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ScienceDirect
Journal of Sound and Vibration 301 (2007) 253–277
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Free vibrations of delaminated unidirectional sandwich panels with a transversely flexible core—a modified Galerkin approach

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Received 23 March 2006; received in revised form 19 September 2006; accepted 4 October 2006
Available online 4 December 2006

Abstract

A theoretical approach for the free vibration analysis of delaminated unidirectional sandwich panels is developed. The theoretical model accounts for the flexibility of the core in the out of plane (vertical) direction and the resulting high-order *displacement, acceleration, and velocity* fields within the core. The analytical approach is based on Hamilton's variational principle along with the high-order unidirectional sandwich panel theory and the modified Galerkin method. The two types of models investigated include delaminated regions with and without contact. The ability of the model to describe the high-order effects such as the pumping phenomenon and the localized effects in the vicinity of the delaminated regions is examined. A numerical example that focuses on the free vibration behavior of simply supported delaminated unidirectional sandwich panels is presented and discussed. A parametric study that examines the influence of the length of the delaminated region, its location, and the mechanical properties of the core material is presented. The numerical results are also compared with finite element analysis and with some special asymptotic cases for which the free vibrations behavior is analytically evaluated. A summary and conclusions close the paper.

1. Introduction

High stiffness and low weight sandwich panels are frequently used today in many structural applications. Recently, the use of sandwich panels made of composite laminated face-sheets and a plastic foam or low strength honeycomb core has become a common practice. These cores are characterized by their shear deformability and flexibility in the horizontal (in-plane) and vertical (out of plane) directions that result in high-order displacement, acceleration, and velocity fields through the height of the core. The shear deformability and the vertical flexibility of the core affect the overall response and cause the panel to change its height and distort its cross section plane under loading. Under dynamic conditions, these effects allow vibration modes that consist of relative displacements between the two face-sheets in the vertical (out of plane) and longitudinal directions to develop.

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2006.10.006

Sandwich structures in general, and especially those constructed with a soft core, are susceptible to partial debonding of the face-sheet from the core at their interface. This effect is illustrated in Fig. 1 and is termed delamination. Such delaminations are usually a result of the layered configuration of the panel, the manufacturing process, the considerable difference between the elastic properties of the face-sheets and the core, adhesive degradation, impact loads, and stress concentration due to localized loading, see Refs. [1–3]. Experimental investigations showed that such delaminations affect the integrity of the panel, reduce its overall stiffness, modify the dynamic behavior of the panel, and lead to a premature failure below the design loads; see Refs. [4–7].

The delaminated surfaces are free of shear stress and may slip longitudinally one with respect to each other. Yet, they can accommodate vertical normal compressive stress if contact between the surfaces exists. In general, the delaminated region is comprised of zones in which the delaminated surfaces maintain vertical contact and accumulate vertical normal compressive stresses (“delamination with contact”), and zones where opening of the interfacial delamination crack occurs and the delaminated surfaces are free of stresses (“delamination without contact”). The differences in the contact conditions have a major influence on the dynamic characteristics of the entire panel.

The free vibration and dynamic behavior of fully bonded incompressible sandwich panels has been extensively investigated, see for example, Refs. [8–11]. However, these studies have neglected the compressibility of the core and the corresponding high-order displacement, velocity, and acceleration effects. This type of analysis is applicable to sandwich panels with traditional incompressible cores; see Refs. [12–15]. On the other hand, in modern panels with a soft core, the high-order effects and the corresponding high-order acceleration and velocity fields affect the static and dynamic response of the panel and cannot be neglected.

Ng [16], Vaswani et al. [17], and Lok and Cheng [18,19] studied the dynamic behavior of fully bonded sandwich panels using equivalent single layer (ESL) beam theory approaches. Using a similar concept, Kant and Mallikarjune [20] and Nayak et al. [21] used the classical lamination theory. This approach was also used by Kant and Swaminathan [22] for the investigation of panels with a soft core. Although the ESL approach and the classical lamination theory yield a satisfactory description of the global behavior of the panel, they can not account for phenomena such as wave-like deformations through the height of the core and a relative displacement between the face-sheets. These effects become even more significant in the case of delaminated panels.

The natural frequencies and vibration modes of delaminated sandwich panels have been investigated in a smaller number of studies. Lee [23], Tracy and Pardoen [24], and Hu et al. [25] examined the free vibration behavior of delaminated sandwich panels. However, they neglected the high-order displacements of the core and assumed that contact between the debonded surfaces does not exist. Kwon and Lannamann [26] used a finite element analysis (FEA) to study the behavior of delaminated sandwich panels due to an impact load taking into account the conditions of with and without contact. On the other hand, the compressibility of the core was ignored. In addition, the different length scales that characterize the sandwich panel (thin face-sheets and thick core), the considerable differences in the mechanical properties, and the singularities near the delaminated region tip significantly affect the FE results. Lok and Cheng [27], Kim and Hwang [28], and Hu and Hwu [29] investigated the free vibration behavior of delaminated sandwich panels based on the “global behavior” approach. Notice that this approach neglects the critical localized effects and their influence on the response of the panel.

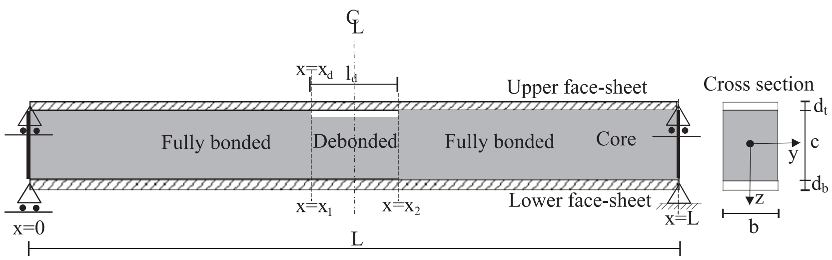

Fig. 1. Simply supported delaminated unidirectional sandwich panel.

Frostig et al. [30] and Frostig [31] have taken into account the compressibility of the core and developed a high-order approach for the static analysis of fully bonded and delaminated unidirectional sandwich panels with a soft core. Frostig and Baruch [32] used the high-order approach to study the free vibrations of fully bonded unidirectional sandwich panels, assuming linear velocities through the height of the core. As a result of these assumptions, the model assumes that core transfers its inertial loads to the face-sheets rather than resisting them by itself. Thus, wave-like behavior of the core and inner vibration modes are not considered. Sokolinsky et al. [33], and Yang and Qiao [34] applied the above approach to a higher-order velocity and acceleration fields in the core. This formulation leads to an inconsistent model that involves time derivatives of the stresses. Frostig and Thomsen [35] presented an analytical dynamic model for fully bonded panels, in which a polynomial distribution of the displacements through the height of the core is adopted. The model predicts wave-like behavior of the core both in the longitudinal and the vertical (out of plane) directions, which is constrained by the fully bonded face-sheets. However, this type of behavior cannot exist in delaminated unidirectional panels, and the above model can not be applied to delaminated unidirectional panels. Schwarts-Givli et al. [36] presented a nonlinear dynamic analysis that accounts for the complex time and space dependent contact conditions at the delaminated region. However, in order to perform such a complex, nonlinear, and time-dependent analysis, a simpler model that can be used for an initial assumption of the response of the structure and estimation of its dynamic characteristics is essential. The combination of the "with contact" and "without contact" models developed in this paper serves this purpose.

The objective of this study is to explore and quantitatively describe the free vibration behavior of delaminated simply supported unidirectional sandwich panels with a "soft" core. A theoretical model that considers the deformability of the core, the high-order displacement, velocity, and acceleration fields in the core, and the rotary inertia of the face-sheets and the core is developed to achieve this goal. The present study uses a formulation that is based on displacements only, and thus it allows a consistent variational formulation. The displacement fields in the core assume polynomial patterns that correspond to the closed form analytical solution of the static case [37]. The delaminated surfaces may be in contact vertically but may slip longitudinally with respect to each other. In terms of the vertical conditions at the delaminated interface, a distinction is made between debonded regions with contact and regions without contact.

The mathematical formulation is based on Hamilton's variational principle and the free vibrations problem is formulated and solved through the modified Galerkin method with a truncated series expansion. The sandwich panel is considered unidirectional and it is simply supported at the face sheets and at the core. Hence, a series expansion that consists of trigonometric function is used to describe the vertical and longitudinal displacement, respectively. The model accounts for the high-order displacement, velocity, and acceleration fields in the core, thus the height of the core may change, and its plane section does not remain planar after deformation. In addition, the model assumes small deformations and a linear elastic behavior, it accounts for the rotatory inertia of the face sheets and the core, and it assumes that the core can resist shear and vertical normal stresses and that its longitudinal stiffness is negligible. Finally, it is assumed that the debonded regions exist prior to the loading and do not grow, and that the contact characteristics at the delaminated interface (with or without contact) are known a priori and do not change with time.

The mathematical model is presented first. Next, the results of the proposed model are verified through a comparison with finite-elements results and with some specific asymptotic cases with analytical results. Numerical examples that focus on the influence of the contact characteristics ("with contact" or "without contact" conditions) on the free vibration response of unidirectional sandwich panels are presented and discussed. The ability of the model to describe the vibrations through the thickness and the stress and displacement fields near the tip of the interfacial delamination crack is described. Finally, a parametric study that examines the influence of the location and length of the debonded region and the effect of the mechanical properties of the core is conducted. In the sequel, a summary and conclusions are presented.

2. Mathematical formulation

A unidirectional sandwich panel that is debonded at one of its face–core interfaces comprise of two types of regions: a "fully bonded" region and a "delaminated" region, see Fig. 1. In the delaminated region, the delaminated surfaces may longitudinally slip one with respect to the other, but can be in contact vertically.

Hence, two types of delaminated regions may exist: a delaminated region with vertical contact or a delaminated region without contact. In the formulation, all of the three characteristic regions, i.e. fully bonded, debonded with contact, and debonded without contact, are combined into a unified model for the entire panel.

2.1. Hamilton's variational principle

The mathematical formulation uses Hamilton's variational principle that reads:

$$\int_{t_1}^{t_2} \delta(T-U)\mathrm{d}t = 0, \tag{1}$$

where T is the kinetic energy, U is the internal potential energy, t is the time coordinate, and δ is the variational operator.

The first variation of the kinetic energy reads:

$$\begin{aligned}\delta T = \int_{t_1}^{t_2}\Bigg[\int_{v_t} &\rho_t(\dot{u}_t(x,z_t,t)\delta\dot{u}_t(x,z_t,t)+\dot{w}_t(x,t)\delta\dot{w}_t(x,t))\,\mathrm{d}v_t\\ &+\int_{v_b}\rho_b(\dot{u}_b(x,z_b,t)\delta\dot{u}_b(x,z_b,t)+\dot{w}_b(x,t)\delta\dot{w}_b(x,t))\,\mathrm{d}v_b\\ &+\int_{v_c}\rho_c(\dot{u}_c(x,z_c,t)\delta\dot{u}_c(x,z_c,t)+\dot{w}_c(x,z_c,t)\delta\dot{w}_c(x,z_c,t))\,\mathrm{d}v_c\Bigg]\,\mathrm{d}t,\end{aligned} \tag{2}$$

where the subscripts t, b, and c refer to the upper and lower face-sheets and the core, respectively; $\rho_i(i=t,b,c)$ is the density of each component; $u_i(x,z_i,t)$, $w_i(x,t)$ $(i=t,b)$ and $u_c(x,z_i,t)$, $w_c(x,z_i,t)$ are the longitudinal and vertical displacements of the face-sheets and the core, respectively; $(\dot{\cdot})$ denotes derivative with respect to time; $v_i=(i=t,b,c)$ are the volume of the various components; and $z_i(i=t,b,c)$ are the vertical coordinates of the upper face sheet, lower face sheet, and the core, respectively, measured from the mid height of each layer downwards.

The first variation of the kinetic energy, after integrating Eq. (2) by parts with respect to time and prescribing the displacements at $t=t_1$ (initial conditions), reads:

$$\begin{aligned}\delta T = -\int_{t_1}^{t_2}\Bigg[\int_0^L\int_{-d_t/2}^{d_t/2} &b\rho_t(\ddot{u}_t(x,z_t,t)\delta u_t(x,z_t,t)+\ddot{w}_t(x,t)\delta w_t(x,t))\mathrm{d}x\,\mathrm{d}z_t\\ +\int_0^L\int_{-d_b/2}^{d_b/2} &b\rho_b(\ddot{u}_b(x,z_b,t)\delta u_b(x,z_b,t)+\ddot{w}_b(x,t)\delta w_b(x,t))\mathrm{d}x\,\mathrm{d}z_b\\ +\int_0^L\int_{-c/2}^{c/2} &b\rho_c(\ddot{u}_c(x,z_c,t)\delta u_c(x,z_c,t)+\ddot{w}_c(x,z_c,t)\delta w_c(x,z_c,t))\mathrm{d}x\,\mathrm{d}z_c\Bigg]\mathrm{d}t,\end{aligned} \tag{3}$$

where L is the length of the panel.

The first variation of the strain energy equals:

$$\begin{aligned}\delta U = \int_{t_1}^{t_2}\Bigg[\int_{v_t} &\sigma_{xxt}(x,z_t,t)\delta\varepsilon_{xxt}(x,z_t,t)\,\mathrm{d}v_t+\int_{v_b}\sigma_{xxb}(x,z_b,t)\delta\varepsilon_{xxb}(x,z_b,t)\,\mathrm{d}v_b\\ &+\int_{v_c}(\tau_c(x,z_c,t)\delta\gamma_c(x,z_c,t)+\sigma_{zz}(x,z_c,t)\delta\varepsilon_{zz}(x,z_c,t)\,\mathrm{d}v_c\Bigg]\,\mathrm{d}t,\end{aligned} \tag{4}$$

where $\sigma_{xxi}(x,z_i,t)$ and $\varepsilon_{xxi}(x,z_i,t)$ $(i=t,\ b)$ are the longitudinal normal stresses and strains, respectively, $\tau_c(x,z_c,t)$ and $\gamma_c(x,z_c,t)$ are the shear stress and shear angle in the core; and $\sigma_{zz}(x,z_c,t)$ and $\varepsilon_{zz}(x,z_c,t)$ are the vertical normal stress and strain in the core. The notations, sign conventions, and the coordinate systems appear in Fig. 2. Correspondingly, throughout the text, the term "vertical" refers to the out of plane response of the panel.

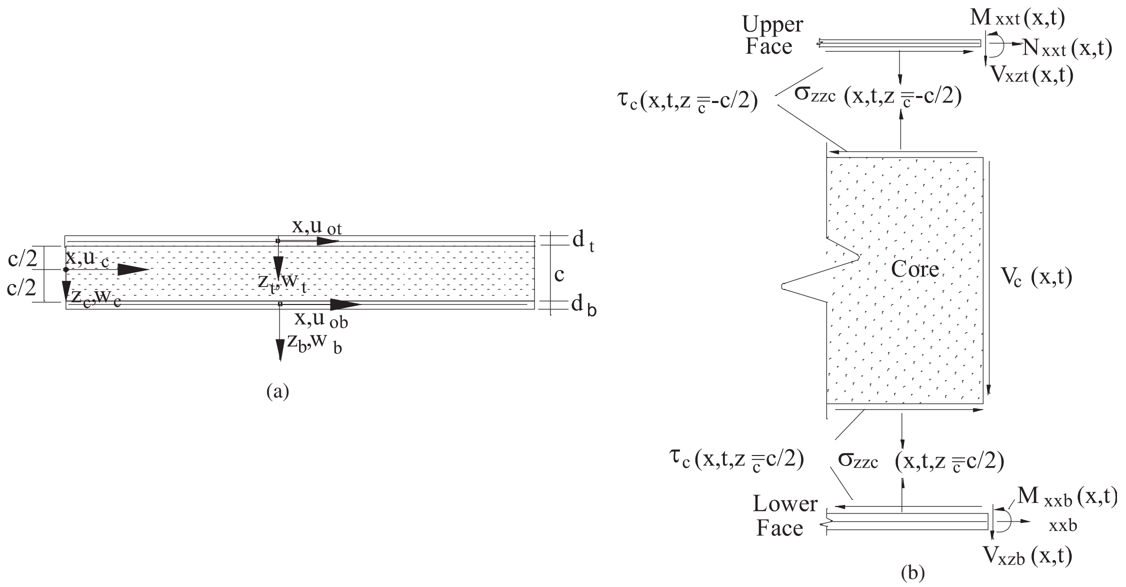

Fig. 2. Notations and sign conventions: (a) geometry displacements and coordinate systems; (b) stresses and internal stress resultants.

2.2. Face-sheets: kinematic and constitutive relations

The kinematic relations for the face-sheets assume small deformations and they are given by ($i = t, b$):

$$u_i(x, t) = u_{oi}(x, t) - w_{i,x}(x, t)z_i, \tag{5}$$

$$\varepsilon_{xxi}(x, t) = u_{oi}(x, t) - z_i w_{i,xx}(x, t), \tag{6}$$

where $u_{oi}(x, t)$, $w_i(x, t)$, and $w_{i,x}(i = t, b)$ are the longitudinal displacements, vertical displacements, and rotation of the reference plane of each face-sheet and $()_{,x}$ denotes a partial derivative with respect to x.

The constitutive relations for the face-sheets use the classical lamination theory and read ($i = t, b$):

$$N_{xxi}(x, t) = b\big(A_{11i}u_{oi,x}(x, t) - B_{11i}w_{i,xx}(x, t)\big), \tag{7}$$

$$M_{xxi}(x, t) = b\big(B_{11i}u_{oi,x}(x, t) - D_{11i}w_{i,xx}(x, t)\big), \tag{8}$$

where b is the width of the panel; A_{11i}, B_{11i} and $D_{11i}(i = t, b)$ are the axial, coupling, and flexural rigidities of the isolated composite laminated face-sheets, respectively; and $N_{xxi}(x, t)$ $(i = t, b)$ and $M_{xxi}(x, t)$ $(i = t, b)$ are the in-plane and the bending moment stress resultants, respectively, at the upper and lower face-sheets. The stress resultants are defined as follows:

$$\big\{N_{xxi}(x, t), M_{xxi}(x, t)\big\} = \int_{-d_i/2}^{d_i/2}\int_0^b \sigma_{xxi}(x, z_i, t)\{1, z_i\}\mathrm{d}y\,\mathrm{d}z_i, \quad (i = t, b). \tag{9}$$

2.3. Core: displacement and stress fields—displacements formulation

The kinematic relation of small deformation of the core read:

$$\gamma_c(x, z_c, t) = u_{c,z}(x, z_c, t) + w_{c,x}(x, z_c, t), \tag{10}$$

$$\varepsilon_{zz}(x, z_c, t) = w_{c,z}(x, z_c, t), \tag{11}$$

where $()_{c,z}$ denotes a partial derivative with respect to z_c, Fig. 2.

The linear elastic constitutive relations for the core are:

$$\sigma_{zz}(x, z_c, t) = E_c \varepsilon_{zz}(x, z_c, t), \tag{12}$$

$$\tau_c(x, z_c, t) = G_c \gamma_c(x, z_c, t), \tag{13}$$

where G_c and E_c are the shear modulus and modulus of elasticity of the core, respectively. Notice that the modulus of elasticity in the longitudinal direction is assumed as null in this analysis.

The longitudinal and vertical displacements of the core are assumed to take a cubic and quadratic polynomial variation in the thickness direction, respectively. Following the closed form analytical solutions of the static case, see Ref. [37], they equal:

$$w_c(x, z_c, t) = w_0(x, t) + w_1(x, t) z_c + w_2(x, t) z_c^2, \tag{14}$$

$$u_c(x, z_c, t) = u_0(x, t) + u_1(x, t) z_c + u_2(x, t) z_c^2 + u_3(x, t) z_c^3, \tag{15}$$

where $u_i(x, t)$ $(i = 0, 1, 2, 3)$ and $w_i(x, t)$ $(i = 0, 1, 2)$ are unknown functions. Hence, the acceleration field of the core reads:

$$\ddot{w}_c(x, z_c, t) = \ddot{w}_0(x, t) + \ddot{w}_1(x, t) z_c + \ddot{w}_2(x, t) z_c^2, \tag{16}$$

$$\ddot{u}_c(x, z_c, t) = \ddot{u}_0(x, t) + \ddot{u}_1(x, t) z_c + \ddot{u}_2(x, t) z_c^2 + \ddot{u}_3(x, t) z_c^3. \tag{17}$$

The distinction between the fully bonded regions and the delaminated ones with and without contact, is defined through the compatibility/debonding conditions at the core–face interface. These conditions govern the displacement and stress fields of the core, see Ref. [37].

In the case of a fully bonded region, the compatibility between the face-sheets and the core is imposed through the following conditions:

$$u_c\left(x, z_c = -\frac{c}{2}, t\right) = u_{ot}(x, t) - \frac{d_t}{2} w_{t,x}(x, t), \tag{18}$$

$$w_c\left(x, z_c = -\frac{c}{2}, t\right) = w_t(x, t), \tag{19}$$

$$u_c\left(x, z_c = \frac{c}{2}, t\right) = u_{ob}(x, t) + \frac{d_b}{2} w_{b,x}(x, t), \tag{20}$$

$$w_c\left(x, z_c = \frac{c}{2}, t\right) = w_b(x, t). \tag{21}$$

In the debonded region, the surfaces are free of shear stresses and may slip longitudinally one with respect to the other. Thus, when the upper face–core interface is debonded, the longitudinal compatibility condition, Eq. (18), is replaced with the following one:

$$\tau_c\left(x, z_c = \frac{c}{2}, t\right) = 0. \tag{22}$$

If vertical contact exists, Eq. (19) is used. If such contact does not exist, it is replaced with the following zero vertical normal stress condition:

$$\sigma_{zzc}\left(x, z_c = -\frac{c}{2}, t\right) = 0. \tag{23}$$

In the lower face–core interface, which is fully bonded, the compatibility conditions are given by Eqs. (20) and (21).

The unknown functions, $w_0(x, t)$, $w_1(x, t)$, $w_2(x, t)$, $u_0(x, t)$, $u_1(x, t)$, $u_2(x, t)$, $u_3(x, t)$, that describe the displacements of the core in Eqs. (14)–(15), are solved for using the relevant set of interfacial compatibility/debonding conditions (fully bonded, debonded with contact, debonded without contact), Eqs. (18)–(23), along with the static equilibrium in the core, and the constitutive relations, Eqs. (12)–(13). In the debonded region, all seven functions describing the displacement fields of the core are expressed in terms of the displacements of the face sheets. On the other hand, in the fully bonded regions, only six functions are be expressed in terms of the face-sheet displacements and the seventh function, $w_0(x,t)$, remains unknown. The formulation of the stresses and displacement fields under static conditions appears in Ref. [37]. For brevity, only the final results, which are augmented to include the time variable, are presented here. These results are presented for the case where the upper core–face interface is debonded whereas similar expressions can be derived when the lower core–face interface is debonded.

Hence, the displacement and stress fields of the core read:

Fully bonded:

$$w_c(x,z_c,t) = \left(1 - \frac{4z_c^2}{c^2}\right)w_0(x,t) + \left(\frac{2}{c^2}z_c^2 - \frac{1}{c}z_c\right)w_t(x,t) + \left(\frac{2}{c^2}z_c^2 + \frac{1}{c}z_c\right)w_b(x,t), \tag{24}$$

$$\begin{aligned} u_c(x,z_c,t) = {} & \left(\frac{1}{2} + \frac{z_c}{c}\right)u_{ob}(x,t) + \left(\frac{1}{2} - \frac{z_c}{c}\right)u_{ot}(x,t) \\ & + \left(\left(-\frac{d_t}{4} - \frac{c}{8}\right) + \left(\frac{1}{6} + \frac{d_t}{2c}\right)z_c + \frac{1}{2c}z_c^2 - \frac{2}{3c^2}z_c^3\right)w_{t,x}(x,t) \\ & + \left(\frac{4z_c^3}{3c^2} - \frac{z_c}{3}\right)w_{0,x}(x,t) + \left(\frac{1}{8}c + \frac{1}{4}d_b + \left(\frac{1}{6} + \frac{d_b}{2c}\right)z_c - \frac{1}{2c}z_c^2 - \frac{2z_c^3}{3c^2}\right)w_{b,x}(x,t), \end{aligned} \tag{25}$$

$$\begin{aligned} \sigma_{zzc}(x,z_c,t) = E_c w_{c,z}(x,z_c,t) = {} & \left(\frac{4z_c}{c^2} + \frac{1}{c}\right)E_c w_b(x,t) + \left(\frac{4z_c}{c^2} - \frac{1}{c}\right)E_c w_t(x,t) \\ & - \frac{8z_c}{c^2}E_c w_0(x,t), \end{aligned} \tag{26}$$

$$\begin{aligned} \tau_c(x,z_c,t) = G_c\big(u_{c,z}(x,z_c,t) + w_{c,x}(x,z_c,t)\big) = {} & -\frac{G_c}{c}u_{ot}(x,t) + \frac{G_c}{c}u_{ob}(x,t) \\ & + \left(\frac{d_b}{2c} + \frac{1}{6}\right)G_c w_{b,x}(x,t) + \left(\frac{d_t}{2c} + \frac{1}{6}\right)G_c w_{t,x}(x,t) + \frac{2}{3}G_c w_{0,x}(x,t). \end{aligned} \tag{27}$$

Debonded region with contact:

$$w_c(x,z_c,t) = \frac{1}{2}(w_b(x,t) + w_t(x,t)) + \left(\frac{w_b(x,t)}{c} - \frac{w_t(x,t)}{c}\right)z_c, \tag{28}$$

$$\begin{aligned} u_c(x,z_c,t) = {} & \left(\frac{d_b w_{b,x}(x,t)}{2} + \frac{3c w_{b,x}(x,t)}{8} + \frac{c w_{t,x}(x,t)}{8} + u_{ob}(x,t)\right) \\ & - \left(\frac{w_{b,x}(x,t)}{2} + \frac{w_{t,x}(x,t)}{2}\right)z_c + \left(\frac{w_{t,x}(x,t)}{2c} - \frac{w_{b,x}(x,t)}{2c}\right)z_c^2, \end{aligned} \tag{29}$$

$$\sigma_{zzc}(x,z_c,t) = \frac{E_c}{c}(w_b(x,t) - w_t(x,t)), \tag{30}$$

$$\tau_c(x,z_c,t) = 0. \tag{31}$$

Debonded region without contact:

$$w_c(x,z_c,t) = w_b(x,t), \tag{32}$$

$$u_c(x,z_c,t) = u_{ob}(x,t) + w_{b,x}(x,t)\left(\frac{d_b}{2} + \frac{c}{2} - z_c\right), \tag{33}$$

$$\sigma_{zzc}(x,z_c,t) = 0, \tag{34}$$

$$\tau_c(x,z_c,t) = 0. \tag{35}$$

2.4. Modified Galerkin approach

The solution procedure proposed uses the modified Galerkin method which is derived here from Hamilton's variational principle. The variational principle is stated explicitly in terms of displacements only and the modified Galerkin method is formulated through the introduction of the trial functions. This formulation leads to an algebraic eigenvalue problem that is solved for the natural frequencies and the vibration modes.

The variational principle expressed in terms of the unknown displacements is defined through the introduction of the kinematic relations, the displacement fields, and the constitutive relations for the face-sheets (Eqs. (5), (6), (7)–(9), respectively), the displacements and stress fields of the core, (Eqs. (24)–(35)), and the corresponding acceleration fields (Eqs. (16)–(17)) into the Hamilton principle, Eqs. (1)–(3). After integrating by parts and some algebraic manipulations, the variational principle is rewritten as the sum of two expressions. The first one, denoted by $\delta\Pi_{\text{field}}$ includes terms that correspond to the response in the field. The second expression, $\delta\Pi_{\text{bound}}$, is related to the boundary conditions and to the continuity conditions. Thus, the variational principle takes the following form:

$$\delta\Pi = \delta\Pi_{\text{field}} + \delta\Pi_{\text{bound}} = 0, \tag{36}$$

where

$$\begin{aligned}\delta\Pi_{\text{field}} = \int_{t_1}^{t_2}\Bigg[\int_0^{x_1}&\big(L_1^{\text{fb}}(\mathbf{Y},\ddot{\mathbf{Y}})\delta u_{ot}(x,t) + L_2^{\text{fb}}(\mathbf{Y},\ddot{\mathbf{Y}})\delta w_t(x,t) + L_3^{\text{fb}}(\mathbf{Y},\ddot{\mathbf{Y}})\delta u_{ob}\\ &+L_4^{\text{fb}}(\mathbf{Y},\ddot{\mathbf{Y}})\delta w_b(x,t) + L_5^{\text{fb}}(\mathbf{Y},\ddot{\mathbf{Y}})\delta w_0(x,t)\big)\,\mathrm{d}x + \int_{x_1}^{x_2}\big(L_1^{\text{del}}(\mathbf{Y},\ddot{\mathbf{Y}})\delta u_{ot}(x,t)\\ &+ L_2^{\text{del}}(\mathbf{Y},\ddot{\mathbf{Y}})\delta w_t(x,t) + L_3^{\text{del}}(\mathbf{Y},\ddot{\mathbf{Y}})\delta u_{ob}(x,t) + L_4^{\text{del}}(\mathbf{Y},\ddot{\mathbf{Y}})\delta w_b(x,t)\big)\,\mathrm{d}x\\ &+\int_{x_2}^{L}\big(L_1^{\text{fb}}(\mathbf{Y},\ddot{\mathbf{Y}})\delta u_{ot}(x,t) + L_2^{\text{fb}}(\mathbf{Y},\ddot{\mathbf{Y}})\delta w_t(x,t) + L_3^{\text{fb}}(\mathbf{Y},\ddot{\mathbf{Y}})\delta u_{ob}(x,t)\\ &+L_4^{\text{fb}}(\mathbf{Y},\ddot{\mathbf{Y}})\delta w_b(x,t) + L_5^{\text{fb}}(\mathbf{Y},\ddot{\mathbf{Y}})\delta w_0(x,t)\Big)\,\mathrm{d}x\Bigg]\,\mathrm{d}t\end{aligned} \tag{37}$$

with $x = x_1, x_2$ defining the coordinates of the edges of the debonded region, see Fig. 1, $L_i^{\text{type}}(\mathbf{Y},\ddot{\mathbf{Y}}), (i = 1..5)$ is a linear differential operator, the index "type" defines the type of the region: i.e. fb refers to a fully bonded region and del refers to a delaminated region, and $\mathbf{Y},\ddot{\mathbf{Y}}$ are the vectors of the unknown displacements and accelerations:

$$\mathbf{Y} = \begin{bmatrix} w_t(x,t)\\ w_b(x,t)\\ u_{ot}(x,t)\\ u_{ob}(x,t)\\ w_0(x,t)\end{bmatrix}, \quad \ddot{\mathbf{Y}} = \begin{bmatrix} \ddot{w}_t(x,t)\\ \ddot{w}_b(x,t)\\ \ddot{u}_{ot}(x,t)\\ \ddot{u}_{ob}(x,t)\\ \ddot{w}_0(x,t)\end{bmatrix}. \tag{38}$$

The linear differential operators $L_i^{\text{type}}(\mathbf{Y},\ddot{\mathbf{Y}}), (i = 1..5)$ equal:

$$\begin{aligned}L_1^{\text{type}}(\mathbf{Y},\ddot{\mathbf{Y}}) = {}&\left(M_t + \frac{\alpha_1 M_c}{3}\right)\ddot{u}_{ot}(x,t) + \frac{\alpha_1 b G_c}{c}u_{ot}(x,t) + \frac{\alpha_1 M_c}{6}\ddot{u}_{ob}(x,t)\\ &- \frac{\alpha_1 b G_c}{c}u_{ob}(x,t) + \alpha_1 M_c\left(\frac{d_b}{12} + \frac{13c}{360}\right)\ddot{w}_{b,x}(x,t) - \alpha_1 b G_c\left(\frac{d_b}{2c} + \frac{1}{6}\right)w_{b,x}(x,t)\\ &- \alpha_1 M_c\left(\frac{d_t}{6} + \frac{17c}{360}\right)\ddot{w}_{t,x}(x,t) - \alpha_1 b G_c\left(\frac{d_t}{2c} + \frac{1}{6}\right)w_{t,x}(x,t) + EA_t u_{ot,xx}(x,t)\\ &+ \frac{M_c c}{90}\ddot{w}_{o,x}(x,t) - \frac{2\alpha_1 b G_c}{3}w_{o,x}(x,t),\end{aligned} \tag{39}$$

$$
\begin{aligned}
L_2^{\text{type}}(\mathbf{Y},\ddot{\mathbf{Y}}) = {} & \left(M_t+\frac{\alpha_3 M_c}{3}+\frac{2\alpha_1 M_c}{15}\right)\ddot{w}_t(x,t)+bE_c\left(\frac{\alpha_3}{c}+\frac{7\alpha_1}{3c}\right)w_t(x,t)\\
& -\left(I_{mt}+\alpha_3\frac{M_c c^2}{20}+\alpha_1\frac{67M_c c^2}{7560}+\alpha_1\frac{17cd_t}{360}+\alpha_1\frac{d_t^2}{12}\right)\ddot{w}_{t,xx}(x,t)\\
& -\alpha_{bG_c1}\left(\frac{d_t}{6}+\frac{c}{36}+\frac{d_t^2}{4c}\right)w_{t,xx}(x,t)+EI_t w_{t,xxxx}(x,t)+\alpha_1\left(\frac{17c}{360}+\frac{d_t}{6}\right)M_c\ddot{u}_{ot,x}(x,t)\\
& +\alpha_1 bG_c\left(\frac{1}{6}+\frac{d_t}{2c}\right)u_{ot,x}(x,t)+M_c\left(\frac{\alpha_3}{6}-\frac{\alpha_1}{30}\right)\ddot{w}_b(x,t)+bE_c\left(\frac{\alpha_1}{3c}-\frac{\alpha_3}{c}\right)w_b(x,t)\\
& \times M_c\left(\frac{\alpha_1 d_t}{12}+\frac{13\alpha_1 c}{360}-\frac{\alpha_3 c}{6}\right)\ddot{u}_{ob,x}(x,t)-bG_c\left(\frac{\alpha_1}{6}+\frac{\alpha_1 d_t}{2c}\right)u_{ob,x}(x,t)\\
& \times M_c\left(\frac{13\alpha_1 d_b c}{720}-\frac{\alpha_3 c d_b}{12}+\frac{13\alpha_1 c d_t}{720}-\frac{3\alpha_3 c^2}{40}+\frac{59\alpha_1 c^2}{7560}+\frac{\alpha_1 d_b d_t}{24}\right)\ddot{w}_{b,xx}(x,t)\\
& -\alpha_1 bG\left(\frac{d_t}{12}+\frac{d_b}{12}+\frac{1}{36}+\frac{d_t d_b}{4c}\right)w_{b,xx}(x,t)+\frac{\alpha_1 M_c}{15}\ddot{w}_0(x,t)\\
& -\frac{8\alpha_1 bE_c}{3c}w_0(x,t)+\alpha_1 M_c\left(\frac{cd_t}{180}+\frac{c^2}{945}\right)\ddot{w}_{0,xx}(x,t)-\alpha_1 bG_c\left(\frac{c}{9}+\frac{d_t}{3}\right)w_{0,xx}(x,t), \qquad (40)
\end{aligned}
$$

$$
\begin{aligned}
L_3^{\text{type}}(\mathbf{Y},\ddot{\mathbf{Y}}) = {} & -EA_b u_{ob,xx}(x,t)+\left(M_b+M_c-\frac{2\alpha_1 M_c}{3}\right)\ddot{u}_{ob}(x,t)+\frac{\alpha_1 bG_c}{c}u_{ob}(x,t)\\
& +\alpha_1\frac{M_c}{6}\ddot{u}_{ot}(x,t)-\frac{\alpha_1 bG_c}{c}u_{ot}(x,t)-\frac{\alpha_1 c}{90}M_c\ddot{w}_{0,x}(x,t)+\frac{2\alpha_1 bG_c}{3}w_{0,x}(x,t)\\
& +M_c\left(\frac{d_b}{2}-\frac{d_b\alpha_1}{3}+\frac{\alpha_2 c}{2}+\frac{17\alpha_1 c}{360}+\frac{\alpha_3 c}{3}\right)\ddot{w}_{b,x}(x,t)+\alpha_1 bG_c\left(\frac{d_b}{2c}+\frac{1}{6}\right)w_{b,x}(x,t)\\
& +M_c\left(\frac{\alpha_3 c}{6}-\frac{13\alpha_1 c}{360}-\frac{\alpha_1 d_t}{12}\right)\ddot{w}_{t,x}(x,t)+\alpha_1 bG_c\left(\frac{d_t}{2c}+\frac{1}{6}\right)w_{t,x}(x,t), \qquad (41)
\end{aligned}
$$

$$
\begin{aligned}
L_4^{\text{type}}(\mathbf{Y},\ddot{\mathbf{Y}}) = {} & EI_b w_{b,xxxx}(x,t)+M_c\left(\frac{\alpha_3}{6}-\frac{\alpha_1}{30}\right)\ddot{w}_t(x,t)+\frac{bE_c}{c}\left(\frac{\alpha_1}{3}-\alpha_3\right)w_t(x,t)\\
& \times\left(M_b+\frac{2\alpha_1 M_c}{15}+\frac{\alpha_3 M_c}{3}+\alpha_2 M_c\right)\ddot{w}_b(x,t)+\frac{bE_c}{c}(\alpha_3+7\alpha_1)w_b(x,t)\\
& -M_c\left(\frac{d_b}{2}-\frac{\alpha_1 d_b}{3}+\frac{\alpha_3 c}{3}+\frac{17\alpha_1 c}{360}+\frac{\alpha_2 c}{2}\right)\ddot{u}_{ob,x}(x,t)-\alpha_1 bG_c\left(\frac{d_b}{2c}+\frac{1}{6}\right)u_{ob,x}(x,t)\\
& -M_c\left(\left(\frac{2\alpha_3}{15}+\frac{67\alpha_1}{7560}+\frac{\alpha_2}{3}\right)c^2-\left(\frac{1}{4}+\frac{\alpha_1}{6}\right)d_b^2-\left(\frac{\alpha_2}{2}+\frac{17\alpha_1}{360}+\frac{\alpha_3}{3}\right)cd_b\right)\ddot{w}_{b,xx}(x,t)\\
& -\alpha_1 G_c b\left(\frac{d_b^2}{4c}+\frac{c}{36}+\frac{d_b}{6}\right)w_{b,xx}(x,t)-I_{mb}\ddot{w}_{b,xx}(x,t)\\
& +\alpha_1 M_c\left(\frac{cd_b}{180}+\frac{c^2}{945}\right)\ddot{w}_{0,xx}(x,t)+\alpha_1 bG_c\left(-\frac{d_b}{3}-\frac{c}{9}\right)w_{0,xx}(x,t)+\alpha_1\frac{M_c}{15}\ddot{w}_0(x,t)\\
& -\alpha_1\frac{8bE_c}{3}w_0(x,t)-\alpha_1 M_c\left(\frac{13c}{360}+\frac{d_b}{12}\right)\ddot{u}_{ot,x}(x,t)+\alpha_1 bG_c\left(\frac{1}{6}+\frac{d_b}{2c}\right)u_{ot,x}(x,t)\\
& +M_c\left(\left(\frac{13\alpha_1}{720}-\frac{\alpha_3}{12}\right)cd_b+\left(\frac{59\alpha_1}{7560}-\frac{3\alpha_3}{40}\right)c^2+\frac{\alpha_1 d_b d_t}{24}+\frac{13\alpha_1 d_t c}{720}\right)\ddot{w}_{t,xx}(x,t)\\
& -\alpha_1 bG_c\left(\frac{d_t}{12}+\frac{d_b}{12}+\frac{c}{36}+\frac{d_b d_t}{4c}\right)w_{t,xx}(x,t), \qquad (42)
\end{aligned}
$$

$$
\begin{aligned}
L_5^{\text{type}}(\mathbf{Y},\ddot{\mathbf{Y}}) = {}& \alpha_1 M_c\left(\frac{c^2}{945}+\frac{d_b c}{180}\right)\ddot{w}_{b,xx}(x,t) - \alpha_1 b G_c\left(\frac{d_b}{3}+\frac{c}{9}\right) w_{b,xx}(x,t) \\
& \times \frac{\alpha_1 M_c}{15}\ddot{w}_t(x,t) - \frac{\alpha_1 8bE_c}{3c} w_t(x,t) + \frac{\alpha_1 M_c}{15}\ddot{w}_b(x,t) + \frac{\alpha_1 8bE_c}{3c} w_b(x,t) \\
& + \alpha_1 M_c\left(\frac{c^2}{945}+\frac{d_t c}{180}\right)\ddot{w}_{t,xx}(x,t) - \alpha_1 b G_c\left(\frac{d_t}{3}+\frac{c}{9}\right) w_{t,xx}(x,t) \\
& - \frac{\alpha_1 M_c c}{90}\ddot{u}_{ot,x}(x,t) + \frac{\alpha_1 2bG_c}{3} u_{ot,x}(x,t) + \frac{\alpha_1 M_c c}{90}\ddot{u}_{ob,x}(x,t) - \frac{\alpha_1 2bG_c}{3} u_{ob,x}(x,t) \\
& + \frac{8\alpha_1 M_c}{15}\ddot{w}_0(x,t) + \frac{\alpha_1 16bE_c}{3c} w_0(x,t) - \frac{2\alpha_1 c^2}{945} M_c \ddot{w}_{0,xx}(x,t) - \frac{\alpha_1 4bG_c c}{9} w_{0,xx}(x,t), \qquad (43)
\end{aligned}
$$

where α_1, α_2 and α_3 are flags that define the type of regions as follows: $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = 0$ for a fully bonded region (fb); $\alpha_2 = 1$ and $\alpha_1 = \alpha_3 = 0$ for a debonded region without contact (del); $\alpha_3 = 1$ and $\alpha_2 = \alpha_1 = 0$ for a debonded region with contact (del), which can accommodate vertical compressive or tensile normal stresses, and $M_i, I_{mi} (i = t, b, c)$ are the mass and the moment of inertia, respectively, of the face-sheets and the core.

The variational expression that corresponds to the terms at the boundaries of the various regions, i.e., at the edges of the unidirectional panel and at the mutual joints between different types of regions, is:

$$
\begin{aligned}
\delta\Pi_{\text{bound}} = \int_{t_1}^{t_2} \Big(& B_1^{\text{fb}}\delta u_{ot}(x,t)\big|_0^{x_1} + B_2^{\text{fb}}\delta w_{t,x}(x,t)\big|_0^{x_1} + B_3^{\text{fb}}\delta w_t(x,t)\big|_0^{x_1} \\
& + B_4^{\text{fb}}\delta u_{ob}(x,t)\big|_0^{x_1} + B_5^{\text{fb}}\delta w_{b,x}(x,t)\big|_0^{x_1} + B_6^{\text{fb}}\delta w_b(x,t)\big|_0^{x_1} + B_7^{\text{fb}}\delta w_0(x,t)\big|_0^{x_1} \\
& + B_1^{\text{del}}\delta u_{ot}(x,t)\big|_{x_1}^{x_2} + B_2^{\text{del}}\delta w_{t,x}(x,t)\big|_{x_1}^{x_2} + B_3^{\text{del}}\delta w_t(x,t)\big|_{x_1}^{x_2} + B_4^{\text{del}}\delta u_{ob}(x,t)\big|_{x_1}^{x_2} \\
& + B_5^{\text{del}}\delta w_{b,x}(x,t)\big|_{x_1}^{x_2} + B_6^{\text{del}}\delta w_b(x,t)\big|_{x_1}^{x_2} + B_1^{\text{fb}}\delta u_{ot}(x,t)\big|_{x_2}^{L} + B_2^{\text{fb}}\delta w_{t,x}(x,t)\big|_{x_2}^{L} \\
& + B_3^{\text{fb}}\delta w_t(x,t)\big|_{x_2}^{L} + B_4^{\text{fb}}\delta u_{ob}(x,t)\big|_{x_2}^{L} + B_5^{\text{fb}}\delta w_{b,x}(x,t)\big|_{x_2}^{L} + B_6^{\text{fb}}\delta w_b(x,t)\big|_{x_2}^{L} \\
& + B_7^{\text{fb}}\delta w_0(x,t)\big|_{x_2}^{L} \Big)\,\mathrm{d}t, \qquad (44)
\end{aligned}
$$

where $B_j^{\text{type}}(\mathbf{Y},\ddot{\mathbf{Y}}), (j = 1..7)$ are linear differential operators that equal:

$$B_1^{\text{type}}(\mathbf{Y},\ddot{\mathbf{Y}}) = EA_t u_{ot,x}(x,t), \qquad (45)$$

$$B_1^{\text{type}}(\mathbf{Y},\ddot{\mathbf{Y}}) = EI_t w_{t,xx}(x,t), \qquad (46)$$

$$
\begin{aligned}
B_3^{\text{type}}(\mathbf{Y},\ddot{\mathbf{Y}}) = {}& M_c\left(\frac{\alpha_3 c}{6} - \frac{13\alpha_1 c}{360} - \frac{\alpha_1 d_t}{12}\right)\ddot{u}_{ob}(x,t) + \alpha_1 b G_c\left(\frac{1}{6}+\frac{d_t}{2c}\right) u_{ob}(x,t) \\
& + M_c\left(\frac{\alpha_1 d_t^2}{12} + \frac{\alpha_3 c^2}{20} + \frac{67\alpha_1 c^2}{7560} + \frac{17\alpha_1 c d_t}{360} + I_{mt}\right)\ddot{w}_{t,x}(x,t) \\
& + \alpha_1 b G_c\left(\frac{d_t}{6} + \frac{c}{36} + \frac{d_t^2}{4c}\right) w_{t,x}(x,t) - EI_t w_{t,xxx}(x,t) - \alpha_1 M_c\left(\frac{17c}{360}+\frac{d_t}{6}\right)\ddot{u}_{ot}(x,t) \\
& - \alpha_1 b G_c\left(\frac{1}{6}+\frac{d_t}{2c}\right) u_{ot}(x,t) + \alpha_1 b G_c\left(\frac{d_t}{3}+\frac{c}{9}\right) w_{0,x}(x,t) \\
& + M_c\left(\frac{3\alpha_3 c^2}{40} - \frac{59\alpha_1 c^2}{7560} - \frac{13\alpha_1 c d_b}{720} - \frac{13\alpha_1 c d_t}{720} - \frac{\alpha_1 d_t d_b}{24} + \frac{\alpha_3 c d_b}{12}\right)\ddot{w}_{b,x}(x,t) \\
& + \alpha_1 b G_c\left(\frac{d_t}{12} + \frac{c}{36} + \frac{d_t d_b}{4c} + \frac{d_b}{12}\right) w_{b,x}(x,t) - \alpha_1 M_c\left(\frac{c^2}{945}+\frac{c d_t}{180}\right)\ddot{w}_{0,x}(x,t), \qquad (47)
\end{aligned}
$$

$$B_4^{\text{type}}(\mathbf{Y},\ddot{\mathbf{Y}}) = EA_b u_{ob,x}(x,t), \qquad (48)$$

$$B_5^{\text{type}}(\mathbf{Y},\ddot{\mathbf{Y}}) = EI_b w_{b,xx}(x,t), \qquad (49)$$

$$
\begin{aligned}
B_6^{\text{type}}(\mathbf{Y},\ddot{\mathbf{Y}}) = & -EI_b w_{b,xxx}(x,t) + I_{mb}\ddot{w}_{b,x}(x,t) + \alpha_1 bG_c\left(\frac{d_b}{6}+\frac{c}{36}+\frac{d_b^2}{4c}\right)w_{b,x}(x,t) \\
& + M_c\left(\frac{d_b^2}{4}-\frac{\alpha_1 d_b^2}{6}+\frac{2\alpha_3 c^2}{15}+\frac{\alpha_2 c^2}{3}+\frac{67\alpha_1 c^2}{7560}+\frac{\alpha_2 c d_b}{2}+\frac{\alpha_3 c d_b}{3}+\frac{17\alpha_1 c d_b}{360}\right)\ddot{w}_{b,x}(x,t) \\
& + M_c\left(\frac{3\alpha_3 c^2}{40}-\frac{59\alpha_1 c^2}{7560}-\frac{13\alpha_1 c d_b}{720}-\frac{13\alpha_1 c d_t}{720}-\frac{\alpha_1 d_t d_b}{24}+\frac{\alpha_3 c d_b}{12}\right)\ddot{w}_{t,x}(x,t) \\
& + \alpha_1 bG_c\left(\frac{d_t}{12}+\frac{c}{36}+\frac{d_t d_b}{4c}+\frac{d_b}{12}\right)w_{t,x}(x,t) + \alpha_1 M_c\left(\frac{13c}{360}+\frac{d_b}{12}\right)\ddot{u}_{ot}(x,t) \\
& - \alpha_1 bG_c\left(\frac{1}{6}+\frac{d_b}{2c}\right)u_{ot}(x,t) + M_c\left(\frac{d_b}{2}-\frac{\alpha_1 d_b}{3}+\frac{\alpha_2 c}{2}+\frac{17\alpha_1 c}{360}+\frac{\alpha_3 c}{3}\right)\ddot{u}_{ob}(x,t) \\
& + \alpha_1 bG_c\left(\frac{1}{6}-\frac{d_b}{2c}\right)u_{ob}(x,t) - \alpha_1 M_c\left(\frac{c^2}{945}+\frac{c d_b}{180}\right)\ddot{w}_{0,x}(x,t) \\
& + \alpha_1 bG_c\left(\frac{d_b}{3}+\frac{c}{9}\right)w_{0,x}(x,t),
\end{aligned}
\tag{50}
$$

$$
\begin{aligned}
B_7^{\text{type}}(\mathbf{Y},\ddot{\mathbf{Y}}) = & -\frac{c}{90}\alpha_1 M_c\ddot{u}_{ob}(x,t) + \frac{2\alpha_1 bG_c}{3}u_{ob}(x,t) + \frac{c}{90}\alpha_1 M_c\ddot{u}_{ot}(x,t) \\
& - \frac{2\alpha_1 bG_c}{3}u_{ot}(x,t) - \alpha_1 M_c\left(\frac{d_t c}{180}+\frac{c^2}{945}\right)\ddot{w}_{t,x}(x,t) + \alpha_1 bG_c\left(\frac{d_t}{3}+\frac{c}{9}\right)w_{t,x}(x,t) \\
& - \alpha_1 M_c\left(\frac{d_b c}{180}+\frac{c^2}{945}\right)\ddot{w}_{b,x}(x,t) + \alpha_1 bG_c\left(\frac{d_b}{3}+\frac{c}{9}\right)w_{b,x}(x,t) + \frac{4\alpha_1 bG_c c}{9}w_{0,x}(x,t) \\
& + \frac{2c^2}{945}\alpha_1 M_c\ddot{w}_{0,x}(x,t).
\end{aligned}
\tag{51}
$$

Note that the distinction between the terms of the fully bonded regions and the terms of the delaminated regions is achieved through the use of the appropriate values of α_1, α_2, and α_3.

The free vibration behavior of the unidirectional delaminated panel is solved using a harmonic behavior in time and expanding the unknown displacements as a truncated series of admissible functions multiplied by unknown constants. These functions satisfy at least the geometrical boundary conditions at the edges of the unidirectional panel. For a panel that is simply supported at the upper and lower face-sheets and at the core the following series expansion is adopted:

$$
\begin{bmatrix} w_t(x,t) \\ w_b(x,t) \\ u_{ot}(x,t) \\ u_{ob}(x,t) \\ w_0(x,t) \end{bmatrix} \approx e^{i\omega t}
\begin{bmatrix}
\sum_{n=1}^{N} Cw_{tn}\sin\frac{n\pi x}{l} \\
\sum_{n=1}^{N} Cw_{bn}\sin\frac{n\pi x}{l} \\
\sum_{n=0}^{N} Cu_{otn}\cos\frac{n\pi x}{l} \\
\sum_{n=0}^{N} Cu_{obn}\cos\frac{n\pi x}{l} \\
\sum_{n=1}^{N} Cw_{0n}\sin\frac{n\pi x}{l}
\end{bmatrix},
\tag{52}
$$

where i is the imaginary unit, ω is the frequency, $C_k(k = w_{tn}, w_{bn}, u_{obn}, u_{otn}, w_{0n})$ are scalar unknowns (amplitudes), and n is the wave number. Notice that in order to allow complete expansion of the functional base by the trial functions, the cosine series also includes the constant ($n = 0$) term. Nevertheless, in the fully bonded unidirectional panel where the solution can be decoupled and can be separately solved for each wave number, the contribution of the terms associated with $n = 0$ vanishes.

The eigenvalue problem is formulated through substituting Eq. (52), into the variational principle, Eq. (36), and integrating in the x direction. Thus it reduces to the following form:

$$\left(-\omega^2\mathbf{M}+\mathbf{K}\right)\mathbf{C}=\mathbf{0}, \tag{53}$$

where ω^2 is the eigenvalue, $\mathbf{C}$ is the eigenvector associated with the unknown constants $C_k(k = w_{tn}, w_{bn}, u_{obn}, u_{otn}, w_{0n})$; $\mathbf{M}$ and $\mathbf{K}$ are the mass and stiffness matrixes of dimension $(5N+2)\times(5N+2)$, respectively; and $\mathbf{0}$ is a null vector of dimension $(5N+2)$.

The $\mathbf{M}$ and $\mathbf{K}$ matrixes comprise of terms related to the behavior within the field, which originate from $\delta\Pi_{\text{field}}$, as well as terms associated with the boundaries and the joints between the bonded and debonded regions. The region boundary terms, which originate from $\delta\Pi_{\text{bound}}$, impose the continuity requirements of the internal stress resultants, which are not identically satisfied by the series expansion of the unknowns. These terms are the result of the variational procedure and they correspond to the terms of the various joints used in the "modified Galerkin method" [38].

In order to describe properly the dynamic behavior of the unidirectional delaminated panel, the series expansion of the displacements must include a sufficient number of terms, which mainly depends on the length of the debonded region and the core to face-sheet stiffness ratio. In the case of a fully bonded panel, the solution is decoupled and can be obtained separately for each wave number, n, which results in five eigenvalues and eigenvectors for each wave number. On the other hand, in the case of a delaminated panel, the trigonometric terms are coupled, and the solution procedure must involve all trigonometric terms simultaneously.

3. Model verification

The model developed here is confirmed through comparison of the natural frequencies and the vibration modes with results of a commercial 2D finite element package, Ansys [39], and with special asymptotic cases with analytical results.

Two cases of simply supported unidirectional sandwich panels that differ in their geometrical and mechanical properties are examined. The density and the elastic properties of the face-sheets and the core in the various panels appear in Table 1. The dimensions of the various components appear in Table 2. Panel A, which is used for comparison with the FE results, consists of a 20 mm long debonded region "without contact" at midspan. The face-sheets are modeled using the BEAM54 element with the offset option enabled, isotropic constitutive behavior, and negligible shear deformations. The PLANE42 four-node element is used for the core. The core is assumed to be linear, elastic, orthotropic with $G_{cyx} = G_{cyz} = 10^{-4}G_{cxz}$ and $E_{cxx} = E_{cyy} = 10^{-4}E_{czz}$, and under plane stress conditions. In the analytical solution, the deflection response is expanded using a truncated series with $N = 100$. The comparison of the natural frequencies determined by the FE analysis and the analytical model appears in Table 3 and, for reference, the natural frequencies for the fully bonded unidirectional panel are presented too in Table 3. The results of the proposed model compare very well with those of the FE results. Yet, some minor differences in the frequencies of the anti-symmetric modes are observed. These differences are attributed to the null longitudinal stiffness of the core in the analytical model versus the small but non-zero stiffness in the FE analysis and to the singularities that occur at the tips of the delaminated region and near the supports. In the FE model, these locations of stress concentration are

Table 1
Material properties of the core and the face-sheets

		Panel A	Panel B
Core	E_c (MPa)	50	85
	G_c (MPa)	21	16
	ρ_c (kg/m^3)	52	100
Face-sheets	$E_f = E_t$, E_b (MPa)	36,000	27,420
	ρ_t, ρ_b (kg/m^3)	4400	1630.9

Table 2
Geometrical properties of the core and the face-sheets

		Panel A	Panel B
Core	c (mm)	19.05	19.05
Face-sheets	d_t (mm)	0.5	0.5
	d_b (mm)	1.0	0.5
Panel	L (mm)	300	300
	b (mm)	20	20

Table 3
Natural frequencies (rad/sec) of fully bonded unidirectional panel and delaminated unidirectional panel ($l_d = 20, x_1 = 140$ mm)

	Delaminated panel			Fully bonded panel		
Mode	Present model[a]	FEA	Difference%	Present model	FEA	Difference %
ω_1	1815.7	1816.15	0.02	1817	1826.9	0.54
ω_2	2408.8	2574	6.45	4450.3	4465.27	0.33
ω_3	6869.0	6895	0.38	7001	7022.7	0.02
ω_4	7206.5	7238	0.44	9494.8	9520.9	0.27
ω_5	11129.9	11187.8	0.51	11952.2	11982.6	0.25

[a]including rotatory inertia.

associated with divergence of the stresses with refinement of the mesh whereas the analytical model is not affected by the singularities.

The next case that is used for verification consists of a symmetric unidirectional panel with a debonded region that asymptotically extends through the entire length of the panel. Here, the behavior of the delaminated panel without contact can be described by the response of two separate simply supported unidirectional panels with different masses. Notice that here, the mass of the core is carried only by the lower bonded face-sheet. Taking advantage of the symmetry of the panel, the response of the delaminated panel with contact may be described by that of two beams with equal mass properties for the symmetrical bending modes with $w_t = w_b$ and by that of two beams that rest on an elastic foundation for the anti-symmetric "pumping" models with $w_t = -w_b$. The first four natural frequencies and the corresponding vibration modes associated with a debonded region "with and without contact" are described in Fig. 3(a) and (b), respectively. The natural frequencies obtained by the proposed model and the analytical expressions of the natural frequencies for the asymptotic cases appear in Table 4. Notice that the first and third modes of the delaminated panel "without contact" correspond to the first and second modes of a beam that has the stiffness of the lower face-sheet and the combined mass of the core and the lower face-sheet. The second and fourth modes of a "without contact" delaminated panel correspond to the first and second modes of a panel with the stiffness and mass of the upper face-sheet alone. The minor differences between the natural frequencies predicted by the proposed model and the analytical results for the asymptotic cases are attributed to the rotatory inertia terms. Thus, the natural frequencies predicted by the proposed model (that accounts for the rotatory inertia terms) are slightly lower than the analytical ones of the asymptotic cases. The first four vibration modes of a unidirectional delaminated panel "with contact" appear in Fig. 3(b) and the corresponding natural frequencies appear in Table 4. These results reveal that all four modes are symmetric with respect to the mid-height surface of the panel with $w_t = w_b$. Thus, they coincide with the response of a simply supported beam that consists of the stiffness of one face sheet and half of the total mass of the sandwich panel. The first "pumping" mode and the first mode associated with longitudinal displacements of the face-sheets are characterized by significantly higher frequencies. Also here, the differences between the proposed model and the asymptotic model are due to the rotatory inertia terms. The influence of these terms becomes more prominent with the increase of the

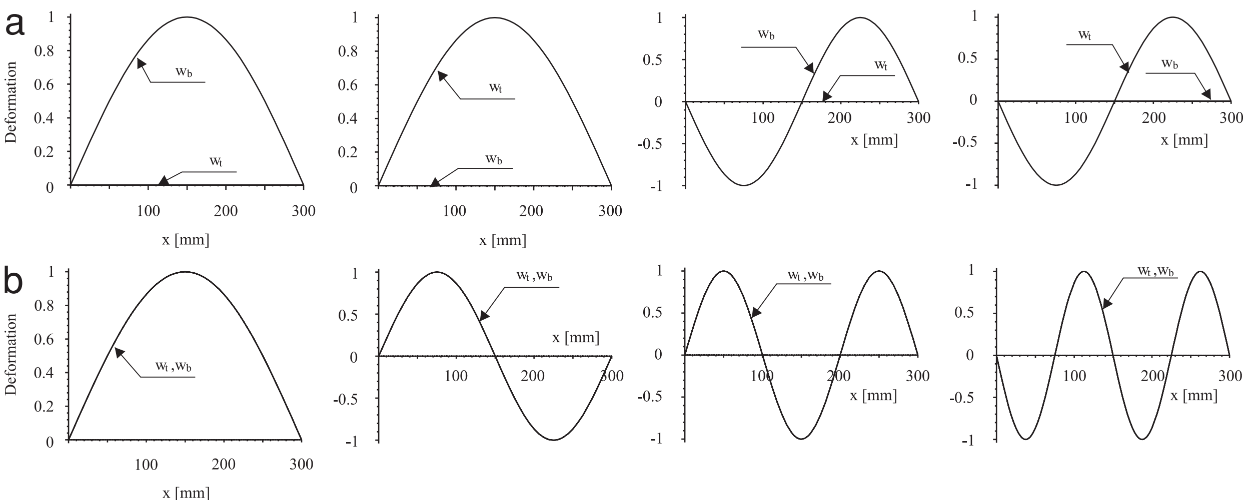

Fig. 3. Vibration modes for panel A with a debonded region extended through the entire length: (a) "without contact" model (b) "with contact" model.

Table 4
Natural frequencies (rad/s) of unidirectional panel with a debonded region that extends through the entire length ($l_d = 300\,\text{mm}$)—with and without contact, comparison between asymptotic cases and present model

	Delaminated region without contact		Delaminated region with contact	
Mode	Present model[a]	Asymptotic case	Present model[a]	Asymptotic case
ω_1	35.363	$\frac{\pi}{l}\sqrt{\frac{E_bI_b}{m_c+m_b}} = 35.533$	43.915	$\frac{\pi}{l}\sqrt{\frac{E_tI_t}{0.5m_c+m_t}} = 44.078$
ω_2	64.901	$\frac{\pi}{l}\sqrt{\frac{E_tI_t}{m_c+m_t}} = 64.902$	173.37	$\frac{4\pi}{l}\sqrt{\frac{E_tI_t}{0.5m_c+m_t}} = 176.31$
ω_3	139.46	$\frac{4\pi}{l}\sqrt{\frac{E_bI_b}{m_c+m_b}} = 142.13$	384.06	$\frac{9\pi}{l}\sqrt{\frac{E_tI_t}{0.5m_c+m_t}} = 396.70$
ω_4	259.60	$\frac{4\pi}{l}\sqrt{\frac{E_tI_t}{m_c+m_t}} = 259.61$	666.71	$\frac{16\pi}{l}\sqrt{\frac{E_tI_t}{0.5m_c+m_t}} = 702.63$

[a]including rotatory inertia.

"waviness" of the vibration mode. Hence, the differences observed in the forth frequency of the "with contact" model are the most notable ones (see Fig. 3(b) and Table 4).

Finally, we can state that the results of the proposed model compare well with those of the FEA and the asymptotic cases.

4. Numerical study

The free vibrations behavior of a simply supported delaminated unidirectional sandwich panel with a "soft" core is studied numerically. The first numerical example focuses on the capabilities of the model and examines the differences between the dynamic behavior of delaminated panels "with and without contact". Emphasis is placed on the vibration modes that are associated with the compressibility of the core and with relative out-of-plane displacements of the face-sheets. The ability of the model to describe the stress and displacement field modes along the unidirectional panel, and especially at the locations of stress concentrations, near the tip of the interfacial delamination crack, is examined too. Finally, a parametric study that investigates the influence of the length and location of the debonded region and the mechanical properties of the core is presented. The layouts of the two unidirectional panels considered appear in Fig. 1 and the geometrical and mechanical properties associated with each case are summarized in Tables 1 and 2, respectively.

4.1. "With Contact" and "Without Contact" models—comparison

The geometrical and mechanical properties considered here correspond to panel A, see Tables 1 and 2, respectively, and the delaminated region is located at mid span at the upper core–face interface. The solution uses truncated series expansion of the displacements along the panel with $N = 100$.

The first five modes and the corresponding natural frequencies associated with a delaminated length of $l_d = 10\,\text{mm}$ appear in Fig. 4. Notice, that $\bar{\omega}_i^{\text{wc}}$ and $\bar{\omega}_i^{\text{woc}}$ are the ith normalized natural frequencies of the "with and without contact" models, respectively. They are normalized with respect to the corresponding frequencies of the fully bonded unidirectional panel. The results reveal that the natural frequencies of the "with contact" case slightly differ from the ones of the "without contact" case. This effect is more prominent for the anti-symmetric modes, where the modes of the two models are almost identical but correspond to different natural frequencies. The modes in the case of a longer delaminated region ($l_d = 80\,\text{mm}$) appear in Fig. 5 and reveal that the response (modes and frequencies) of the "with contact" and "without contact" case are quiet different. This observation clarifies that in the cases associated with relatively long delaminated region, the assumption of contact conditions at the debonded interface critically affects the results.

The influence of the contact conditions at the debonded region on the free vibration of a unidirectional sandwich panel with a 20 mm long delaminated region at mid-span has been also compared for two different core materials. The first core is relatively stiff with $E_c = 0.095E_t$, while the second one is very soft with $E_c = 0.001E_t$. The mass densities and the shear moduli of the cores equal: $\rho_c\,(\text{kg/m}^3) = 1.0805E_c$ (E_c in MPa), $G_c = 0.4E_c$, respectively, see Ref. [40] and the geometrical properties are those of Panel A, see Tables 1 and 2. The first five natural frequencies and the corresponding vibration modes appear in Figs. 6 and 7 for the relatively stiff and for the very soft cores, respectively. The results reveal that for the panels with the stiff core, the differences between the natural frequencies of the "with contact" and "without contact" models are considerable, while in the case of unidirectional panels with the very soft core, the natural frequencies and corresponding vibration modes almost coincide. This effect is caused by the limited ability of the soft core to effectively transfer loads from one face-sheet to the other. Notice also that in the case of a relatively stiff core, the modes may change their order between the two models, see, for example, the fifth and sixth modes described in Fig. 6. Hence, it implies that in spite of the relatively small length of the delaminated regions, the contact conditions at the debonded interface affect the dynamic response of the panel.

The existence of modes involving a relative displacement between the two face-sheets ("pumping modes") is demonstrated in Ref. [32] for the fully bonded unidirectional panel and it occurs at the 32 mode number. However, when a delaminated region exists, pumping modes that correspond to lower frequencies may appear as a local phenomenon at the delaminated region. Furthermore, as illustrated in Fig. 5 for panel A with a mid-span delaminated region of 80 mm long, pumping modes are more likely to appear in cases where there is no contact between the delaminated surfaces. Another important factor that dominates the existence of the pumping modes at lower modes is the length of the debonded region. In panels with longer delaminated regions, the pumping modes correspond to lower natural frequencies. This relation is demonstrated in Figs. 4 and 5 where pumping modes do not exist within the lowest five vibration modes of panels with small delaminated region lengths, but appear when longer debonded regions are considered. Fig. 6 further reveals in that the case of panels with the relatively stiff cores ($E_c = 0.095E_t$), the pumping modes may occur in the lower modes, even for small delaminated region lengths. This is not the case when the panel is made of a very soft core ($E_c = 0.001E_t$), see Fig. 7.

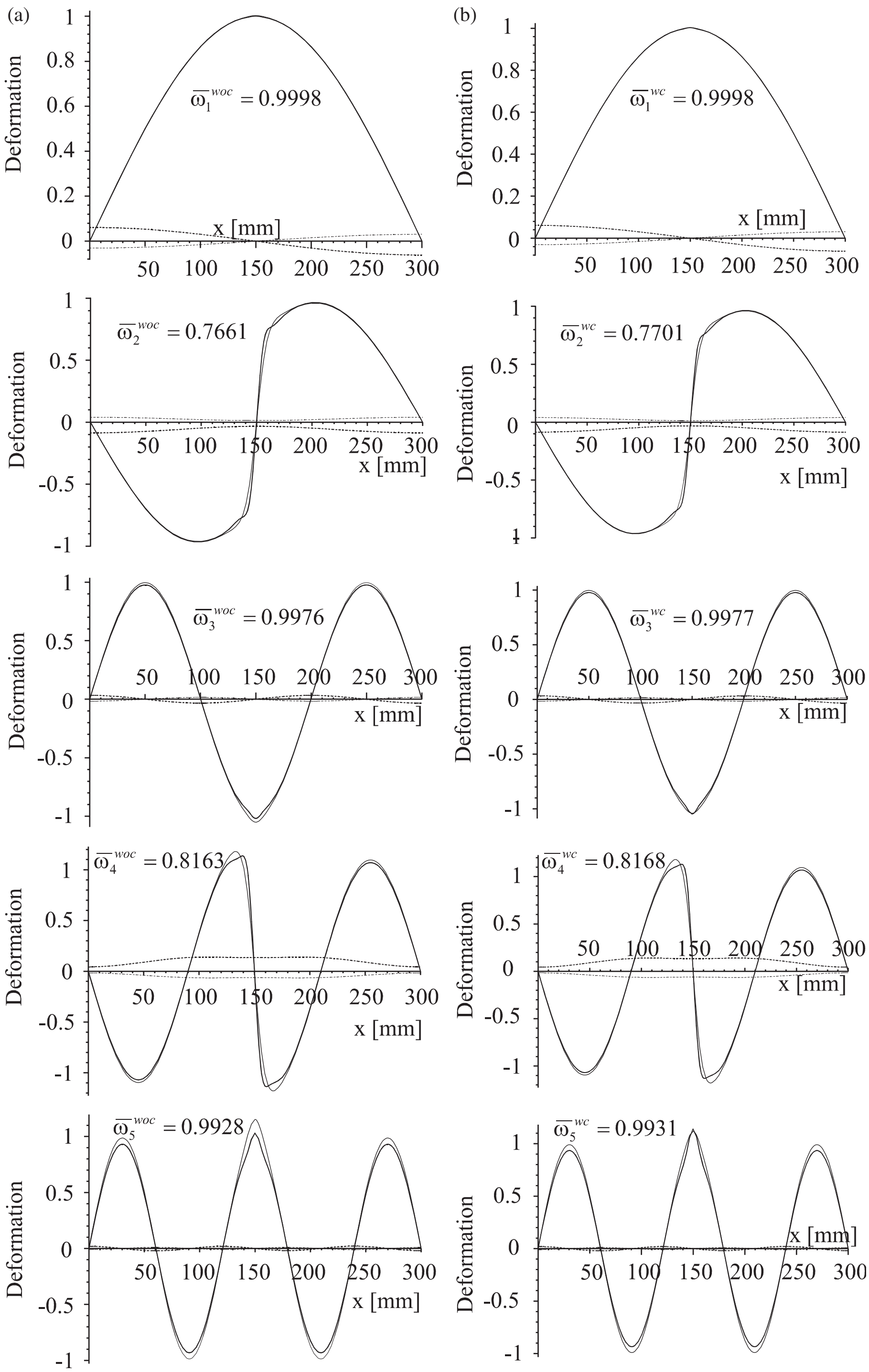

Fig. 4. First five modes and normalized natural frequencies for panel A with $l_d = 20$ mm at midspan: (a) ''without contact'' model (b) ''with contact'' model. Legend: ————— w_t, ————— w_b, -·-·-·- u_{ot}, ------------- u_{ob}.

One of the most important aspects of the structural response of the delaminated unidirectional sandwich panel is the stress concentrations that occur in the vicinity of the tips of the delaminated region. The vertical normal stresses mode at the upper core–face interface and the shear stresses in the core corresponding to

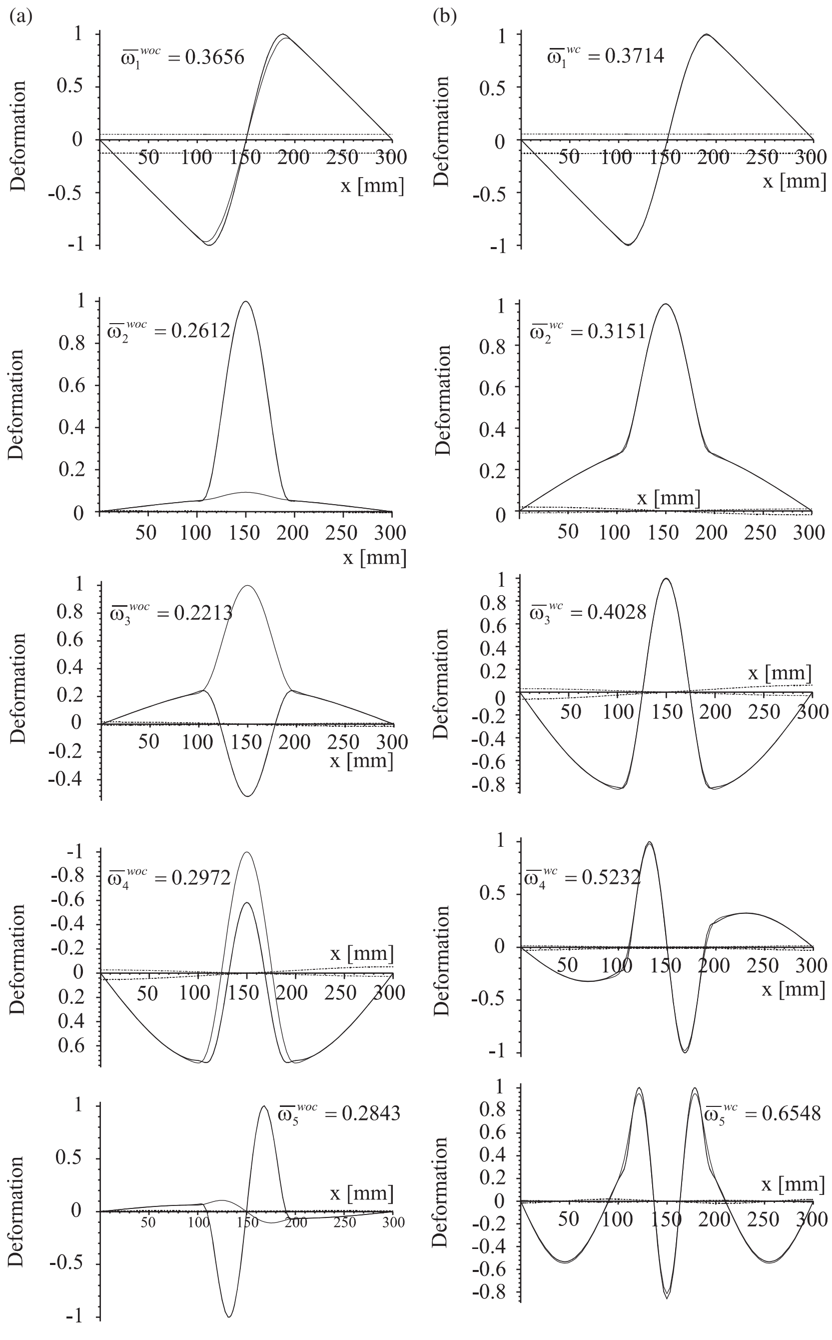

Fig. 5. First five modes and normalized natural frequencies for panel A with $l_d = 80$ mm at midspan: (a) "without contact" model (b) "with contact" model. Legend: ——— w_t, ——— w_b, -·-·-·- u_{ot}, ------- u_{ob}.

the first and second modes of panel A with $l_d = 80$ mm appear in Figs. 8 and 9, respectively. The results include the behavior associated with the "with contact" model and the "without contact" model and are compared with the fully bonded case. In all cases, the modes are normalized with respect to a unit maximal deflection (w_t, w_b, u_{ot} or u_{ob}). It is seen that the first mode of the fully bonded unidirectional panel is symmetric, see Fig. 8(a), but for the delaminated panel, with and without contact, it is anti-symmetric, see Figs. 8(d) and 8(g), respectively. An opposite behavior is detected for the second mode, see Fig. 9. Here, the

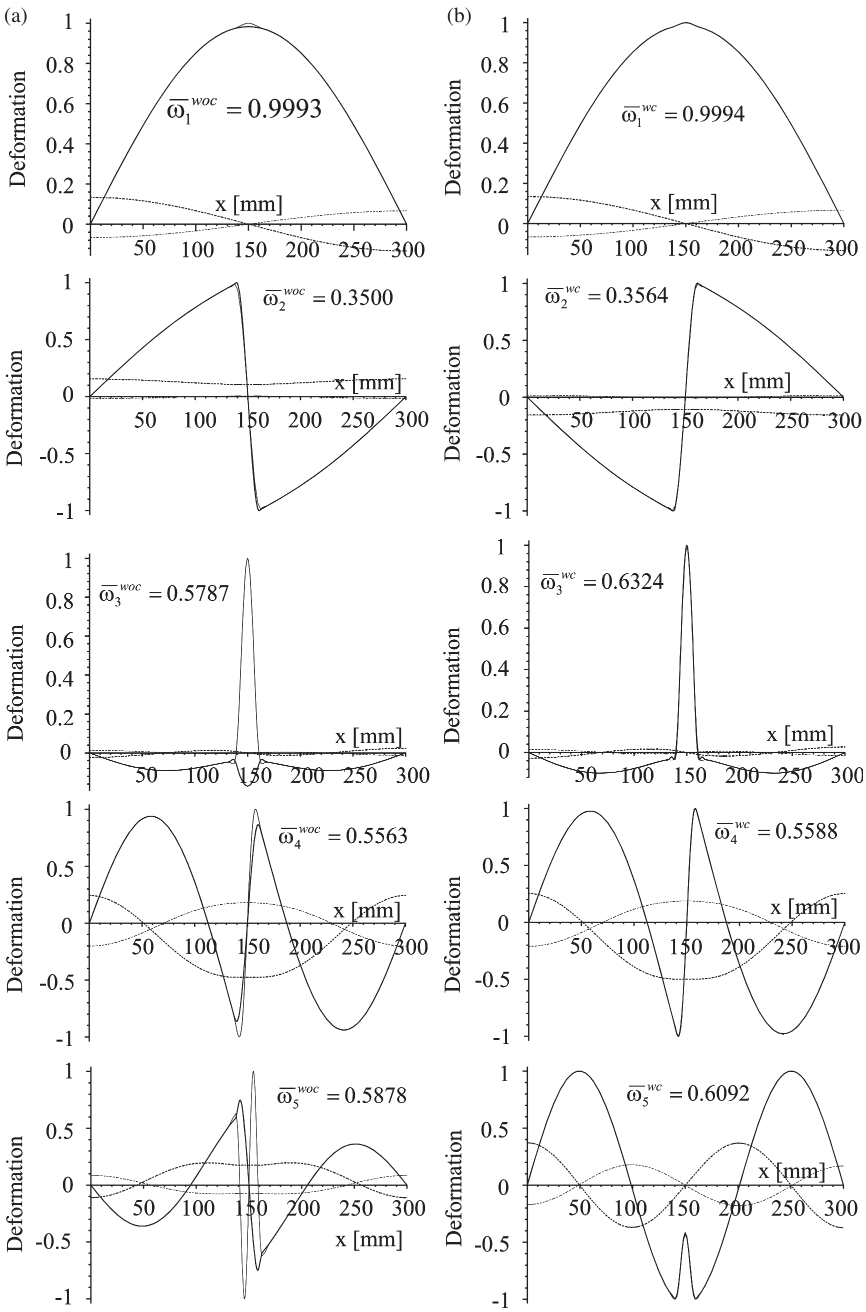

Fig. 6. First five modes and normalized natural frequencies for panel A with a soft core and with $l_d = 20$ mm at midspan: (a) "without contact" model (b) "with contact" model. Legend: ——— w_t, ——— w_b, - - - u_{ot}, - - - u_{ob}.

second mode of the fully bonded panel is an anti-symmetric one, see Fig. 9(a), while that of the delaminated panel is symmetric, see Figs. 9(d) and 9(g). In general, Figs. 8 and 9 reveal that the existence of a delaminated region leads to a change in the mode shapes, a sharp increase in the shear stresses in the vicinity of the tips of the debonded region, see Figs. 8(c), (f), and (i) and 9(c), (f), and (i), and the development of large, yet finite, vertical normal stresses, see Figs. 8(b), (e), and (h) and 9(b), (e), and (h).

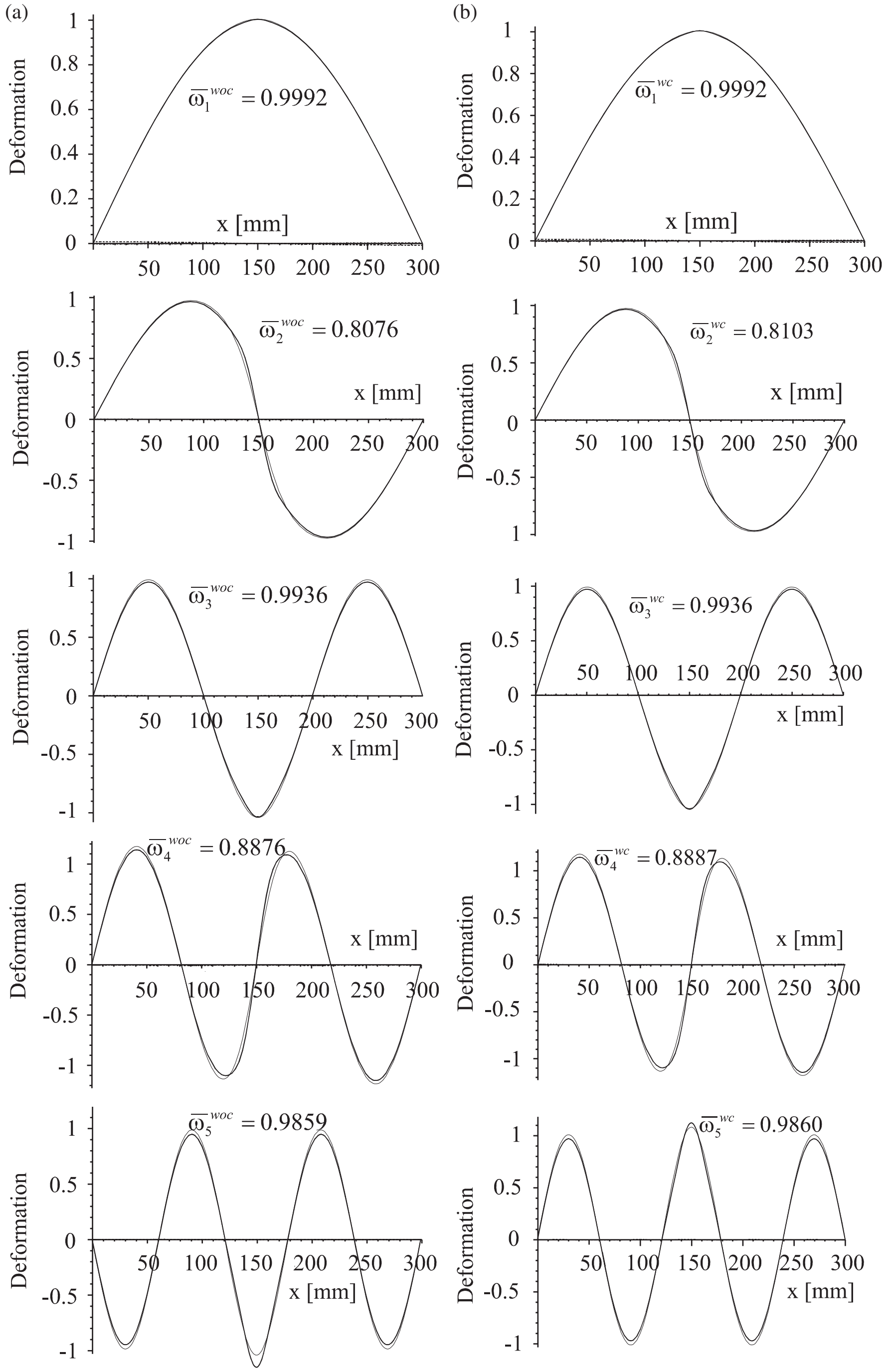

Fig. 7. First five modes and normalized natural frequencies for panel A with a stiff core and with $l_d = 20$ mm at midspan: (a) ''without contact'' model (b) ''with contact'' model. Legend: ———————— w_t, ———————— w_b, -·-·-—-·-·-—-·-·- u_{ot}, ------------------------ u_{ob}.

4.2. Parametric study

The parametric study investigates the influence of the delaminated region length, its location, and the mechanical properties of the core on the free vibrations behavior of a delaminated simply supported

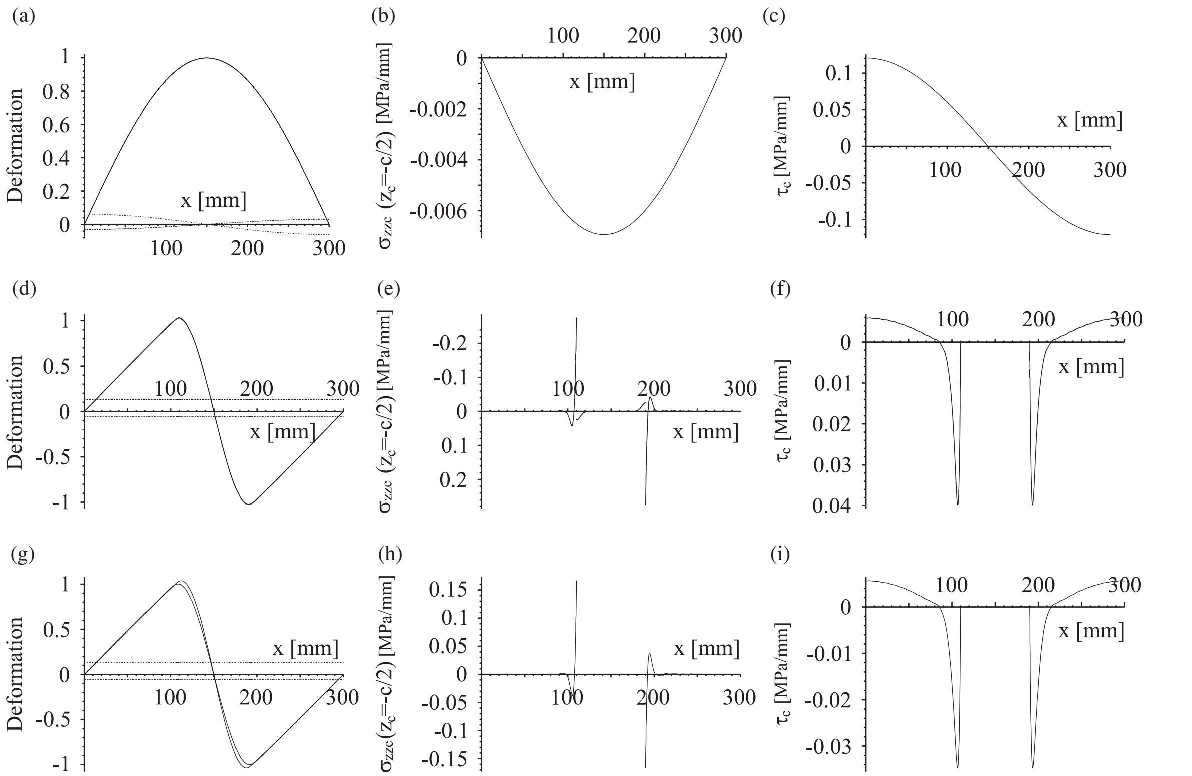

Fig. 8. First mode with shear and vertical normal stresses in fully bonded and delaminated unidirectional panels ($l_d = 80$ mm, delamination at midspan): fully bonded: (a) displacements; (b) vertical normal stresses; (c) shear stresses; delaminated unidirectional panel (with contact): (d) displacements; (e) vertical normal stresses; (f) shear stresses; delaminated unidirectional panel (without contact): (g) displacements; (h) vertical normal stresses; (i) shear stresses. Legend: ———— w_t, ———— w_b, - - - - - u_{ot}, - - - - - u_{ob}.

unidirectional sandwich panel. The influence of the various parameters on the differences between the two cases of debonded region "with and without contact" is also examined.

The influence of the length of the delaminated region on the first five natural frequencies of the "with contact" and "without contact" models appears in Fig. 10. The results reveal that, as expected, the natural frequencies drastically decrease as the delaminated length increases. Moreover, this effect is pronounced in the "without contact" case. Interestingly, the results in Fig. 10 indicate that the dependence of the frequency on the delaminated region length is not necessarily monotonic. This phenomenon is attributed to the changes in the shape of the vibration modes and the transition between symmetric and anti-symmetric modes for different delaminated lengths, see Fig. 5.

The influence of the location, x_d, of 20 mm long delaminated regions without contact along the panel on the first five natural frequencies appear in Fig. 11 (x_d, is the x coordinate of the left edge of the debonded region). The results reveal that the influence of the location of the delaminated region differs from one mode to the other. For example, the first natural frequency is mostly affected when the debonded region is located near the supports. On the other hand, the second natural frequency is mostly affected when the debonded region is located close to mid-span. The third frequency is affected when x_d is about one third of the panel's length. In general, the results indicate that the influence of the delaminated region is more prominent when it is found in the vicinity of locations that correspond to high shear stress in the core of the fully bonded panel. Notice, that the presence of the delaminated region critically damages the ability of the core to transfer shear and the collaboration between the two face sheets.

Finally, the influence of the mechanical properties of the core on the free vibrations of a unidirectional sandwich panel with a 20 mm long delaminated region at mid-span is investigated. The results for the "with

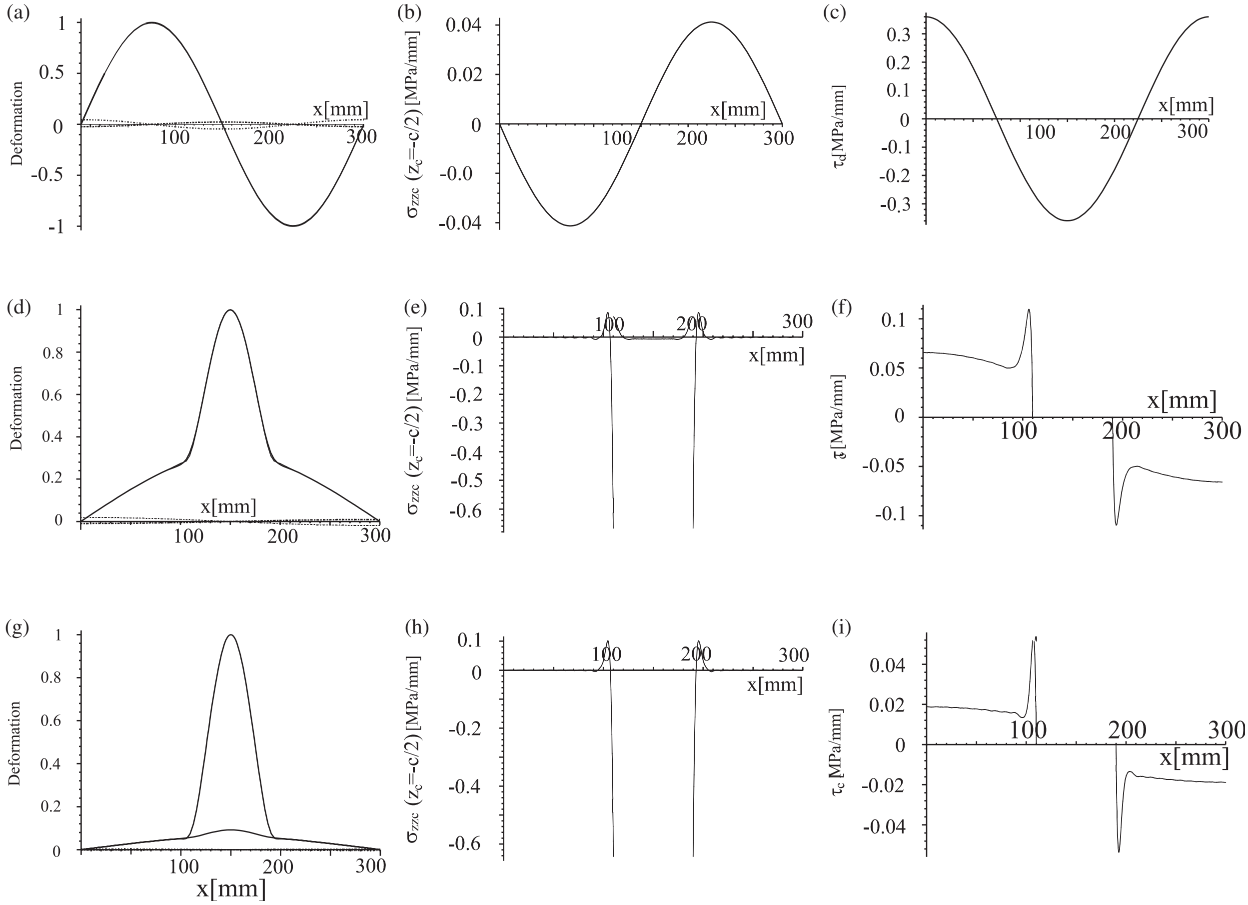

Fig. 9. Second mode with shear and vertical normal stresses in fully bonded and delaminated unidirectional panels ($l_d = 80$ mm, delamination at midspan): fully bonded: (a) displacements; (b) vertical normal stresses; (c) shear stresses; delaminated unidirectional panel (with contact): (d) displacements; (e) vertical normal stresses; (f) shear stresses; delaminated unidirectional panel (without contact): (g) displacements; (h) vertical normal stresses; (i) shear stresses. Legend: ———— w_t, ———— w_b, - - -—- - - u_{ot}, - - - - - - - - u_{ob}.

and without contact'' models are presented in Fig. 12. The non-dimensional frequencies are normalized with respect to the corresponding frequencies of the fully bonded panel A (with $E_c = 0.0525E_t$, see Table 1). The mass and shear modulus of the core equal: ρ_c (kg/m^3) $= 1.0805E_c$ (E_c in MPa), $G_c = 0.4E_c$, see Ref. [40]. The results are presented for values of E_c/E_f in the range of 0.0001 to 0.1, which covers the typical range of moduli ratios for cores made of foam or nomex honeycomb. The results in Fig. 12 reveal that for relatively flexible cores, the natural frequencies increases with the increase of the core moduli. On the other hand, for relatively stiff core materials, the opposite trend is observed. The comparison of the results of the ''with contact'' model to those of the ''without contact'' model, reveals that the first and second natural frequencies are similar in both models, whereas the other frequencies are different. This behavior is a result of the changes in the mode shapes and of the development of ''pumping'' vibration modes, see Fig. 6. The curves also reveal that the decrease in the value of the natural frequencies is more prominent for the stiffer cores.

5. Summary and conclusion

A consistent variational high-order approach and a modified Galerkin procedure for the free vibrations analysis of delaminated unidirectional sandwich panels with a flexible core have been presented. The mathematical formulation, which is based on Hamilton's variational principle and the high-order sandwich

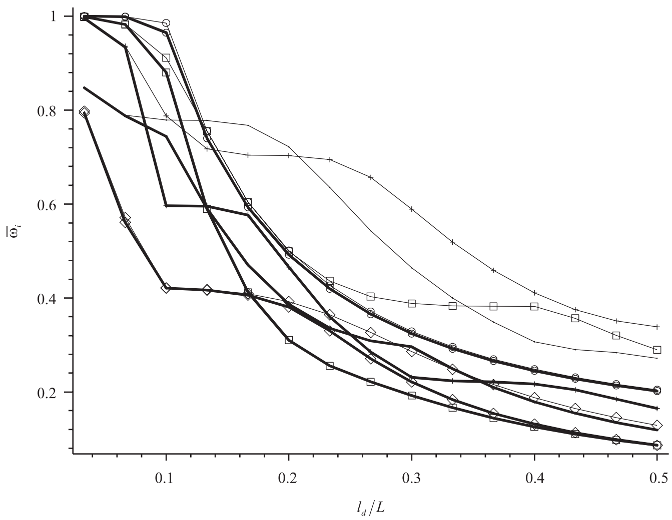

Fig. 10. First five natural frequencies versus delamination length (delamination located at mid span). Legend: ———-without contact, ———-with contact, ω_1-o—o—o—o, ω_2-◇—◇—◇—◇, ω_3-□—□—□—□, ω_4-———, ω_5-+—+—+—+.

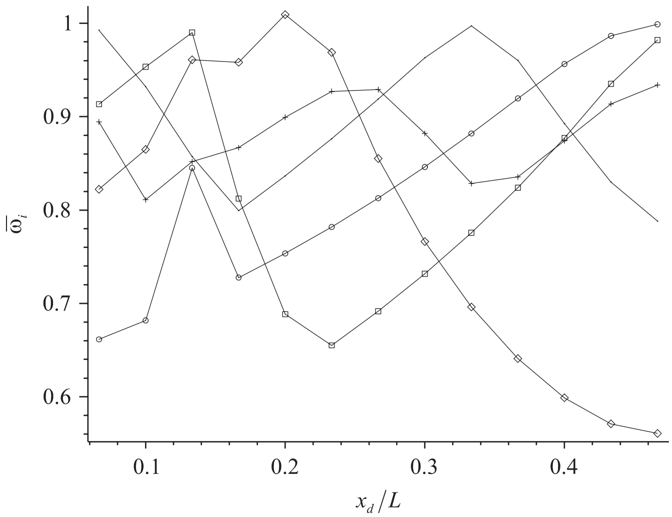

Fig. 11. First five natural frequencies versus the location of a 20 mm delamination ("with contact" model). Legend: ω_1-o—o—o—o, ω_2-◇—◇—◇—◇, ω_3-□—□—□—□, ω_4-———, ω_5-+—+—+—+.

panel theory approach, makes a distinction between bonded regions, debonded regions with contact and debonded regions without contact. The Hamilton principle is expressed in terms of displacements only, leading to a consistent variational formulation that avoids the use of a mixed stress/displacement formulation.

The free vibrations analysis has been solved by assuming a harmonic behavior in time, expanding the unknown displacements as a truncated series of admissible functions that satisfy at least the geometrical boundary conditions at the edges of the panel, and employing the modified Galerkin's method. Notice that with various trial functions, the proposed approach can be applied to any boundary conditions, including those which are different for each face sheet and the core at the same section.

The longitudinal and vertical displacements of the core have taken a cubic and quadratic polynomial variation through the height, respectively, in accordance with closed form analytical solutions of the static case. The compressibility and shear deformability of the core, which leads to nonlinear displacements, velocity, and acceleration fields in the core and allow the development of modes with relative displacement

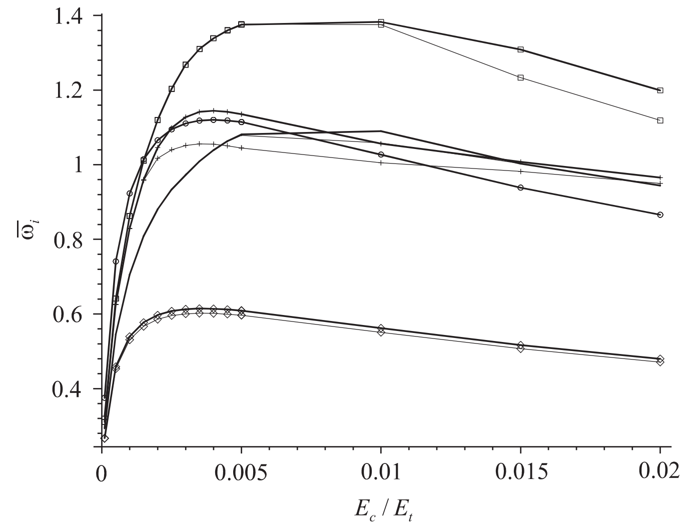

Fig. 12. First five natural frequencies (normalized with respect to the corresponding frequencies of the same unidirectional panel with the core of panel A), versus core–face elastic moduli ratio where $\rho_c\,(\mathrm{kg/m^3}) = 1.0805E_c\,(\mathrm{MPa})$ and $G_c = 0.4E_c$. Legend: ———-without contact, ———-with contact, ω_1-, ω_2-, ω_3-, ω_4-, ω_5-.

between the two face-sheets in the vertical and longitudinal directions, "pumping" modes, has also been considered. The natural frequencies and vibration modes determined by the proposed model have been verified through comparison with FE results as well as with special asymptotic cases.

The numerical study has focused on the natural frequencies, the vibration modes, and the corresponding stress field mode in the core. The results have revealed that the assumption with regards to the contact conditions at the debonded interface has a major influence on the natural frequencies and the vibration modes. It has been found that the contact conditions (either "with contact" or "without contact") at the delaminated surfaces has a significant influence in cases involving large delaminated regions or stiff core materials. In these cases, the different contact cases are associated with different mode shapes. The results have also demonstrated the ability of the model to reveal modes involving relative displacement between the two face-sheets, either in the vertical or the horizontal direction such as the "pumping" modes. Notice, that in the fully bonded panel, this mode occurs at higher mode number, while in the case of delaminated panels it occurs at lower modes. Hence, the ability to predict and quantify these effects, which is beyond the capabilities of models that use "equivalent" single layer, lamination, or the classical incompressible core theories, is essential for the analysis and assessment of delaminated sandwich panels. The numerical study has also described stress and displacement fields along the panel and, in particular, at points with high stresses, such as at the tip of the interfacial delaminatied crack. Contrary to the FEA, which has difficulties handling the singular points at the interfacial crack tip, the results of the proposed model demonstrate its ability to cope with the stress concentrations that develop near this point.

Finally, a parametric study that examines the influence of the length and the location of the debonded region and the mechanical properties of the core on the free vibrations behavior of delaminated unidirectional sandwich panel has been presented. The study has quantitatively revealed that the influence of the delaminated region on the dynamic characteristics is amplified as the length of the debonded region is increased. It has also been observed that the location of the debonded region dictates which modes are affected.

The model presented in this paper provides a general approach for the free vibrations analysis of the delaminated unidirectional sandwich panel. The "with contact" and "without contact" models can be used for the initial assessment of the range of response of the panel under realistic time and space-dependent contact conditions. Thus, the model developed here further provides a basis for an inclusive, fully nonlinear, and dynamic analysis that accounts for the complex contact conditions within the delaminated panel.

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