. 2 and 3, two veering regions can be observed in which two curves approach each other in the vicinities of $k_5/k_1 = 1.2$ and $k_5/k_1 = 2.5$. In the veering regions, two curves get close to each other and may cross or rapidly diverge. In both cases, rapid changes take place in the derivatives of the eigensolutions. It can also be observed from Fig. 2 that the system is stable since the real parts of all eigenvalues are negative. The units of both real and imaginary parts in Figs. 2 and 3 are (rad/s). It should be noted that the numerical values of the system parameters of the example have been chosen so that the system exhibits veering regions at which the second-order eigenderivatives are significant.

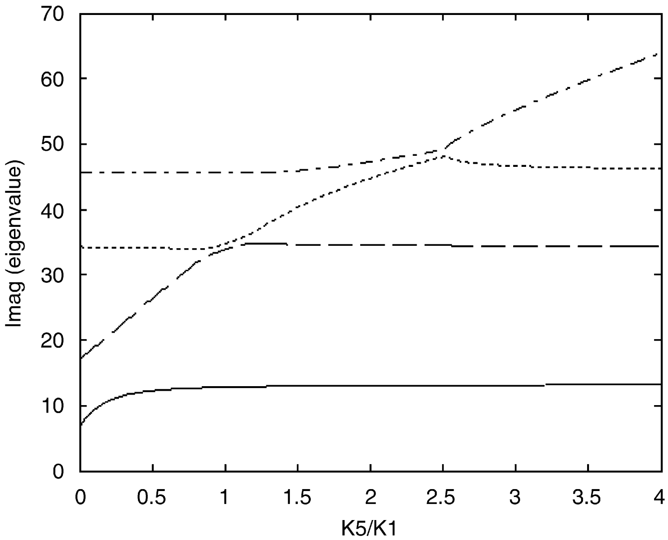

Fig. 3. The imaginary parts of the eigenvalues for the 4-dof system. —— Eigenvalue 1, – – – eigenvalue 2, ----- eigenvalue 3, —-— eigenvalue 4.

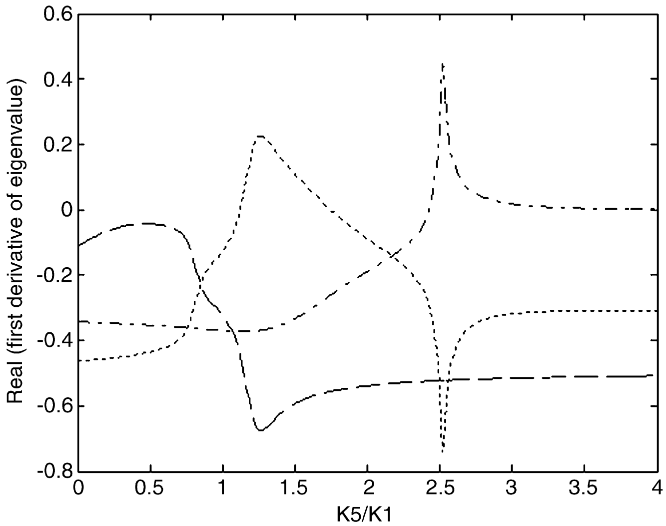

Fig. 4. The real parts of the first derivatives of the eigenvalues with respect to the parameter c_1 for the 4-dof system. – – – Eigenvalue 2, ----- eigenvalue 3, —-— eigenvalue 4.

Figs. 4 and 5 show, respectively, the real and the imaginary parts of the first derivative of the second, third and fourth eigenvalues. The derivative of the first eigenvalue is not plotted since it is roughly independent of the design parameter c_1 for all values of the parameter k_5. An important variation in the real and imaginary parts of the first-order derivatives of the third and fourth eigenvalues is observed in the vicinity of $k_5/k_1 = 2.5$. Furthermore, another significant variation in the imaginary parts of the second and third eigenvalue derivatives is also noted in the neighborhood of $k_5/k_1 = 1.2$ as shown in Fig. 5. These large variations indicate that these three eigenvalues are very sensitive to the design parameter c_1 in the vicinities of $k_5/k_1 = 1.2$ and $k_5/k_1 = 2.5$. In the vicinities of $k_5/k_1 = 2.5$, the second derivatives of the third and fourth eigenvalues are not negligible as illustrated by Figs. 6 and 7. In this region, both real and imaginary parts of the second derivatives have values of the same order of magnitude as the first derivatives. Finally, it should be noted that the units of

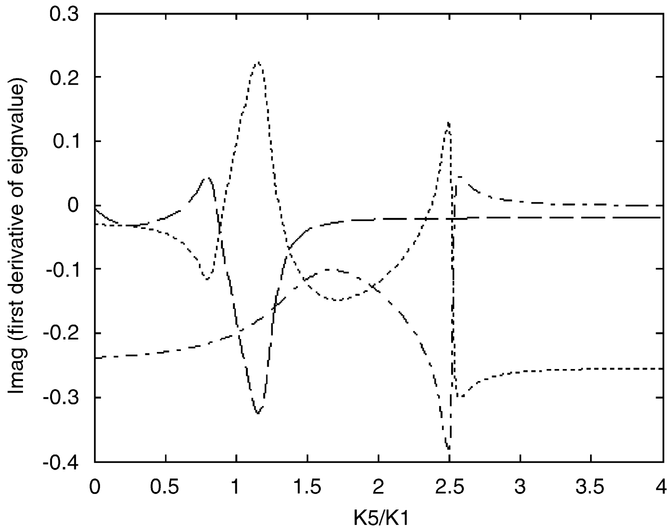

Fig. 5. The imaginary parts of the first derivatives of the eigenvalues with respect to the parameter c_1 for the 4-dof system. – – – Eigenvalue 2, ----- eigenvalue 3, —-— eigenvalue 4.

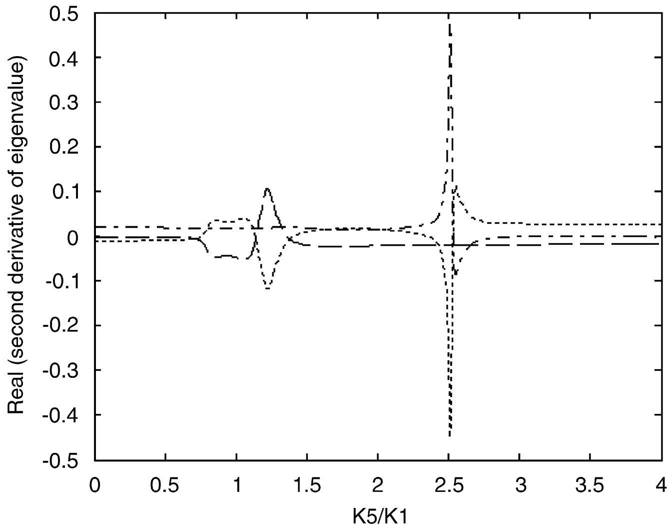

Fig. 6. The real parts of the second derivatives of the eigenvalues with respect to the parameter c_1 for the 4-dof system. – – – Eigenvalue 2, ----- eigenvalue 3, —-— eigenvalue 4.

the real and imaginary parts of the first derivatives are (rad/s) $\times$ (m/N s) whereas the units of the second derivatives are (rad/s) $\times$ (m/N s)^2.

7.2. An asymmetric damped system example

As an example of asymmetric damped systems, a two-disk rotor is considered. The rotor, depicted in Fig. 8, consists of two identical thin rigid disks and a flexible shaft supported at its ends by two bearings. The rotor system is modeled using the finite element method. The rotor shaft is discretized into 4 identical shaft elements using a Euler–Bernoulli beam model in which the gyroscopic and the rotary inertia effects are taken into account. The disks are fixed at the second and fourth nodes. The shaft has a diameter $d = 0.05\,\text{m}$, a length

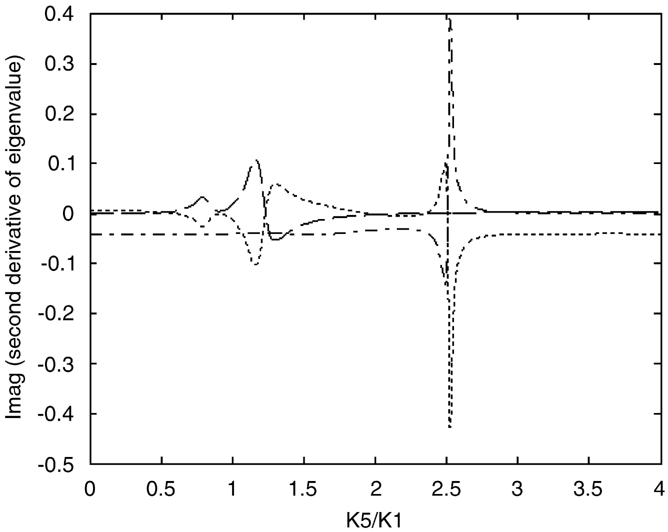

Fig. 7. The imaginary parts of the second derivatives of the eigenvalues with respect to the parameter c_1 for the 4-dof system. – – – Eigenvalue 2, ----- eigenvalue 3, —— eigenvalue 4.

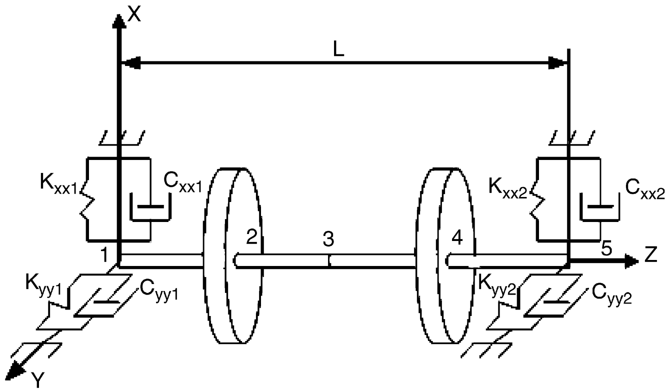

Fig. 8. A schematic of the 20-dof rotor example.

$L = 0.6\,\text{m}$ a Young's modulus $E = 2.1 \times 10^{11}\,\text{N/m}^2$, a mass density $7850\,\text{kg/m}^3$ and a Poisson's ratio $\nu = 0.3$. Each disk has an external diameter $D = 0.4\,\text{m}$ and a thickness $e = 20 \times 10^{-3}\,\text{m}$. Both disks have the same material as the shaft. The two bearings are modeled using springs and dashpots with coefficients defined in Fig. 8. The spring stiffness coefficients have the following numerical values:

$$K_{xx1} = 2.6 \times 10^6, \;\; K_{yy1} = 1.2 \times 10^6, \;\; K_{xx2} = 2.1 \times 10^6, \;\; K_{yy2} = 1.3 \times 10^6\,\text{N/m}.$$

All dashpots have the same damping coefficients $c = 10^3\,\text{N s/m}$. The coupling stiffness and damping coefficients of the bearings are assumed to be negligible. All shaft elements have the same length $l = 0.15\,\text{m}$. Thus, the system has 5 nodes. Each node has four degrees of freedom: the transverse displacements and the rotations in the XZ and YZ planes. Therefore, the model has 20 degrees of freedom, and the vector of degrees of freedom is partitioned as

$$\mathbf{q} = \begin{Bmatrix} \mathbf{q}_{XZ} \\ \mathbf{q}_{YZ} \end{Bmatrix}.$$

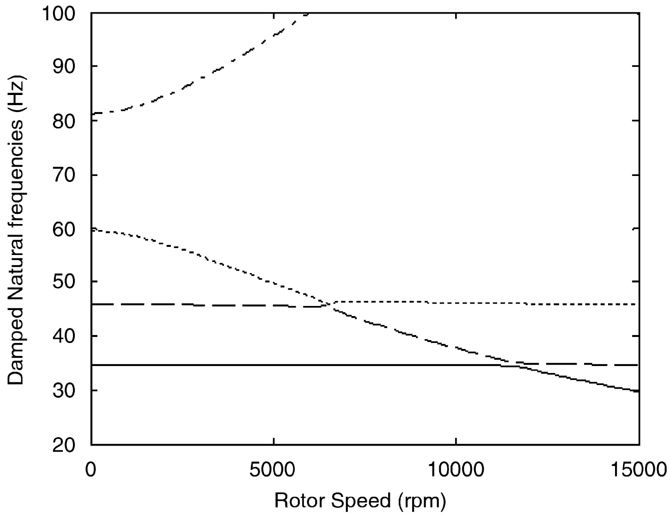

Fig. 9. The Campbell diagram for the rotor example. —— Eigenvalue 1, – – – eigenvalue 2, ----- eigenvalue 3, –·– eigenvalue 4.

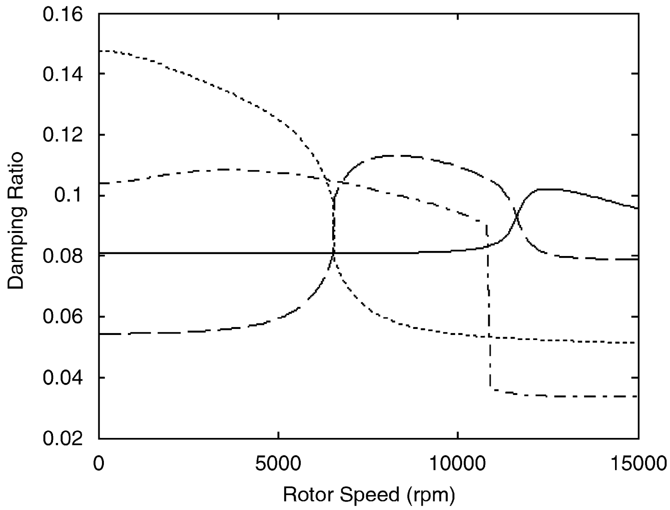

Fig. 10. The damping ratios of the first four eigenvalues of the rotor example. —— Eigenvalue 1, – – – eigenvalue 2, ----- eigenvalue 3, –·– eigenvalue 4.

The degrees of freedom are ordered as follows:

$$\mathbf{q}^{\mathrm{T}} = \left\{ x_1 \quad \theta_{y1} \quad x_2 \quad \theta_{y2} \quad \ldots \quad x_5 \quad \theta_{y5} \quad y_1 \quad \theta_{x1} \quad y_2 \quad \theta_{x2} \quad \ldots \quad y_5 \quad \theta_{x5} \right\}^{\mathrm{T}}. \tag{63}$$

The rotor model is therefore a damped asymmetric system, due to the gyroscopic effects of the disks and the shaft as well as the damping of the bearing dashpots. The Campbell diagram for the first four damped natural frequencies is shown in Fig. 9 for a rotor speed range from 0 to 15,000 rev/min. Fig. 9 shows also that the gyroscopic effect increases with the rotational speed especially for the fourth mode. Due to the asymmetric bearing stiffness, the frequencies are not equal at the lateral planes at zero speed, as shown in the Campbell diagram. Fig. 9 shows also two veering regions, where two natural frequencies become close, at about 6500 and 11700 rev/min. The variation of the damping ratio with the rotor speed for the first four modes is presented in Fig. 10. The damping ratio ζ_i is computed from the ith eigenvalue λ_i which is expressed in the

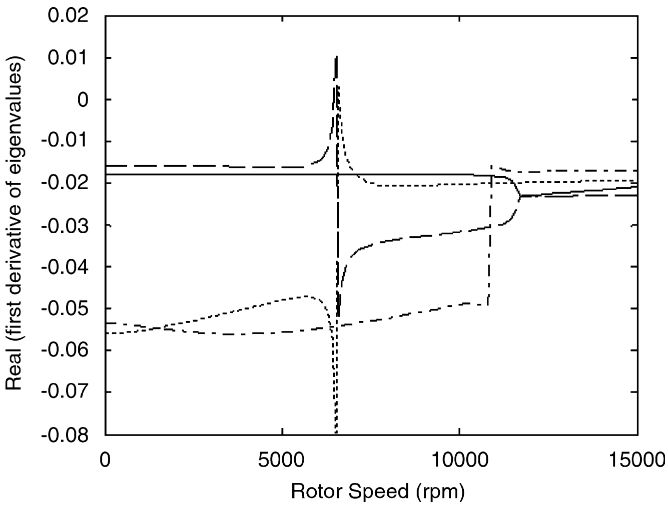

Fig. 11. The real parts of the first derivatives of the eigenvalues of the rotor example. —— Eigenvalue 1, – – – eigenvalue 2, ----- eigenvalue 3, —–— eigenvalue 4.

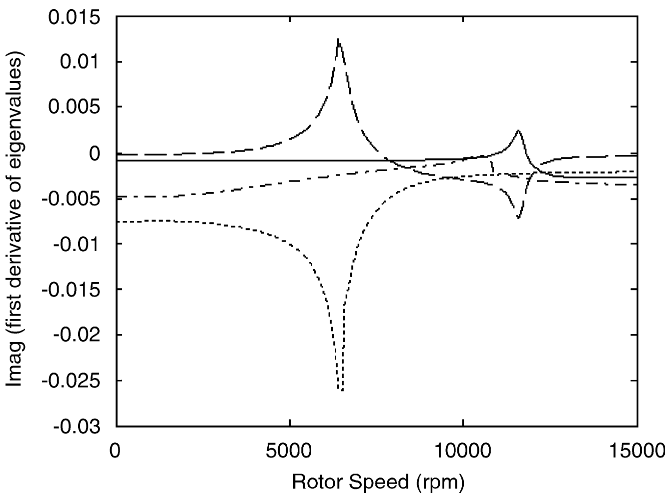

Fig. 12. The imaginary parts of the first derivatives of the eigenvalues of the rotor example. —— Eigenvalue 1, – – – eigenvalue 2, ----- eigenvalue 3, —–— eigenvalue 4.

following form:

$$\lambda_i = -\omega_{0i}\zeta_i + j\omega_{0i}\sqrt{1-\zeta_i^2} = -\omega_{0i}\zeta_i + j\omega_i, \tag{64}$$

where ω_{0i} and ω_i are, respectively, the undamped and damped natural frequency and $j = \sqrt{-1}$. Large variations of the damping ratios are also observed in the vicinity of the same two veering regions in Fig. 10. All the displayed damping ratios in Fig. 10 are positive; other computed damping ratios are also positive which indicate the stability of the system.

Since damping has an influence on real and imaginary parts of the eigenvalues, these two parts are plotted for the first and second derivative of the eigenvalues as a function of rotor speed in Figs. 11–14. The real and the imaginary parts of the first derivatives of the first four eigenvalues are presented respectively in Figs. 11 and 12. These two plots are obtained by programming directly Eq. (9). Note that the first derivatives have large values in the neighborhood of the first veering region, at about 6500 rev/min. The real and imaginary parts of the second-order derivative of the eigenvalues are given in Figs. 13 and 14. Eq. (24) is applied to

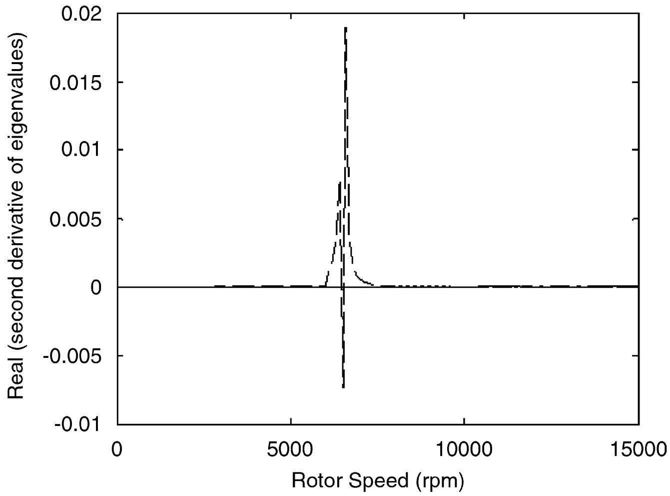

Fig. 13. The real parts of the second derivatives of the eigenvalues of the rotor example. —— Eigenvalue 1, – – – eigenvalue 2, ----- eigenvalue 3, —-— eigenvalue 4.

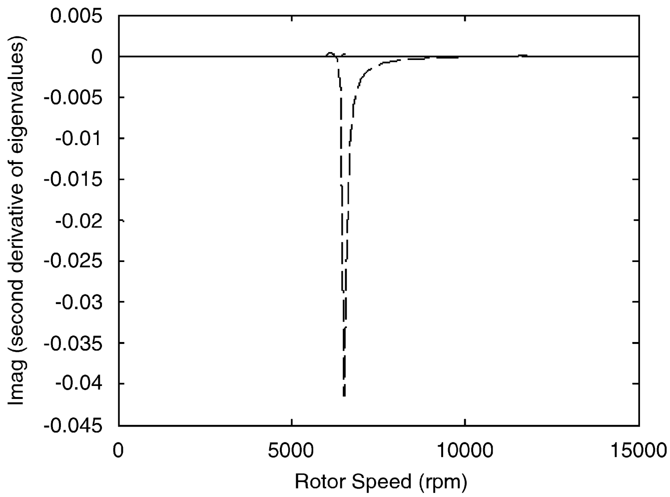

Fig. 14. The imaginary parts of the second derivatives of the eigenvalues of the rotor example. —— Eigenvalue 1, – – – eigenvalue 2, ----- eigenvalue 3, —-— eigenvalue 4.

Table 2
The first four eigenvalues and their first and second derivatives

Eigenvalues, λ_i	$\mathrm{d}\lambda_i/\mathrm{d}c$	$\mathrm{d}^2\lambda_i/\mathrm{d}c^2$
$-17.671+217.94i$	$-1{,}7771\times10^{-2}-8{,}7138\times10^{-4}i$	$-2{,}4760\times10^{-7}-8{,}8971\times10^{-7}\mathrm{i}$
$-23.017+287.50i$	$1{,}0679\times10^{-2}+1{,}1046\times10^{-2}i$	$-7{,}3550\times10^{-3}-4{,}1392\times10^{-2}i$
$-28.614+288.24i$	$-7{,}9569\times10^{-2}-2{,}6063\times10^{-2}i$	$-1{,}7995\times10^{-4}+1{,}8508\times10^{-4}i$
$-67.702+642.04i$	$-5{,}4224\times10^{-2}-2{,}4413\times10^{-3}i$	$-1{,}1826\times10^{-6}-4{,}7987\times10^{-6}i$

obtain these results. Both real and imaginary parts of the second derivative of the second eigenvalue are significant in the vicinity of the first veering region.

To demonstrate the calculation of the eigensolution derivatives at a particular rotating speed, a suitable rotor speed of 6500 rev/min from the first veering region is chosen. Table 2 shows the first four eigenvalues of the rotor system and their first and second derivatives with respect to the dashpot damping coefficient. It can

Table 3
The second right eigenvector and its first and second derivatives

dof	u_2	du_2/dc	d^2u_2/dc^2
1	$1{,}2582\times10^{-3}+7{,}8858\times10^{-3}i$	$3{,}9833\times10^{-3}-1{,}8984\times10^{-3}i$	$-1{,}1103\times10^{-2}-6{,}2666\times10^{-3}i$
2	$1{,}8602\times10^{-2}+1{,}4997\times10^{-3}i$	$-7{,}6850\times10^{-3}-8{,}0562\times10^{-3}i$	$-9{,}3955\times10^{-3}+3{,}0131\times10^{-2}i$
3	$4{,}0400\times10^{-3}+7{,}9337\times10^{-3}i$	$2{,}7382\times10^{-3}-3{,}0748\times10^{-3}i$	$-1{,}2283\times10^{-2}-1{,}5816\times10^{-3}i$
4	$1{,}8435\times10^{-2}-2{,}0184\times10^{-3}i$	$-9{,}5224\times10^{-3}-7{,}4205\times10^{-3}i$	$-4{,}8398\times10^{-3}+3{,}3420\times10^{-2}i$
…	…	…	…
17	$-3{,}2360\times10^{-3}-3{,}2815\times10^{-3}i$	$-7{,}5998\times10^{-4}+3{,}3321\times10^{-3}i$	$8{,}8887\times10^{-3}-3{,}0922\times10^{-3}i$
18	$-1{,}8620\times10^{-2}-1{,}9926\times10^{-2}i$	$-5{,}0321\times10^{-3}+1{,}9677\times10^{-2}i$	$5{,}3251\times10^{-2}-1{,}6972\times10^{-2}i$
19	$-5{,}9332\times10^{-3}-6{,}1122\times10^{-3}i$	$-1{,}4542\times10^{-3}+6{,}1604\times10^{-3}i$	$1{,}6500\times10^{-2}-5{,}5995\times10^{-3}i$
20	$-1{,}7656\times10^{-2}-1{,}8337\times10^{-2}i$	$-4{,}4233\times10^{-3}+1{,}8437\times10^{-2}i$	$4{,}9469\times10^{-2}-1{,}6584\times10^{-2}i$

Table 4
The second left eigenvector and its first and second derivatives

dof	v_2	dv_2/dc	d^2v_2/dc^2
1	$-1{,}2582\times10^{-3}-7{,}8858\times10^{-3}i$	$-3{,}9833\times10^{-3}+1{,}8984\times10^{-3}i$	$1{,}1103\times10^{-2}+6{,}2666\times10^{-3}i$
2	$-1{,}8602\times10^{-2}-1{,}4997\times10^{-3}i$	$7{,}6850\times10^{-3}+8{,}0562\times10^{-3}i$	$9{,}3955\times10^{-3}-3{,}0131\times10^{-2}i$
3	$-4{,}0400\times10^{-3}-7{,}9337\times10^{-3}i$	$-2{,}7382\times10^{-3}+3{,}0748\times10^{-3}i$	$1{,}2283\times10^{-2}+1{,}5816\times10^{-3}i$
4	$-1{,}8435\times10^{-2}+2{,}0184\times10^{-3}i$	$9{,}5224\times10^{-3}+7{,}4205\times10^{-3}i$	$4{,}8398\times10^{-3}-3{,}3420\times10^{-2}i$
…	…	…	…
17	$-3{,}2360\times10^{-3}-3{,}2815\times10^{-3}i$	$-7{,}5998\times10^{-4}+3{,}3321\times10^{-3}i$	$8{,}8887\times10^{-3}-3{,}0922\times10^{-3}i$
18	$-1{,}8620\times10^{-2}-1{,}9926\times10^{-2}i$	$-5{,}0321\times10^{-3}+1{,}9677\times10^{-2}i$	$5{,}3251\times10^{-2}-1{,}6972\times10^{-2}i$
19	$-5{,}9332\times10^{-3}-6{,}1122\times10^{-3}i$	$-1{,}4542\times10^{-3}+6{,}1604\times10^{-3}i$	$1{,}6500\times10^{-2}-5{,}5995\times10^{-3}i$
20	$-1{,}7656\times10^{-2}-1{,}8337\times10^{-2}i$	$-4{,}4233\times10^{-3}+1{,}8437\times10^{-2}i$	$4{,}9469\times10^{-2}-1{,}6584\times10^{-2}i$

be noted that the second derivative of the second eigenvalue is significant and therefore is not negligible at this rotor speed. Tables 3 and 4 present the first and second derivatives of some components of the second right, respectively, left eigenvector. It can also be noted that the components of the second derivative of these vectors are of the same order of magnitude, for both real and imaginary parts, as those of the first derivative. The efficiency of Nelson's method for the computation of second-order derivatives can be compared to the modal method given in reference [12]. To compute higher order eigenvector sensitivity, the modal method in its basic form requires the computation of all lower order eigensolutions and their associated derivatives. Nelson's method, however, can be used to compute the sensitivity of a single mode using only the associated eigensolution and its lower order derivatives. To illustrate the efficiency of Nelson's approach, the CPU time required to compute all eigensensitivities of this example for both methods have been checked using the same computer and the same computing accuracy. The computation time required by the modal method has been found to be approximately sixteen times the time required using Neslon's method.

8. Conclusion

The method for computing the first and second order eigensolution derivatives of symmetric damped systems using Nelson's approach is first reviewed. Nelson's approach is then extended to asymmetric damped systems. Contrary to the modal method, the computation of the derivatives of the eigensolution using the presented approach requires only the eigensolution to be derived. First- and second-order eigensensitivities of symmetric damped systems are treated as special cases of the general derived approach. Two numerical examples have been used to illustrate the application of the method and to show that the second-order derivatives of the eigensolution compared to the first derivatives are not always negligible. The computation efficiency of Nelson's method has also been highlighted.

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