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On the natural vibrations of linear structures with constraints

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Abstract

The undamped natural vibrations of a constrained linear structure are given by the solutions to a generalized eigenvalue problem derived from the equations of motion for the constrained system involving Lagrangian multipliers. The eigenvalue problem derived is defined by the mass matrix of the unconstrained structure and a non-symmetric and singular stiffness matrix for the constrained system. The character of the solution of the eigenvalue problem of the constrained system is stated and proved in a Theorem. Applications of the constrained eigenvalue problem to some simple structures are demonstrated. Finally a condition for the calculation of the damped natural vibrations for the constrained structure in terms of the undamped mode shapes is formulated.

1. Introduction

One of the most elementary subjects that need to be considered by designers of today is the dynamic behavior of a structure exposed to an external excitation. It can be described by the modal properties of the structure which are characterized by the natural frequencies and the corresponding mode shapes.

If an analysis reveals that the structure has unwanted modal properties, there is a need to modify the structure in order to obtain desired modal properties. A solution to this problem, where changes in mass- or stiffness-properties aim at a change in the performance of the structural resonances, is referred to as a structural modification. One way to obtain such a modification may be to introduce constraints on the mechanical structure. This technique has been investigated by several authors using Lagrangian multipliers or other approaches, see for instance, Refs. [1–4] and references cited therein. A survey of this research field is given in the review article by Kerstens [5].

In this paper the Lagrangian multiplier technique is used. The undamped natural vibrations of a constrained linear structure are given by the solutions to a generalized eigenvalue problem derived from the equations of motion for the constrained system involving Lagrangian multipliers. The eigenvalue problem derived is defined by the mass matrix of the unconstrained structure and a non-symmetric and singular stiffness matrix for the constrained system.

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The main objective of this paper, an investigation of the consequences of the Lagrangian multiplier approach, is of a theoretical nature. The resulting dynamical equation (Eq. (4.10)) will certainly not be the most efficient formulation from the numerical point of view but it is a logical consequence of the Lagrangian approach and it may give some additional insight into the general character of the solution to the constrained problem. The non-symmetric and singular character of the obtained stiffness matrix is a consequence of the fact that the original coordinates are retained. This paper demonstrates how constrained modes are obtained by using certain projection operators associated with the so-called constraint matrix. The numerical examples submitted, serve as simple illustrations of the contents of Theorem 1.

The major contributions of this paper are the statements made in Theorems 1 and 2, together with their proofs. Statements similar to the content of Theorem 1 may be found in the literature, see Refs. [2,4], but the proof here is new and it displays in detail the mathematical structure of the solution to the constrained problem. Theorem 2, which is of some novelty, contains a necessary and sufficient condition for the calculation of the damped natural vibrations for the constrained structure in terms of the undamped mode shapes and it also opens for some new questions concerning the constrained damped vibration problem.

2. Notation

In this paper $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ the set of complex numbers. The set of n-dimensional, real column vectors is denoted by $\mathbb{R}^n \equiv \mathbb{R}^{n\times 1}$ and the null vector in $\mathbb{R}^n$ is written $\mathbf{0}_n$. $\mathbb{R}^{m\times n}$ denotes the set of real matrices of order $m \times n$ with the null matrix written $\mathbf{0}_{m\times n}$. If $\mathbf{A} \in \mathbb{R}^{m\times n}$ then $\mathbf{A}^{\mathrm{T}} \in \mathbb{R}^{n\times m}$ is the transpose of $\mathbf{A}$. The rank of a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ is written rank($\mathbf{A}$). If $\mathbf{A}$ is a square matrix, i.e., $n = m$, then det($\mathbf{A}$) denotes the determinant of $\mathbf{A}$ and if det($\mathbf{A}$)$\neq$0 then $\mathbf{A}^{-1}$ denotes its inverse. $\mathbf{I}_{n\times n}$ denotes the identity matrix in $\mathbb{R}^{n\times n}$. Let $\mathbf{A} \in \mathbb{R}^{m\times n}$, then the following linear spaces associated with $\mathbf{A}$ will be employed:

$$\mathrm{range}(\mathbf{A}) = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{x} = \mathbf{A}\mathbf{u},\ \mathbf{u} \in \mathbb{R}^m\}, \quad \mathrm{kernel}(\mathbf{A}) = \{\mathbf{u} \in \mathbb{R}^n | \ \mathbf{A}\mathbf{u} = \mathbf{0}_m\}.$$

If $\mathscr{V}$ is a linear subspace of $\mathbb{R}^n$ then the dimension of $\mathscr{V}$ is denoted dim($\mathscr{V}$) and the orthogonal complement of $\mathscr{V}$, $\mathscr{V}^{\perp}$, is defined by

$$\mathscr{V}^{\perp} = \left\{\mathbf{y} \in \mathbb{R}^n \middle| \mathbf{y}^{\mathrm{T}}\mathbf{x} = 0,\ \forall \mathbf{x} \in \mathscr{V}\right\}.$$

3. Preliminaries

Consider the free, undamped vibrations of an n-degree-of-freedom (dof) undamped mechanical structure. This is modelled by the system of linear, second-order differential equations

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}_n \tag{3.1}$$

with a mass matrix $\mathbf{M} \in \mathbb{R}^{n\times n}$ which, throughout this paper, is assumed to be symmetric and positive definite, and a stiffness matrix $\mathbf{K} \in \mathbb{R}^{n\times n}$ which is assumed to be symmetric and positive semi-definite. The configuration coordinates of the structure are given by the vector $\mathbf{q} = [q_1 q_2 \ldots q_n]^{\mathrm{T}} \in \mathbb{R}^n$, $\mathbf{q} = \mathbf{q}(t)$. A solution to Eq. (3.1) is given by

$$\mathbf{q} = \mathbf{x} \sin \omega t, \tag{3.2}$$

where $\mathbf{x}$ and ω satisfy the linear system of equations defining the generalized eigenvalue problem

$$(-\omega^2\mathbf{M} + \mathbf{K})\mathbf{x} = 0 \tag{3.3}$$

and $\mathbf{x} \in \mathbb{R}^n$ is a constant vector, here referred to as the mode shape.

The existence of non-trivial solutions, $\mathbf{x}\neq\mathbf{0}$, to Eq. (3.3) requires that the angular frequency ω satisfies the secular equation

$$\det(-\omega^2\mathbf{M} + \mathbf{K}) = 0, \tag{3.4}$$

where the roots of Eq. (3.4), the natural frequencies of the structure, and the corresponding mode shape vectors are denoted by

$$0 \leqslant \omega_1^2 \leqslant \omega_2^2 \leqslant \cdots \leqslant \omega_n^2 \quad \text{and} \quad \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n, \tag{3.5}$$

respectively, so that

$$(-\omega_i^2 \mathbf{M} + \mathbf{K})\mathbf{x}_i = \mathbf{0}, \quad i = 1, 2, \ldots, n \tag{3.6}$$

is satisfied. The mode shapes $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ may be chosen as linearly independent, satisfying $\mathbf{x}_i^{\mathrm{T}}\mathbf{M}\mathbf{x}_j = 0$, $i \neq j$. It is convenient to assemble the mode shape vectors in the non-singular modal matrix $\mathbf{X} = [\mathbf{x}_1\ \mathbf{x}_2 \ldots \mathbf{x}_n]$, and the corresponding natural frequencies in the diagonal spectral matrix $\mathbf{\Omega}^2 = \mathrm{diag}(\omega_1^2\ \omega_2^2 \ldots \omega_n^2)$. A normalization of the modal matrix with respect to the mass matrix is obtained using the requirement $\mathbf{X}^{\mathrm{T}}\mathbf{M}\mathbf{X} = \mathbf{I}_{n\times n}$ nd then $\mathbf{X}^{\mathrm{T}}\mathbf{K}\mathbf{X} = \mathbf{\Omega}^2$. We will subsequently refer to the structure discussed above as the *unconstrained structure*.

4. The constrained structure

A set of *linear constraints* on the unconstrained structure is now introduced. These are defined by the following m, $1 \leqslant m \leqslant n$, independent linear equations

$$\begin{cases} a_{11}q_1 + a_{12}q_2 + \cdots + a_{1n}q_n = 0, \\ a_{21}q_1 + a_{22}q_2 + \cdots + a_{2n}q_n = 0, \\ \qquad\vdots \\ a_{m1}q_1 + a_{m2}q_2 + \cdots + a_{mn}q_n = 0, \end{cases} \tag{4.1}$$

or in more compact notation

$$\mathbf{A}\mathbf{q} = \mathbf{0}_m, \tag{4.2}$$

where the constant matrix $\mathbf{A} = [a_{ij}] \in \mathbb{R}^{m\times n}$ is assumed to be of full rank, i.e. $\mathrm{rank}(\mathbf{A}) = m$. The configuration space of the constrained system is the linear subspace $\mathscr{C}$ in $\mathbb{R}^n$, defined by $\mathscr{C} = \mathrm{kernel}(\mathbf{A}) = (\mathrm{range}(\mathbf{A}^{\mathrm{T}}))^{\perp}$ where $\dim(\mathscr{C}) = k = n - m$.

In this paper the problem of calculating the natural vibrations of a mechanical structure defined by the mass matrix $\mathbf{M}$ and the stiffness matrix $\mathbf{K}$ and subjected to the constraints defined by Eq. (4.2) will be studied. A direct approach to this problem would be to solve for m of the coordinates $q_1, q_2, \ldots, q_n$ in terms of the remaining $n-m$ and thereafter calculate the mass and stiffness matrices corresponding to the reduced structure described by these $k = n - m$ independent configuration coordinates. This could, for instance, be done by a rearrangement of the q-coordinates so that the constraint matrix $\mathbf{A}$ may be written $\mathbf{A} = [\mathbf{A}_1\mathbf{A}_2] \in \mathbb{R}^{m\times n}$, c.f. Ref. [6], where $\mathbf{A}_1 \in \mathbb{R}^{m\times m}$ is non-singular and $\mathbf{A}_2 \in \mathbb{R}^{m\times k}$. This is always possible since $\mathbf{A}$ has full rank. The constraint condition may then be written

$$\mathbf{A}\mathbf{q} = [\mathbf{A}_1\mathbf{A}_2]\begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix} = \mathbf{0}, \tag{4.3}$$

where $\mathbf{q}_1 \in \mathbb{R}^m$ and $\mathbf{q}_2 \in \mathbb{R}^k$. This implies

$$\mathbf{M}_r\ddot{\mathbf{q}}_2 + \mathbf{K}_r\mathbf{q}_2 = 0, \tag{4.4}$$

where the mass- and stiffness-matrices of the reduced system are defined by

$$\mathbf{M}_r = \mathbf{R}^{\mathrm{T}}\mathbf{M}\mathbf{R} \in \mathbb{R}^{k\times k}, \quad \mathbf{K}_r = \mathbf{R}^{\mathrm{T}}\mathbf{K}\mathbf{R} \in \mathbb{R}^{k\times k} \quad \text{and} \quad \mathbf{R} = \begin{bmatrix} -\mathbf{A}_1^{-1}\mathbf{A}_2 \\ \mathbf{I}_{k\times k} \end{bmatrix}. \tag{4.5}$$

The matrix $\mathbf{M}_r$ is, obviously, symmetric and positive definite, and $\mathbf{K}_r$ is symmetric and positive semi-definite.

In the following discussion, however, we will stay with the original set of generalized coordinates $\mathbf{q}$. The equations of motion for the constrained system may then be written as

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}, \tag{4.6}$$

where Lagrangian multipliers $\boldsymbol{\lambda} = \boldsymbol{\lambda}(t) = [\lambda_1\lambda_2 \ldots \lambda_m]^{\mathrm{T}} \in \mathbb{R}^{m\times 1}$ have been introduced, cf. Ref. [7]. From Eqs. (4.2) and (4.6) it follows that

$$\mathbf{0}_m = \mathbf{A}\ddot{\mathbf{q}} = -\mathbf{A}\mathbf{M}^{-1}\mathbf{K}\mathbf{q} + \boldsymbol{\Gamma}\boldsymbol{\lambda}, \tag{4.7}$$

where

$$\boldsymbol{\Gamma} = \mathbf{A}\mathbf{M}^{-1}\mathbf{A}^{\mathrm{T}} \in \mathbb{R}^{m\times m} \tag{4.8}$$

is a symmetric and positive definite matrix. The symmetry is obvious and the positive definiteness follows from the following argument. If $\mathbf{u} \in \mathbb{R}^m$, then

$$\mathbf{u}^{\mathrm{T}}\boldsymbol{\Gamma}\mathbf{u} = (\mathbf{A}^{\mathrm{T}}\mathbf{u})^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{u} > 0 \Leftrightarrow \mathbf{A}^{\mathrm{T}}\mathbf{u} \neq \mathbf{0} \Leftrightarrow \mathbf{u} \neq \mathbf{0}$$

since $\mathbf{A}$ has full rank and $\mathbf{M}^{-1}$ is positive definite. Since $\boldsymbol{\Gamma}$ is non-singular, $\boldsymbol{\lambda}$, in Eq. (4.6), can be obtained from Eq. (4.8), i.e.

$$\boldsymbol{\lambda} = \boldsymbol{\Gamma}^{-1}\mathbf{A}\mathbf{M}^{-1}\mathbf{K}\mathbf{q}. \tag{4.9}$$

Substituting this into Eq. (4.6), gives the following equation:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}_c\mathbf{q} = \mathbf{0}, \tag{4.10}$$

where

$$\mathbf{K}_c = \mathbf{Q}\mathbf{K} \tag{4.11}$$

is the constrained stiffness matrix and

$$\mathbf{Q} = \mathbf{I}_{n\times n} - \mathbf{P}, \tag{4.12}$$

where

$$\mathbf{P} = \mathbf{A}^{\mathrm{T}}\boldsymbol{\Gamma}^{-1}\mathbf{A}\mathbf{M}^{-1}. \tag{4.13}$$

Remark 1. Note that if the mechanical structure is *completely constrained*, i.e. if $m = n$, then $\mathbf{A} \in \mathbb{R}^{n\times n}$ is non-singular, and from Eqs. (4.8) and (4.12) $\mathbf{P} = \mathbf{I}_{n\times n}$ and, then, $\mathbf{K}_c = \mathbf{0}_{n\times n}$.

Remark 2. An equation similar to Eq. (4.10) is presented in Refs. [2,4].

Proposition 1. $\mathbf{P}$ *and* $\mathbf{Q}$ *are projections and* $\mathbf{P}\neq\mathbf{0}$. $\mathrm{range}(\mathbf{A}^{\mathrm{T}})$ *is a linear subspace of* $\mathrm{range}(\mathbf{P})$.

Proof. From Eqs. (4.12) and (4.8) $\mathbf{P} = \mathbf{A}^{\mathrm{T}}\boldsymbol{\Gamma}^{-1}\mathbf{A}\mathbf{M}^{-1} = \mathbf{A}^{\mathrm{T}}(\mathbf{A}\mathbf{M}^{-1}\mathbf{A}^{\mathrm{T}})^{-1}\mathbf{A}\mathbf{M}^{-1}$ and, then

$$\mathbf{P}^2 = \mathbf{A}^{\mathrm{T}}(\mathbf{A}\mathbf{M}^{-1}\mathbf{A}^{\mathrm{T}})^{-1}\mathbf{A}\mathbf{M}^{-1}\mathbf{A}^{\mathrm{T}}(\mathbf{A}\mathbf{M}^{-1}\mathbf{A}^{\mathrm{T}})^{-1}\mathbf{A}\mathbf{M}^{-1} = \mathbf{A}^{\mathrm{T}}(\mathbf{A}\mathbf{M}^{-1}\mathbf{A}^{\mathrm{T}})^{-1}\mathbf{A}\mathbf{M}^{-1} = \mathbf{P}$$

and

$$\mathbf{Q}^2 = (\mathbf{I}_{n\times n} - \mathbf{P})^2 = \mathbf{I}_{n\times n} - 2\mathbf{P} + \mathbf{P}^2 = \mathbf{I}_{n\times n} - \mathbf{P} = \mathbf{Q}$$

demonstrating that $\mathbf{P}$ and $\mathbf{Q}$ are projections. For the second part of the Proposition, take $\mathbf{x} \in \mathrm{range}(\mathbf{A}^{\mathrm{T}})$ then $\mathbf{x} = \mathbf{A}^{\mathrm{T}}\mathbf{u}$, $\mathbf{u} \in \mathbb{R}^m$ and

$$\mathbf{P}\mathbf{x} = \mathbf{P}\mathbf{A}^{\mathrm{T}}\mathbf{u} = \mathbf{A}^{\mathrm{T}}(\mathbf{A}\mathbf{M}^{-1}\mathbf{A}^{\mathrm{T}})^{-1}\mathbf{A}\mathbf{M}^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{u} = \mathbf{A}^{\mathrm{T}}\mathbf{u} = \mathbf{x}$$

and consequently, $\mathbf{x} \in \mathrm{range}(\mathbf{P})$. Since $\mathrm{rank}(\mathbf{A}^{\mathrm{T}}) = m \geqslant 1$ it is concluded that $\mathbf{P}\neq\mathbf{0}$. $\square$

Proposition 2. $\mathbf{K}_c$ *is singular and, in general, non-symmetric.*

Proof. From Eq. (4.11) it follows that $\det(\mathbf{K}_c) = \det(\mathbf{Q})\det(\mathbf{K}) = 0$ since $\mathbf{Q}$ is a projection, not equal to the identity, and then obviously $\det(\mathbf{Q}) = 0$. The non-symmetry is obtained by the following argument; Let $\mathbf{x}_i$ and $\mathbf{x}_j$ be arbitrary mode shape vectors of the unconstrained structure. Then

$$\begin{aligned}\mathbf{x}_j^{\mathrm{T}}\mathbf{K}_c\mathbf{x}_i &= \mathbf{x}_j^{\mathrm{T}}\mathbf{K}\mathbf{x}_i - \mathbf{x}_j^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\boldsymbol{\Gamma}^{-1}\mathbf{A}\mathbf{M}^{-1}\mathbf{K}\mathbf{x}_i = \mathbf{x}_j^{\mathrm{T}}\mathbf{K}\mathbf{x}_i - \omega_i^2\mathbf{x}_j^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\boldsymbol{\Gamma}^{-1}\mathbf{A}\mathbf{x}_i \\ &\neq \mathbf{x}_i^{\mathrm{T}}\mathbf{K}\mathbf{x}_j - \omega_j^2\mathbf{x}_i^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\boldsymbol{\Gamma}^{-1}\mathbf{A}\mathbf{x}_j = \mathbf{x}_i^{\mathrm{T}}\mathbf{K}\mathbf{x}_j - \mathbf{x}_i^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\boldsymbol{\Gamma}^{-1}\mathbf{A}\mathbf{M}^{-1}\mathbf{K}\mathbf{x}_j = \mathbf{x}_i^{\mathrm{T}}\mathbf{K}_c\mathbf{x}_j,\end{aligned}$$

since in general, $\omega_i^2 \neq \omega_j^2,\ i \neq j$. □

The projection $\mathbf{P}$ is not orthogonal but, according to Proposition 3 below, an orthogonal projection may be obtained through a similarity transformation of $\mathbf{P}$, using the mass matrix.

Proposition 3. $\boldsymbol{\Pi} = \mathbf{M}^{-1/2}\mathbf{P}\mathbf{M}^{1/2}$ *is an orthogonal projection and*

$$\mathrm{range}(\boldsymbol{\Pi}) = \mathrm{range}(\mathbf{M}^{-1/2}\mathbf{A}^{\mathrm{T}}). \tag{4.14}$$

The following result, cf. Heath [8], leads to a proof of Proposition 3.

Lemma 1. *Let* $\mathbf{L} \in \mathbb{R}^{n\times m}$ *be a linear mapping. Then*

$$\mathbf{L}(\mathbf{L}^{\mathrm{T}}\mathbf{L})^{-1}\mathbf{L}^{\mathrm{T}} \in \mathbb{R}^{n\times n} \tag{4.15}$$

is an orthogonal projection and $\mathrm{range}(\mathbf{L}(\mathbf{L}^{\mathrm{T}}\mathbf{L})^{-1}\mathbf{L}^{\mathrm{T}}) = \mathrm{range}(\mathbf{L})$.

Proof of Lemma 1. Obviously $\mathbf{L}(\mathbf{L}^{\mathrm{T}}\mathbf{L})^{-1}\mathbf{L}^{\mathrm{T}}$ is symmetric and

$$(\mathbf{L}(\mathbf{L}^{\mathrm{T}}\mathbf{L})^{-1}\mathbf{L}^{\mathrm{T}})^2 = \mathbf{L}(\mathbf{L}^{\mathrm{T}}\mathbf{L})^{-1}\mathbf{L}^{\mathrm{T}}\mathbf{L}(\mathbf{L}^{\mathrm{T}}\mathbf{L})^{-1}\mathbf{L}^{\mathrm{T}} = \mathbf{L}(\mathbf{L}^{\mathrm{T}}\mathbf{L})^{-1}\mathbf{L}^{\mathrm{T}}.$$

Assume that $\mathbf{x} \in \mathrm{range}(\mathbf{L})$. Then $\mathbf{x} = \mathbf{L}\mathbf{u}$ for some $\mathbf{u} \in \mathbb{R}^m$ and

$$\mathbf{L}(\mathbf{L}^{\mathrm{T}}\mathbf{L})^{-1}\mathbf{L}^{\mathrm{T}}\mathbf{x} = \mathbf{L}(\mathbf{L}^{\mathrm{T}}\mathbf{L})^{-1}\mathbf{L}^{\mathrm{T}}\mathbf{L}\mathbf{u} = \mathbf{L}\mathbf{u} = \mathbf{x} \Rightarrow \mathbf{x} \in \mathrm{range}(\mathbf{L}(\mathbf{L}^{\mathrm{T}}\mathbf{L})^{-1}\mathbf{L}^{\mathrm{T}}).$$

On the other hand, assume that $\mathbf{x} \in \mathrm{range}(\mathbf{L}(\mathbf{L}^{\mathrm{T}}\mathbf{L})^{-1}\mathbf{L}^{\mathrm{T}})$. Then

$$\mathbf{L}(\mathbf{L}^{\mathrm{T}}\mathbf{L})^{-1}\mathbf{L}^{\mathrm{T}}\mathbf{x} = \mathbf{x} \Rightarrow \mathbf{x}^{\mathrm{T}} = \mathbf{x}^{\mathrm{T}}\mathbf{L}(\mathbf{L}^{\mathrm{T}}\mathbf{L})^{-1}\mathbf{L}^{\mathrm{T}},$$

and if $\mathbf{y} \in \mathrm{kernel}(\mathbf{L}^{\mathrm{T}})$ then $\mathbf{x}^{\mathrm{T}}\mathbf{y} = \mathbf{x}^{\mathrm{T}}\mathbf{L}(\mathbf{L}^{\mathrm{T}}\mathbf{L})^{-1}\mathbf{L}^{\mathrm{T}}\mathbf{y} = 0$, which means that

$$\mathbf{x} \in (\mathrm{kernel}(\mathbf{L}^{\mathrm{T}}))^{\perp} = \mathrm{range}(\mathbf{L})$$

and this concludes the proof of the Lemma. □

Proof of Proposition 3. By taking $\mathbf{L} = (\mathbf{A}\mathbf{M}^{-1/2})^{\mathrm{T}}$ the projection $\boldsymbol{\Pi}$ may be written as

$$\begin{aligned}\boldsymbol{\Pi} &= \mathbf{M}^{-1/2}\mathbf{P}\mathbf{M}^{1/2} = \mathbf{M}^{-1/2}\mathbf{A}^{\mathrm{T}}\boldsymbol{\Gamma}^{-1}\mathbf{A}\mathbf{M}^{-1/2} = (\mathbf{A}\mathbf{M}^{-1/2})^{\mathrm{T}}(\mathbf{A}\mathbf{M}^{-1/2}(\mathbf{A}\mathbf{M}^{-1/2})^{\mathrm{T}})^{-1}\mathbf{A}\mathbf{M}^{-1/2} \\ &= \mathbf{L}(\mathbf{L}^{\mathrm{T}}\,\mathbf{L})^{-1}\mathbf{L}^{\mathrm{T}}\end{aligned}$$

and the Proposition follows from Lemma 1. □

The free vibrations, according to Eq. (3.2), of the constrained structure are determined by the eigenvalue problem

$$(-\omega_c^2\mathbf{M} + \mathbf{K}_c)\mathbf{x} = \mathbf{0} \tag{4.16}$$

and the constraint condition

$$\mathbf{A}\mathbf{x} = \mathbf{0}_m. \tag{4.17}$$

Note that $\mathbf{0}_m = \mathbf{A}\mathbf{q} = \mathbf{A}\mathbf{x} \sin \omega t, \ \forall t, \ \omega \neq 0 \Leftrightarrow \mathbf{A}\,\mathbf{x} = \mathbf{0}_m$. In the case of a completely constrained structure, $\mathbf{K}_c = \mathbf{0}_{n\times n}$, and condition (4.16) gives

$$-\omega_c^2 \mathbf{M}\mathbf{x} = \mathbf{0}, \tag{4.18}$$

with the conclusion that if $\mathbf{x} \neq \mathbf{0}$ then $\omega_c^2 = 0$. This corresponds to a rigid body mode and the solution to (4.10) may be written

$$\mathbf{q} = \mathbf{a}t + \mathbf{b}. \tag{4.19}$$

However, the constraint condition requires that $\mathbf{A}\mathbf{q} = \mathbf{0}$ and since $\mathbf{A}$, in this case, is non-singular it trivially follows that $\mathbf{a} = \mathbf{b} = \mathbf{0}$.

Let the solution to the eigenvalue problem (4.16), i.e. the natural frequencies of the constrained structure and the corresponding mode shapes, be denoted by

$$\omega_{c,1}^2, \omega_{c,2}^2, \ldots, \omega_{c,n}^2 \quad \text{and} \quad \mathbf{x}_{c,1}, \mathbf{x}_{c,2}, \ldots, \mathbf{x}_{c,n}, \tag{4.20}$$

respectively. The mode shape vectors of the constrained structure may be collected in the modal matrix $\mathbf{X}_c = [\mathbf{x}_{c,1}\ \mathbf{x}_{c,2} \ldots \mathbf{x}_{c,n}]$ and the corresponding natural frequencies in the spectral matrix $\mathbf{\Omega}_c^2 = \mathrm{diag}(\omega_{c,1}^2\ \omega_{c,2}^2 \ldots \omega_{c,n}^2)$.

Theorem 1. *If* $\mathbf{K}$ *is positive definite then the natural frequencies of the constrained structure,* $\omega_{c,i}^2$, $1 \leqslant i \leqslant n$, *are real and non-negative. The m first natural frequencies are equal to zero*

$$\omega_{c,1}^2 = \omega_{c,2}^2 = \cdots = \omega_{c,m}^2 = 0 \tag{4.21}$$

and the following k natural frequencies are positive

$$0 < \omega_{c,m+1}^2 \leqslant \omega_{c,m+2}^2 \leqslant \cdots \leqslant \omega_{c,n}^2. \tag{4.22}$$

The modal matrix $\mathbf{X}_c$ *is non-singular, and if the natural frequencies in Eq.* (4.22) *all are separated, i.e.*

$$\omega_{c,m+1}^2 < \omega_{c,m+2}^2 < \cdots < \omega_{c,n}^2 \tag{4.23}$$

then

$$\mathbf{X}_c^{\mathrm{T}} \mathbf{M} \mathbf{X}_c = \begin{bmatrix} \mathbf{G} & \mathbf{H}^{\mathrm{T}} \\ \mathbf{H} & \mathbf{I}_{k\times k} \end{bmatrix} \tag{4.24}$$

where $k = n - m$, $\mathbf{G} \in \mathbb{R}^{m\times m}$ *is symmetric and* $\mathbf{H} \in \mathbb{R}^{k\times m}$. *Furthermore*

$$\mathbf{A}\mathbf{x}_{c,i} \neq \mathbf{0}, \ \ i = 1, \ldots, m \ \ \textit{and} \ \ \mathbf{A}\mathbf{x}_{c,m+i} = \mathbf{0}, \ \ i = 1, \ldots, k, \tag{4.25}$$

i.e. the mode shape vectors $\mathbf{x}_{c,m+1}, \mathbf{x}_{c,m+2}, \ldots, \mathbf{x}_{c,n}$ *satisfy the constraint condition but the mode shape vectors* $\mathbf{x}_{c,1}, \mathbf{x}_{c,2}, \ldots, \mathbf{x}_{c,m}$ *do not.*

Proof. By performing the coordinate transformation $\mathbf{q} = \mathbf{M}^{-1/2}\mathbf{z}$ the equations of motion (4.10) may be written

$$\ddot{\mathbf{z}} + \mathbf{K}_{M,c}\mathbf{z} = \mathbf{0}, \tag{4.26}$$

where

$$\mathbf{K}_{M,c} = \mathbf{\Phi}\mathbf{K}_M, \tag{4.27}$$

$\mathbf{\Phi} = \mathbf{I} - \mathbf{\Pi}$, $\mathbf{K}_M = \mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}$. Assuming a solution $\mathbf{z} = \mathbf{z}(t) = \mathbf{w} \sin \omega t$ the necessary condition

$$(-\omega_c^2 \mathbf{I} + \mathbf{K}_{M,c})\mathbf{w} = \mathbf{0} \tag{4.28}$$

is obtained, where $\mathbf{w}\neq\mathbf{0}$ is a constant vector in $\mathbb{R}^n$. From Eq. (4.17) we have the constraint requirement $\mathbf{A}\mathbf{M}^{-1/2}\mathbf{w}=\mathbf{0}_m$. The secular equation, corresponding to Eq. (4.28), reads

$$\det(-\omega_c^2\mathbf{I}+\mathbf{K}_{M,c})=0. \tag{4.29}$$

Since $\det(\mathbf{K}_{M,c})=0$, $\omega_c^2=0$ is a solution to Eq. (4.29) and thus

$$\mathbf{K}_{M,c}\mathbf{w}=\mathbf{\Phi}\mathbf{K}_M\mathbf{w}=\mathbf{\Phi}\mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}\mathbf{w}=\mathbf{0}. \tag{4.30}$$

This implies that $\mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}\mathbf{w}\in\text{range}(\mathbf{M}^{-1/2}\mathbf{A}^{\mathrm{T}})$ and, consequently, $\mathbf{K}\mathbf{M}^{-1/2}\mathbf{w}\in\text{range}(\mathbf{A}^{\mathrm{T}})$, which means that $\mathbf{x}=\mathbf{M}^{-1/2}\mathbf{w}=\mathbf{K}^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{u}\in\mathbb{R}^n$, $\mathbf{u}\in\mathbb{R}^m$. Now, if

$$\mathbf{A}^{\mathrm{T}}=[\mathbf{a}_1\ \mathbf{a}_2\ldots\mathbf{a}_m],\quad \mathbf{a}_i=[a_{i1}\ a_{i2}\ldots a_{in}]^{\mathrm{T}}$$

linear independent solutions to (4.30) can be chosen as

$$\mathbf{w}_i=\mathbf{M}^{1/2}\mathbf{K}^{-1}\mathbf{a}_i,\quad i=1,\ldots,m \tag{4.31}$$

Due to the assumption that $\mathbf{K}$ is positive definite it follows that $\mathbf{a}_i^{\mathrm{T}}\mathbf{K}^{-1}\mathbf{a}_i>0$ and this implies that the constraint requirement $\mathbf{A}\mathbf{x}_i=\mathbf{A}\mathbf{M}^{-1/2}\mathbf{w}_i=\mathbf{0}_m,\ \ i=1,\ldots,m$ is not satisfied, since $\mathbf{A}\mathbf{M}^{-1/2}\mathbf{w}_i=\mathbf{A}\mathbf{K}^{-1}\mathbf{a}_i\neq\mathbf{0}_m$.

If $\omega_c^2\neq0$, operating with $\mathbf{\Pi}$ on Eq. (4.28) gives

$$\mathbf{\Pi}\mathbf{w}=\mathbf{0}\quad\text{and thus}\quad\mathbf{\Phi}\mathbf{w}=\mathbf{w} \tag{4.32}$$

and hence, from Eq. (4.28) it follows that

$$(-\omega_c^2\mathbf{I}+\mathbf{\Psi})\mathbf{w}=\mathbf{0}, \tag{4.33}$$

where

$$\mathbf{\Psi}=\mathbf{K}_{M,c}\mathbf{\Phi}=\mathbf{\Phi}\mathbf{K}_M\mathbf{\Phi}=\mathbf{\Phi}\mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}\mathbf{\Phi}\in\mathbb{R}^{n\times n} \tag{4.34}$$

is obviously symmetric and positive semi-definite, which implies $\omega_c^2\in\mathbb{R}$ and $\omega_c^2\geqslant0$. On the other hand by assuming that $\mathbf{w}\neq\mathbf{0}$, $\omega_c^2\neq0$ is a solution to the eigenvalue problem Eq. (4.33) and by operating with $\mathbf{\Pi}$ on Eq. (4.33) one obtains $\mathbf{\Pi}\mathbf{w}=\mathbf{0}$ and consequently $\mathbf{\Phi}\mathbf{w}=\mathbf{w}$ and this inserted into Eq. (4.33) proves Eq. (4.28). We may thus conclude that $\mathbf{w}\neq\mathbf{0}$, $\omega^2>0$ is a solution to the eigenvalue problem (4.28) if and only if it is a solution to Eq. (4.33). Now

$$\mathbf{w}\in\text{kernel}(\mathbf{\Phi})\Rightarrow\mathbf{\Psi}\mathbf{w}=\mathbf{0} \tag{4.35}$$

and thus, since $\dim(\text{kernel}(\mathbf{\Phi}))=\dim(\text{range}(\mathbf{M}^{-1/2}\mathbf{A}^{\mathrm{T}}))=m$, there are m linearly independent eigenvectors to $\mathbf{\Psi}$ corresponding to the eigenvalue zero. Note however that, in general, $\mathbf{w}\in\text{kernel}(\mathbf{\Phi})\nRightarrow\mathbf{K}_{M,c}\mathbf{w}=\mathbf{0}$. Since $\mathbf{\Psi}$ is symmetric there are k additional linearly independent eigenvectors $\mathbf{w}_{m+1},\mathbf{w}_{m+2},\ldots,\mathbf{w}_{m+k}\in\text{range}(\mathbf{\Phi})$ corresponding to the eigenvalues (4.22), i.e.

$$\mathbf{\Psi}\mathbf{w}_{m+i}=\omega_{c,m+i}^2\mathbf{w}_{m+i},\quad i=1,\ldots,k. \tag{4.36}$$

From Eq. (4.23) it follows that these vectors are uniquely defined, up to an arbitrary real constant, and due to the symmetry of $\mathbf{\Psi}$ they may be chosen as orthonormal, i.e.

$$\mathbf{w}_{m+i}^{\mathrm{T}}\mathbf{w}_{m+j}=\delta_{ij},\quad 1\leqslant i,\ j\leqslant k. \tag{4.37}$$

Furthermore

$$\begin{aligned}\omega_{c,m+i}^2&=\mathbf{w}_{m+i}^{\mathrm{T}}\mathbf{\Psi}\mathbf{w}_{m+i}=\mathbf{w}_{m+i}^{\mathrm{T}}\mathbf{\Phi}\mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}\mathbf{\Phi}\mathbf{w}_{m+i}\\&=(\mathbf{M}^{-1/2}\mathbf{w}_{m+i})^{\mathrm{T}}\mathbf{K}\mathbf{M}^{-1/2}\mathbf{w}_{m+i}>0,\quad i=1,\ldots,k\end{aligned} \tag{4.38}$$

since $\mathbf{K}$ is assumed to be positive definite. From

$$\text{kernel}(\mathbf{A}\mathbf{M}^{-1/2})=(\text{range}(\mathbf{M}^{-1/2}\mathbf{A}^{\mathrm{T}})^{\perp})=\text{range}(\mathbf{\Phi}) \tag{4.39}$$

we may conclude that

$$\mathbf{A}\mathbf{M}^{-1/2}\mathbf{w}_{m+i} = \mathbf{0}_m, \quad i = 1, \ldots, k. \tag{4.40}$$

Thus, by taking

$$\begin{gathered}\mathbf{x}_{c,1} = \mathbf{K}^{-1}\mathbf{a}_1, \quad \mathbf{x}_{c,2} = \mathbf{K}^{-1}\mathbf{a}_2, \ldots, \mathbf{x}_{c,m} = \mathbf{K}^{-1}\mathbf{a}_m, \\ \mathbf{x}_{c,m+1} = \mathbf{M}^{-1/2}\mathbf{w}_{m+1}, \quad \mathbf{x}_{c,m+2} = \mathbf{M}^{-1/2}\mathbf{w}_{m+2}, \ldots, \mathbf{x}_{c,m+k} = \mathbf{M}^{-1/2}\mathbf{w}_{m+k}\end{gathered} \tag{4.41}$$

it follows that $\mathbf{x}_{c,i}, \quad i = 1, \ldots, n$ are linearly independent and

$$\begin{gathered}(-\omega_{c,i}^2\mathbf{M} + \mathbf{K}_c)\mathbf{x}_{c,i} = \mathbf{0}, \\ \mathbf{A}\mathbf{x}_{c,i} \neq \mathbf{0}, \quad i = 1, \ldots, m \text{ and } \mathbf{A}\mathbf{x}_{c,m+i} = \mathbf{0}, \ i = 1, \ldots, k\end{gathered} \tag{4.42}$$

and

$$\mathbf{x}_{c,m+i}^{\mathrm{T}}\mathbf{M}\mathbf{x}_{c,m+j} = \mathbf{w}_{m+i}^{\mathrm{T}}\mathbf{M}^{-1/2}\mathbf{M}\mathbf{M}^{-1/2}\mathbf{w}_{m+j} = \mathbf{w}_{m+i}^{\mathrm{T}}\mathbf{w}_{m+j} = \delta_{ij}, \quad 1 \leqslant i,j \leqslant k. \tag{4.43}$$

The modal matrix of the eigenvalue problem (4.10), under the presumption that $\mathbf{K}$ is positive definite, is therefore given by

$$\mathbf{X}_c = [\mathbf{x}_{c,1} \ldots \mathbf{x}_{c,m} \ \ \mathbf{x}_{c,m+1} \ldots \mathbf{x}_{c,n}] = [\mathbf{K}^{-1}\mathbf{a}_1 \ldots \mathbf{K}^{-1}\mathbf{a}_m \ \ \mathbf{M}^{-1/2}\mathbf{w}_{m+1} \ldots \mathbf{M}^{-1/2}\mathbf{w}_{m+k}] \tag{4.44}$$

and thus

$$\mathbf{M}\mathbf{X}_c = [\mathbf{M}\mathbf{K}^{-1}\mathbf{a}_1 \ldots \mathbf{M}\mathbf{K}^{-1}\mathbf{a}_m \ \ \mathbf{M}^{1/2}\mathbf{w}_{m+1} \ldots \mathbf{M}^{1/2}\mathbf{w}_{m+k}] \tag{4.45}$$

which gives

$$\mathbf{X}_c^{\mathrm{T}}\mathbf{M}\mathbf{X}_c = \begin{bmatrix} \mathbf{G} & \mathbf{H}^{\mathrm{T}} \\ \mathbf{H} & \mathbf{I}_{k\times k} \end{bmatrix}, \tag{4.46}$$

where

$$\mathbf{G} = \begin{bmatrix} \mathbf{a}_1^{\mathrm{T}}\mathbf{K}^{-1}\mathbf{M}\mathbf{K}^{-1}\mathbf{a}_1 & \ldots & \mathbf{a}_1^{\mathrm{T}}\mathbf{K}^{-1}\mathbf{M}\mathbf{K}^{-1}\mathbf{a}_m \\ \vdots & \ddots & \vdots \\ \mathbf{a}_m^{\mathrm{T}}\mathbf{K}^{-1}\mathbf{M}\mathbf{K}^{-1}\mathbf{a}_1 & \ldots & \mathbf{a}_1^{\mathrm{T}}\mathbf{K}^{-1}\mathbf{M}\mathbf{K}^{-1}\mathbf{a}_m \end{bmatrix} \in \mathbb{R}^{m\times m}, \quad \mathbf{G}^{\mathrm{T}} = \mathbf{G}, \tag{4.47}$$

$$\mathbf{H} = \begin{bmatrix} \mathbf{w}_{m+1}^{\mathrm{T}}\mathbf{M}^{1/2}\mathbf{K}^{-1}\mathbf{a}_1 & \ldots & \mathbf{w}_{m+1}^{\mathrm{T}}\mathbf{M}^{1/2}\mathbf{K}^{-1}\mathbf{a}_m \\ \vdots & \ddots & \vdots \\ \mathbf{w}_{m+k}^{\mathrm{T}}\mathbf{M}^{1/2}\mathbf{K}^{-1}\mathbf{a}_1 & \ldots & \mathbf{w}_{m+k}^{\mathrm{T}}\mathbf{M}^{1/2}\mathbf{K}^{-1}\mathbf{a}_m \end{bmatrix} \in \mathbb{R}^{k\times m}. \tag{4.48}$$

This concludes the proof. □

The first m modes, with natural frequencies all equal to zero do not satisfy the constraint condition, i.e. $\mathbf{A}\mathbf{x}_{c,i} \neq \mathbf{0}, \ i = 1, \ldots, m$ and should therefore be ignored. The remaining positive natural frequencies and their corresponding mode shape vectors

$$0 < \omega_{c,m+1}^2 < \omega_{c,m+2}^2 < \cdots < \omega_{c,n}^2 \quad \text{and} \quad \mathbf{x}_{c,m+1}, \mathbf{x}_{c,m+2}, \ldots, \mathbf{x}_{c,n} \tag{4.49}$$

represent the vibration modes of the constrained structure.

Remark 1. The requirement in Eq. (4.23) may, of course, be lifted since the matrix $\boldsymbol{\Psi}$ is symmetric. The mode shapes in Eq. (4.49) will then, however, not be uniquely defined.

Remark 2. The requirement in the theorem, that the stiffness matrix is positive definite, may also be lifted, resulting in the possibility of constrained rigid body modes with the spectrum

$$0 = \omega^2_{c,m+1} = \omega^2_{c,m+2} = \cdots = \omega^2_{c,m+r} \leqslant \omega^2_{c,m+r} \leqslant \cdots \leqslant \omega^2_{c,n}, \tag{4.50}$$

where the first r natural frequencies, $0 \leqslant r \leqslant k$, correspond to rigid body modes. The representation (4.24) will then be replaced by

$$\mathbf{X}_c^{\mathrm{T}} \mathbf{M} \mathbf{X}_c = \begin{bmatrix} \mathbf{G} & \mathbf{H}^{\mathrm{T}} \\ \mathbf{H} & \mathbf{J} \end{bmatrix}, \tag{4.51}$$

where

$$\mathbf{J} = \begin{bmatrix} \mathbf{0}_{r \times r} & \mathbf{0}_{r \times (k-r)} \\ \mathbf{0}_{(k-r) \times r} & \mathbf{I}_{(k-r) \times (k-r)} \end{bmatrix}. \tag{4.52}$$

Remark 3. Note that the set $\mathbf{s}_1 = \mathbf{S}\mathbf{w}_1, \mathbf{s}_2 = \mathbf{S}\mathbf{w}_2, \ldots, \mathbf{s}_m = \mathbf{S}\mathbf{w}_m$, where $\mathbf{S} \in \mathbb{R}^{n \times n}$ is any non-singular matrix, may replace $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$, defined by Eq. (4.31), as a solution to the eigenvalue problem (4.28) with $\omega_c^2 = 0$.

Remark 4. Due to the separation theorem by Lord Rayleigh [9], we may conclude that the eigenvalues ω_c^2 of the constrained structure are bracketed by those of the unconstrained structure according to

$$\omega_i^2 \leqslant \omega^2_{c,m+i} \leqslant \omega^2_{i+m}, \quad i = 1, 2, \ldots, n-m. \tag{4.53}$$

5. Examples

In all the examples below the modal matrix is normalized with respect to the mass matrix.

Example 1. The following data is given for an unconstrained structure with three dofs, $n = 3$, see Fig. 1.

$$\mathbf{M} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1.0 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 3000 & -1000 & -1000 \\ -1000 & 3000 & -1000 \\ -1000 & -1000 & 3000 \end{bmatrix},$$

with $m_1 = 0.5\,\mathrm{kg}$, $m_2 = 1.0\,\mathrm{kg}$, $m_3 = 1.5\,\mathrm{kg}$, $k_1 = \cdots = k_6 = 1000\,\mathrm{N\,m^{-1}}$. The corresponding modal and spectral matrices are determined to

$$\mathbf{X} = \begin{bmatrix} 0.4639 & 0.2181 & -1.3181 \\ 0.5361 & 0.7819 & 0.3181 \\ 0.6351 & -0.4932 & 0.1419 \end{bmatrix}, \quad \mathbf{\Omega}^2 = \mathrm{diag}(950 \quad 3352 \quad 6698).$$

A constrained system is now defined by $\mathbf{A} = [0 \quad 1 \quad -1]$ and the constraint

$$\mathbf{A}\mathbf{q} = [0 \quad 1 \quad -1] \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = q_2 - q_3 = 0$$

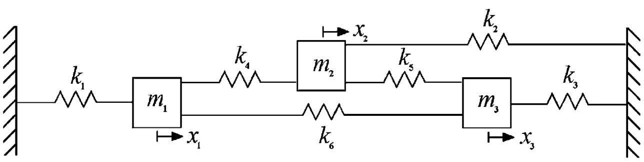

Fig. 1. The unconstrained structure of Example 1.

is rigidly connecting the 2nd and 3rd coordinates which is equivalent to letting $k_5 \to \infty$. From Eq. (4.11) the constrained stiffness matrix

$$\mathbf{K}_c = \begin{bmatrix} 3000 & -1000 & -1000 \\ -800 & 800 & 800 \\ -1200 & 1200 & 1200 \end{bmatrix}$$

is obtained. The corresponding modal and spectral matrices are

$$\mathbf{X}_c = \begin{bmatrix} 0 & 0.4734 & 1.3326 \\ 0.6325 & 0.5960 & -0.2117 \\ -0.6325 & 0.5960 & -0.2117 \end{bmatrix}, \quad \mathbf{\Omega}_c^2 = \mathrm{diag}(0 \quad 964 \quad 6635),$$

where the first column in $\mathbf{X}_c$ and the first diagonal element in $\mathbf{\Omega}_c^2$, $\Omega_{c,11}^2 = 0$, should be ignored since $\mathbf{A}\mathbf{x}_{c,1} \neq 0$ Note that $\mathbf{A}\mathbf{x}_{c,2} = \mathbf{A}\mathbf{x}_{c,3} = 0$ and that the Rayleigh separation theorem is fulfilled.

Example 2. The following data, given for an unconstrained structure with six dofs, $n = 6$, see Fig. 2, corresponds to a linear array of equal masses and springs.

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 \\ 0 & 0 & 0 & m & 0 & 0 \\ 0 & 0 & 0 & 0 & m & 0 \\ 0 & 0 & 0 & 0 & 0 & m \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 2k & -k & 0 & 0 & 0 & 0 \\ -k & 2k & -k & 0 & 0 & 0 \\ 0 & -k & 2k & -k & 0 & 0 \\ 0 & 0 & -k & 2k & -k & 0 \\ 0 & 0 & 0 & -k & 2k & -k \\ 0 & 0 & 0 & 0 & -k & 2k \end{bmatrix},$$

where $m = 2$ and $k = 1000$. The corresponding modal and spectral matrices are

$$\mathbf{X} = \begin{bmatrix} 0.1640 & -0.2955 & -0.3685 & 0.3685 & 0.2955 & 0.1640 \\ 0.2955 & -0.3685 & -0.1640 & -0.1640 & -0.3685 & -0.2955 \\ 0.3685 & -0.1640 & 0.2955 & -0.2955 & 0.1640 & 0.3685 \\ 0.3685 & 0.1640 & 0.2955 & 0.2955 & 0.1640 & -0.3685 \\ 0.2955 & 0.3685 & -0.1640 & 0.1640 & -0.3685 & 0.2955 \\ 0.1640 & 0.2955 & -0.3685 & -0.3685 & 0.2955 & -0.1640 \end{bmatrix},$$

$$\mathbf{\Omega}^2 = \mathrm{diag}(99 \quad 377 \quad 777 \quad 1223 \quad 1623 \quad 1901).$$

With a constrained structure defined by

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix},$$

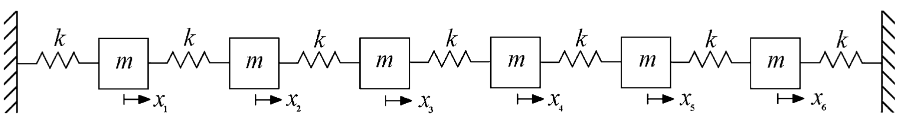

Fig. 2. The unconstrained structure of Example 2.

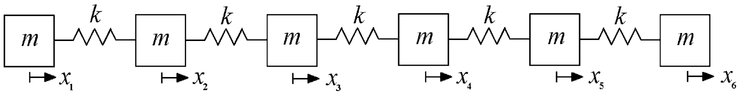

Fig. 3. The unconstrained structure of Example 3.

the corresponding modal and spectral matrices are

$$\mathbf{X}_c = \begin{bmatrix} 0.3525 & 0.3509 & 0.2513 & 0.2279 & 0.1027 & 0.3536 \\ 0.4983 & 0.0390 & 0.1155 & 0.2279 & 0.1027 & -0.3536 \\ 0.1823 & -0.3898 & -0.1697 & 0.4558 & 0.2055 & 0 \\ 0.2624 & -0.4288 & 0.2512 & 0.2279 & 0.1027 & 0.3536 \\ 0.1531 & -0.1949 & 0.3531 & 0.2906 & -0.6446 & 0 \\ 0.0437 & 0.0390 & 0.4549 & 0.2279 & 0.1027 & -0.3536 \end{bmatrix},$$

$$\mathbf{\Omega}_c^2 = \mathrm{diag}(0 \quad 0 \quad 0 \quad 216 \quad 1159 \quad 1250),$$

where the three first columns in $\mathbf{X}_c$ and the three first diagonal elements in $\mathbf{\Omega}_c^2$; $\Omega_{c,11}^2 = \Omega_{c,22}^2 = \Omega_{c,33}^2 = 0$, should be ignored.

Example 3. The free–free structure in Fig. 3 has the original mass and stiffness matrices

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 \\ 0 & 0 & 0 & m & 0 & 0 \\ 0 & 0 & 0 & 0 & m & 0 \\ 0 & 0 & 0 & 0 & 0 & m \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} k & -k & 0 & 0 & 0 & 0 \\ -k & 2k & -k & 0 & 0 & 0 \\ 0 & -k & 2k & -k & 0 & 0 \\ 0 & 0 & -k & 2k & -k & 0 \\ 0 & 0 & 0 & -k & 2k & -k \\ 0 & 0 & 0 & 0 & -k & k \end{bmatrix},$$

where $m = 2$ and $k = 1000$. The spectral matrix is given by

$$\mathbf{\Omega}^2 = \mathrm{diag}(0 \quad 134 \quad 500 \quad 1000 \quad 1500 \quad 1866).$$

The first natural frequency is equal to zero and the corresponding mode shape corresponds to a rigid body translation of the structure. Using the same constraints as in the previous example the following modal and spectral matrices are obtained:

$$\mathbf{X}_c = \begin{bmatrix} -0.2603 & 0.6124 & 0.4148 & -0.2324 & 0.3536 & 0.0923 \\ -0.3900 & 0.2041 & 0.4318 & -0.2324 & -0.3536 & 0.0923 \\ -0.3638 & 0 & 0.2787 & -0.4647 & 0 & 0.1845 \\ -0.3377 & -0.2041 & 0.1255 & -0.2324 & 0.3536 & 0.0923 \\ -0.1819 & 0 & -0.0446 & -0.2610 & 0 & -0.6572 \\ -0.0261 & 0.2041 & -0.2147 & -0.2324 & -0.3536 & 0.0923 \end{bmatrix},$$

$$\mathbf{\Omega}_c^2 = \mathrm{diag}(0 \quad 0 \quad 0 \quad 109.6 \quad 1000 \quad 1140.4).$$

It should be observed that there are three non-zero natural frequencies for the constrained structure. The constrained structure is not connected to its surroundings and is, in this sense, like the unconstrained structure a 'free–free' structure. It could thus be expected that a zero natural frequency corresponding to a rigid body mode would appear. This is, however, not the case. The structure has three non-zero natural frequencies. This is due to the fact that, in this case, the rigid body mode $\mathbf{x}_{c,r} = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]^{\mathrm{T}}$ does not belong to the configuration space of the constrained structure since $\mathbf{Ax}_{c,r} = [1 \quad 0 \quad 0]^{\mathrm{T}}$. This seemingly anomalous behaviour reflects the fact that the constraints used in this example may be somewhat artificial and hard to interpret physically.

5. Structures with viscous damping

The formulation (4.10) may be extended to linear systems containing viscous damping. The equation governing free vibrations of the unconstrained system is assumed to be given by

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}, \tag{6.1}$$

where $\mathbf{C}$ is the damping matrix, generally symmetric and positive semi-definite. The corresponding eigenvalue problem then reads

$$(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K})\mathbf{x} = \mathbf{0}, \tag{6.2}$$

where $s \in \mathbb{C}$. If we introduce the constraint (4.2), Eq. (6.1) will be replaced by

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}, \tag{6.3}$$

and from Eq. (6.3) and the constraint condition (4.2) it follows that

$$\mathbf{0}_m = \mathbf{A}\ddot{\mathbf{q}} = -\mathbf{A}\mathbf{M}^{-1}\mathbf{C}\dot{\mathbf{q}} - \mathbf{A}\mathbf{M}^{-1}\mathbf{K}\mathbf{q} + \boldsymbol{\Gamma}\boldsymbol{\lambda}, \tag{6.4}$$

where the matrix $\boldsymbol{\Gamma}$ is defined in Eq. (4.8). Consequently

$$\boldsymbol{\lambda} = \boldsymbol{\Gamma}^{-1}\mathbf{A}\mathbf{M}^{-1}\mathbf{C}\dot{\mathbf{q}} + \boldsymbol{\Gamma}^{-1}\mathbf{A}\mathbf{M}^{-1}\mathbf{K}\mathbf{q} \tag{6.5}$$

and by eliminating $\boldsymbol{\lambda}$ in Eq. (6.3) the following equation of motion is obtained:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}_c\dot{\mathbf{q}} + \mathbf{K}_c\mathbf{q} = \mathbf{0}, \tag{6.6}$$

where $\mathbf{K}_c$ is given by Eq. (4.11) and $\mathbf{C}_c$ is analogously defined by

$$\mathbf{C}_c = \mathbf{Q}\mathbf{C}. \tag{6.7}$$

The eigenvalue problem corresponding to Eq. (6.6) then reads

$$(s_c^2\mathbf{M} + s_c\mathbf{C}_c + \mathbf{K}_c)\mathbf{x} = \mathbf{0}. \tag{6.8}$$

Let $\mathbf{X}_c = [\mathbf{x}_{c,1}\mathbf{x}_{c,2}\ldots\mathbf{x}_{c,n}]$ denote the modal matrix and $\boldsymbol{\Omega}_c^2 = \mathrm{diag}(\omega_{c,1}^2\omega_{c,2}^2\ldots\omega_{c,n}^2)$ the spectral matrix corresponding to the eigenvalue problem with zero damping, $\mathbf{C} = \mathbf{0}$, i.e. the constrained undamped natural modes according to Theorem 1.

Theorem 2. *If $\mathbf{C}$ is symmetric and positive semi-definite then $\mathbf{X}_c$ will be the modal matrix of the eigenvalue problem* (6.6), *i.e.*

$$(s_{c,i}^2\mathbf{M} + s_{c,i}\mathbf{C}_c + \mathbf{K}_c)\mathbf{x}_{c,i} = \mathbf{0}, \tag{6.9}$$

if and only if

$$\boldsymbol{\Phi}\mathbf{M}^{-1/2}\mathbf{C}\mathbf{M}^{-1/2}\boldsymbol{\Phi}\mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}\boldsymbol{\Phi} = \boldsymbol{\Phi}\mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}\boldsymbol{\Phi}\mathbf{M}^{-1/2}\mathbf{C}\mathbf{M}^{-1/2}\boldsymbol{\Phi}, \tag{6.10}$$

where

$$\boldsymbol{\Phi} = \mathbf{I} - \boldsymbol{\Pi} = \mathbf{I} - \mathbf{M}^{-1/2}\mathbf{P}\mathbf{M}^{1/2} = \mathbf{I} - \mathbf{M}^{-1/2}\mathbf{A}^{\mathrm{T}}(\mathbf{A}\mathbf{M}^{-1}\mathbf{A}^{\mathrm{T}})^{-1}\mathbf{A}\mathbf{M}^{-1}\mathbf{M}^{1/2}. \tag{6.11}$$

The eigenvalues, $s_{c,i}$, are given by

$$s_{c,i} = s_{c,i}^1 = s_{c,i}^2 = 0, \quad i = 1, \ldots, m, \tag{6.12}$$

$$s_{c,i} = \begin{cases} s^1_{c,i} = -\dfrac{\mathbf{x}^{\mathrm{T}}_{c,i}\mathbf{C}\mathbf{x}_{c,i}}{2} + \sqrt{\dfrac{(\mathbf{x}^{\mathrm{T}}_{c,i}\mathbf{C}\mathbf{x}_{c,i})^2}{4} - \omega^2_{c,i}}, \\ s^2_{c,i} = -\dfrac{\mathbf{x}^{\mathrm{T}}_{c,i}\mathbf{C}\mathbf{x}_{c,i}}{2} - \sqrt{\dfrac{(\mathbf{x}^{\mathrm{T}}_{c,i}\mathbf{C}\mathbf{x}_{c,i})^2}{4} - \omega^2_{c,i}}, \end{cases} \quad i = m+1, \ldots, n. \tag{6.13}$$

Proof. If $s = 0$ then $\mathbf{K}_{M,c}\mathbf{w} = \mathbf{0}$, with solutions $\mathbf{w}_i = \mathbf{M}^{1/2}\mathbf{K}^{-1}\mathbf{a}_i,\ \ i = 1, \ldots, m$ according to Eq. (4.31). The eigenvalue problem, corresponding to Eq. (4.33), may be written as

$$(s^2\mathbf{I} + s\mathbf{\Delta} + \mathbf{\Psi})\mathbf{w} = \mathbf{0}, \tag{6.14}$$

where $\mathbf{\Psi}$ is defined by Eq. (4.34) and

$$\mathbf{\Delta} = \mathbf{\Phi}\mathbf{M}^{-1/2}\mathbf{C}\mathbf{M}^{-1/2}\mathbf{\Phi} \in \mathbb{R}^{n\times n} \tag{6.15}$$

is a symmetric and positive semi-definite matrix. Now

$$\mathbf{\Phi}\mathbf{M}^{-1/2}\mathbf{C}\mathbf{M}^{-1/2}\mathbf{\Phi}\mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}\mathbf{\Phi} = \mathbf{\Phi}\mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}\mathbf{\Phi}\mathbf{M}^{-1/2}\mathbf{C}\mathbf{M}^{-1/2}\mathbf{\Phi} \Leftrightarrow \mathbf{\Delta}\mathbf{\Psi} = \mathbf{\Psi}\mathbf{\Delta}, \tag{6.16}$$

and condition (6.10) is then equivalent to the condition that the symmetric matrices $\mathbf{\Delta}$ and $\mathbf{\Psi}$ commute, which is true if and only if they have a complete orthonormal set of common eigenvectors, cf. Ref. [10]. From the proof of Theorem 1

$$\mathbf{\Psi}\mathbf{w}_i = \omega^2_{c,m+i}\mathbf{w}_{m+i}, \quad i = 1, \ldots, k. \tag{6.17}$$

Take

$$\mathbf{\Delta}\mathbf{w}_i = \delta^2_{c,m+i}\mathbf{w}_{m+i}, \quad i = 1, \ldots, k, \quad \delta^2_{c,i} \geqslant 0 \tag{6.18}$$

and then by combining Eqs. (6.14), (6.17) and (6.18)

$$(s^2\mathbf{I} + s\mathbf{\Delta} + \mathbf{\Psi})\mathbf{w}_i = (s^2 + s\delta^2_{c,i} + \omega^2_{c,i})\mathbf{w}_i = \mathbf{0}. \tag{6.19}$$

Consequently $\mathbf{w}_i,\ \ i = 1, \ldots, n$ will constitute a complete set of eigenvectors to Eq. (6.14) with the corresponding eigenvalues

$$\begin{aligned} s &= s_i = \pm 0, \quad i = 1, \ldots, m, \\ s &= s_i = -\frac{\delta^2_{c,i}}{2} \pm \sqrt{\frac{\delta^4_{c,i}}{4} - \omega^2_{c,i}}, \quad i = m+1, \ldots, n. \end{aligned} \tag{6.20}$$

From Eqs. (6.18), (6.15) and (4.42) it follows that

$$\begin{aligned} \delta^2_{c,i} &= \frac{\mathbf{w}^{\mathrm{T}}_i\mathbf{\Delta}\mathbf{w}_i}{\mathbf{w}^{\mathrm{T}}_i\mathbf{w}_i} = \mathbf{w}^{\mathrm{T}}_i\mathbf{\Delta}\mathbf{w}_i = \mathbf{w}^{\mathrm{T}}_i\mathbf{\Phi}\mathbf{M}^{-1/2}\mathbf{C}\mathbf{M}^{-1/2}\mathbf{\Phi}\mathbf{w}_i = \mathbf{w}^{\mathrm{T}}_i\mathbf{M}^{-1/2}\mathbf{C}\mathbf{M}^{-1/2}\mathbf{w}_i = \mathbf{x}^{\mathrm{T}}_{c,i}\mathbf{C}\mathbf{x}_{c,i}, \\ i &= m+1, \ldots, n \end{aligned} \tag{6.21}$$

This proves the Theorem. □

Remark 1. For Rayleigh damping, i.e. if $\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K},\ \ \alpha, \beta \in \mathbb{R}$, it is a simple matter to demonstrate that condition (6.10) is satisfied for all constraint matrices $\mathbf{A}$.

Remark 2. If the unconstrained system fulfills the requirement

$$\mathbf{C}\mathbf{M}^{-1}\mathbf{K} = \mathbf{K}\mathbf{M}^{-1}\mathbf{C} \tag{6.22}$$

then the modal matrix for eigenvalue problem (6.2) is given by the modal matrix for the un-damped structure, i.e. $\mathbf{X} = [\mathbf{x}_1\mathbf{x}_2 \ldots \mathbf{x}_n]$. It is conjectured that condition (6.22), in general, does not necessarily imply condition

(6.10). It is, however, evident that there are structures and accompanying constraints where (6.22) does not hold but where Eq. (6.10) does hold.

6. Concluding remarks

In this paper it has been demonstrated that the undamped natural vibrations of a constrained linear structure may be calculated using a generalized eigenvalue problem derived from the equations of motion for the constrained system involving Lagrangian multipliers. The eigenvalue problem derived is defined by the mass matrix of the unconstrained structure and a non-symmetric and singular stiffness matrix for the constrained system.

A condition for the calculation of the damped natural vibrations in terms of the undamped mode shapes was also formulated for the constrained structure. This property may, for some constraints, be an inheritance from the unconstrained structure. For other constraints it may appear as a new property. This raises the following questions. How can constraints be imposed on the damped structure in order for the modal matrix to be equal to the undamped modal matrix? Is there, for a given structure, a certain class of constraints that will satisfy this property? How are these constraints characterized?

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References

[1] J.G.M. Kerstens, Vibration of complex structures: the modal constraint method, *Journal of Sound and Vibration* 76 (1981) 467–480.
[2] J.G.M. Kerstens, Vibration of modified discrete systems: the modal constraint method, *Journal of Sound and Vibration* 83 (1) (1982) 81–92.
[3] E.H. Dowell, Free vibrations of linear structure with arbitrary support conditions, *Journal of Applied Mechanics* 38 (1971) 560–595.
[4] S.M. Yang, Modal analysis of structures with holonomic constraints, *AIAA Journal* 30 (10) (1992) 2526–2531.
[5] J.G.M. Kerstens, A review of methods for constrained eigenvalue problems, *HERON* 50 (2) (2005) 109–132.
[6] M. Geradin, D. Rixen, *Mechanical Vibrations*, Wiley, New York, 1997.
[7] C. Lanczos, *The Variational Principles of Mechanics*, Dover Publications Inc., New York, 1986.
[8] M.T. Heath, *Scientific Computing. An introductory survey*, McGraw-Hill, New York, 2002.
[9] Lord Rayleigh, *The Theory of Sound, 2 volumes*, Dover Publications, New York, 1945.
[10] F.R. Gantmacher, *The Theory of Matrices*, Chelsea Publishing Company, 1960.