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An alternative formulation of the boundary value problem for the Timoshenko beam and Mindlin plate

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Abstract

The traditional formulations of the boundary value problems for the Timoshenko beam and Mindlin plate theories do not allow the bending and shearing deflections to be determined uniquely. An alternative formulation is proposed in which the bending deflection is regarded as a fundamental variable in place of the angle of rotation due to bending. Using the total deflection as an accompanying variable, the governing equations and boundary conditions can be derived on the basis of Hamilton's principle. This formulation is shown to afford unique results for the bending and shearing deflections, with natural frequencies equal to or higher than those determined by traditional methods for certain boundary conditions. The proposed formulation represents a deductive approach to determining the total deflection, providing consistency for both dynamic and static analyses.

1. Introduction

Timoshenko's beam theory [1,2] and Mindlin's plate theory [3] are well-known theories used to analyze the dynamic behavior of beams and flat plates. These theories take into account rotary inertia and deformation due to shearing in addition to bending deformation, and are generally referred to as first-order shear deformation theories. Nevertheless, these theories are only an approximation of reality, requiring a shear coefficient to satisfy the constitutive relationship between shear stress and shear strain. To eliminate this kind of approximation, many higher-order shear deformation theories have been investigated [4,5], and the proposed theories have generally been successful in avoiding the need for hypotheses [4]. However, as analyses using higher-order shear deformation theories are not feasible in a practical sense, it remains necessary to establish a usable general theory. Timoshenko–Mindlin theory remains widely used as a simple and highly useful approach that considers a sufficient suite of factors to yield a reasonable physical, and even quantitative, picture of wave travel. This theory thus continues to be the focus of much research [6–9].

In conventional static analyses for beams [10–14], bending and shearing deflections are recognized as simple physical entities, where the shearing deflection is obtained independently using inherent boundary conditions

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and simply added to the bending deflection to give the total deflection. However, such an approach is not deductive and thus produces inconsistent results. The boundary conditions, based on physical reality, dictate that the bending and shearing deflections become zero at clamped and simply supported ends. These features are also unexceptional in Timoshenko's and Donnell's textbooks [10,12] as part of the traditional methodology of static analysis.

As discussed in this paper, however, the bending and shearing deflections in the boundary value problem of the dynamic Timoshenko beam and Mindlin plate theories cannot be determined uniquely, and neither deflection can be defined specifically for the Mindlin plate. Thus, if the conventional Timoshenko beam and Mindlin plate theories are applied to static problems, the traditional concept of deformation in static analyses cannot be obtained, precluding solution by a deductive methodology. To the best of the authors' knowledge, the only theory that affords consistent results is Shimpi's static plate theory [15], which is a recent theory that considers the effect of higher-order shearing in the thickness direction of the plate.

Jacobsen and Ayre [16] describe the bending and shearing deflections as being determined independently in the Timoshenko beam theory, allowing the two deflections to be simply added to give the total deflection. However, no concrete examples clearly verifying this notion have been shown. Anderson expressed the governing equations for the Timoshenko beam using the bending deflection and shearing deflection explicitly [17], and concluded that these governing equations as well as the boundary conditions can be successfully derived using Hamilton's principle [18]. However, the implied boundary conditions are the same as those of the traditional theory, and thus Anderson's theory does not differ substantially from the conventional approach.

The present study proposes an alternative formulation of the boundary value problem for the Timoshenko beam and Mindlin plate theories. By adopting the bending and shearing deflections as independently recognizable physical entities as in the traditional static analyses but taking the bending deflection and total deflection as fundamental variables, the bending and shearing deflections are shown to be uniquely determinable. The boundary condition models for clamped and simply supported ends for beams and flat plates can thus be obtained with physical reality.

It should be emphasized that the main aim of the proposed formulation is not to improve the accuracy of analysis but to introduce some flexibility with respect to the traditional concept of deformation in static analyses and enhance the physical reality of first-order shear deformation theory for beams and flat plates. The proposed formulation may not be equivalent physically or mathematically with the traditional formulation, despite the targeted continuum structure elements being the same. The main difference between the alternative concept of deformation and the traditional approach is the basic premise that both the bending and shearing deflections should be recognized as physical entities, that the two entities can be distinguished, and that both deflections should be assigned zero values at supported ends (i.e., clamped and simply supported ends). This concept is proposed from the viewpoint of maintaining consistency with conventional static analyses, which take account of shear in addition to bending. The proposed deformation concept for structure elements imposes a more restrictive degree of system deformation in comparison with the traditional approach, and as a result the two formulations exhibit differences in both the calculated natural frequencies and mode shapes for a range of boundary conditions.

2. Traditional formulations

To facilitate investigation of the relationship between the orders and boundary conditions of traditional and alternative models, the traditional formulations of the boundary value problem for an isotropic and uniform Timoshenko beam and Mindlin plate are presented here briefly. The governing equations and boundary conditions are all derived from Hamilton's principle as follows:

$$\delta \int_{t_1}^{t_2} (T - U)\,\mathrm{d}t = 0. \tag{1}$$

Here, T and U are the kinetic and potential energies, t is time, and δ is the variation. External forces are not considered.

2.1. Timoshenko beam

Fig. 1 shows a schematic of the traditional concept of deformation for the Timoshenko beam [1,2], where w is the total transverse deflection caused by both bending and shear. The following relation holds for this system:

$$\frac{\partial w}{\partial x} = \phi + \beta, \tag{2}$$

where ϕ is the angle of rotation due to bending, β is the angle of distortion due to shear, and x is the axial coordinate of the beam. Calculating the kinetic and potential energies with respect to w and ϕ as fundamental variables and applying Hamilton's principle, the governing equations and boundary conditions can be derived as follows [19]:

$$\begin{aligned} &\rho A \frac{\partial^2 w}{\partial t^2} - k' GA \left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \phi}{\partial x} \right) = 0, \\ &\rho I \frac{\partial^2 \phi}{\partial t^2} - EI \frac{\partial^2 \phi}{\partial x^2} - k' GA \left(\frac{\partial w}{\partial x} - \phi \right) = 0, \end{aligned} \tag{3}$$

$$\begin{aligned} &\left[EI \frac{\partial \phi}{\partial x} \delta\phi \right]_0^\ell = 0, \\ &\left[k' GA \left(\frac{\partial w}{\partial x} - \phi \right) \delta w \right]_0^\ell = 0. \end{aligned} \tag{4}$$

Here, ρ is density, E and G are the Young's and shear moduli, ℓ, A and I are the length, cross-sectional area and moment of inertia of the beam, and k' is a numerical modification factor that depends on the shape of the cross-section (i.e., the shear coefficient).

A boundary condition expressed as $[A\delta B]_0^\ell = 0$ implies that $A = 0$ or B must be assigned at the end of the beam, i.e., $x = 0$ and ℓ. The expressions of boundary conditions for simply supported, clamped and free ends in traditional formulation are clearly defined in Ref. [19].

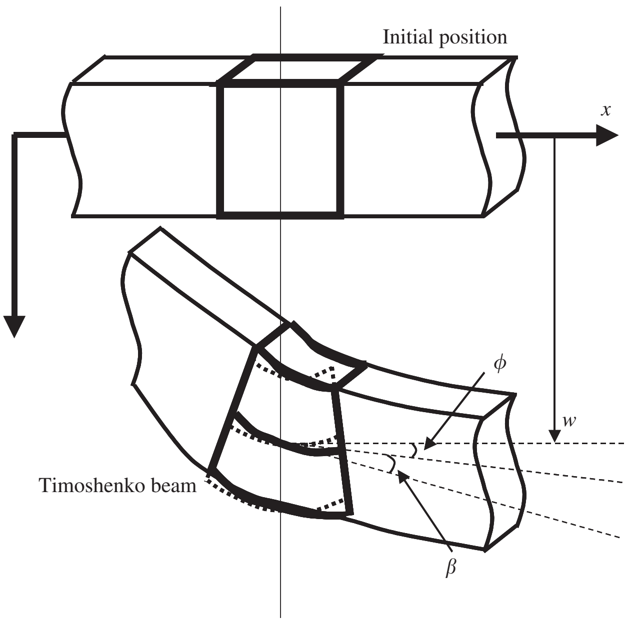

Fig. 1. Schematic of traditional concept of deformation for Timoshenko's beam.

The separation of variables with respect to t is given by

$$w = W(x)\,\mathrm{e}^{\mathrm{i}pt}, \quad \phi = \Phi(x)\,\mathrm{e}^{\mathrm{i}pt}, \tag{5}$$

where p is the harmonic angular frequency and $\mathrm{i} = \sqrt{-1}$. The following solution to the modified homogeneous Eq. (3) is then assumed:

$$W = \bar{W}\,\mathrm{e}^{(\lambda/\ell)x}, \quad \Phi = \bar{\Phi}\,\mathrm{e}^{(\lambda/\ell)x}. \tag{6}$$

The characteristic equation for λ is obtained from the condition that a solution of the form of Eq. (6) may exist, as follows:

$$\left(\frac{\lambda}{\ell}\right)^4 + \rho\left(\frac{1}{E}+\frac{1}{k'G}\right)p^2\left(\frac{\lambda}{\ell}\right)^2 + \rho^2\left(\frac{1}{E}\frac{1}{k'G}\right)p^2\left(p^2-\frac{k'GA}{\rho I}\right) = 0. \tag{7}$$

Solving Eq. (7) yields the roots λ_i ($i = 1, 2$), as given by

$$\begin{aligned}\left(\frac{\lambda_1}{\ell}\right)^2 &= -\frac{1}{2}\rho\left(\frac{1}{E}+\frac{1}{k'G}\right)p^2 + \sqrt{\frac{1}{4}\rho^2\left(\frac{1}{E}-\frac{1}{k'G}\right)^2 p^4 + \frac{\rho A}{EI}p^2},\\ \left(\frac{\lambda_2}{\ell}\right)^2 &= -\frac{1}{2}\rho\left(\frac{1}{E}+\frac{1}{k'G}\right)p^2 - \sqrt{\frac{1}{4}\rho^2\left(\frac{1}{E}-\frac{1}{k'G}\right)^2 p^4 + \frac{\rho A}{EI}p^2}.\end{aligned} \tag{8}$$

Thus, the general solution for W and Φ is given by

$$\begin{aligned}W(x) &= C_1\,\mathrm{e}^{(\lambda_1/\ell)x} + C_2\,\mathrm{e}^{(-\lambda_1/\ell)x} + C_3\,\mathrm{e}^{(\lambda_2/\ell)x} + C_4\,\mathrm{e}^{(-\lambda_2/\ell)x},\\ \Phi(x) &= B_1C_1\,\mathrm{e}^{(\lambda_1/\ell)x} - B_1\,C_2\,\mathrm{e}^{(-\lambda_1/\ell)x} + B_2C_3\,\mathrm{e}^{(\lambda_2/\ell)x} - B_2C_4\,\mathrm{e}^{(-\lambda_2/\ell)x},\end{aligned} \tag{9}$$

where

$$B_1 = \frac{\ell}{\lambda_1}\left\{\left(\frac{\lambda_1}{\ell}\right)^2 + \frac{\rho}{k'G}p^2\right\}, \quad B_2 = \frac{\ell}{\lambda_2}\left\{\left(\frac{\lambda_2}{\ell}\right)^2 + \frac{\rho}{k'G}p^2\right\}.$$

Here, C_i ($i = 1$–4) are the integral constants determined from the four boundary conditions (Eq. (4)) at the ends of a beam ($x = 0$ and ℓ).

2.2. Mindlin plate

Consider a uniform and isotropic rectangular plate with edge lengths a and b and thickness h. The coordinate axes x and y are taken along two adjacent edges. In the Mindlin plate theory [3], the fundamental variables are w, ϕ_x and ϕ_y, where w is the total transverse deflection of the mid-plane, and ϕ_x and ϕ_y are the rotations of a transverse normal about the x- and y-axis. In the Mindlin theory, the kinetic and potential energies are expressed as

$$T = \frac{1}{2}\int_0^b\int_0^a\left[\rho h\left(\frac{\partial w}{\partial t}\right)^2 + \frac{\rho h^3}{12}\left\{\left(\frac{\partial \phi_x}{\partial t}\right)^2 + \left(\frac{\partial \phi_y}{\partial t}\right)^2\right\}\right]\mathrm{d}x\,\mathrm{d}y, \tag{10}$$

$$\begin{aligned}U = \frac{1}{2}\int_0^b\int_0^a\Bigg[&D\left\{\left(\frac{\partial \phi_x}{\partial x}\right)^2 + \left(\frac{\partial \phi_y}{\partial y}\right)^2 + 2\nu\frac{\partial \phi_x}{\partial x}\frac{\partial \phi_y}{\partial y}\right\}\\ &+\frac{1-\nu}{2}D\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}\right)^2 + k'Gh\left\{\left(\frac{\partial w}{\partial x}-\phi_x\right)^2 + \left(\frac{\partial w}{\partial y}-\phi_y\right)^2\right\}\Bigg]\mathrm{d}x\,\mathrm{d}y,\end{aligned} \tag{11}$$

where

$$D = \frac{Eh^3}{12(1-v^2)}$$

and v is the Poisson's ratio.

Applying Eqs. (10) and (11) to Hamilton's principle, the three governing equations and twelve boundary conditions are derived as follows:

$$\begin{aligned}
&\rho h \frac{\partial^2 w}{\partial t^2} - k' G h \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} - \frac{\partial \phi_x}{\partial x} - \frac{\partial \phi_y}{\partial y} \right) = 0, \\
&\frac{\rho h^3}{12} \frac{\partial^2 \phi_x}{\partial t^2} - D \left(\frac{\partial^2 \phi_x}{\partial x^2} + \frac{1+v}{2} \frac{\partial^2 \phi_y}{\partial x \partial y} + \frac{1-v}{2} \frac{\partial^2 \phi_x}{\partial y^2} \right) - k' G h \left(\frac{\partial w}{\partial x} - \phi_x \right) = 0, \\
&\frac{\rho h^3}{12} \frac{\partial^2 \phi_y}{\partial t^2} - D \left(\frac{\partial^2 \phi_y}{\partial y^2} + \frac{1+v}{2} \frac{\partial^2 \phi_x}{\partial x \partial y} + \frac{1-v}{2} \frac{\partial^2 \phi_y}{\partial x^2} \right) - k' G h \left(\frac{\partial w}{\partial y} - \phi_y \right) = 0,
\end{aligned} \tag{12}$$

$$\begin{aligned}
&\left[k' G h \left(\frac{\partial w}{\partial x} - \phi_x \right) \delta w \right]_0^a = 0, \\
&\left[k' G h \left(\frac{\partial w}{\partial y} - \phi_y \right) \delta w \right]_0^b = 0, \\
&\left[D \left(\frac{\partial \phi_x}{\partial x} + v \frac{\partial \phi_y}{\partial y} \right) \delta \phi_x \right]_0^a = 0, \\
&\left[D \left(\frac{\partial \phi_y}{\partial y} + v \frac{\partial \phi_x}{\partial x} \right) \delta \phi_y \right]_0^b = 0, \\
&\left[\frac{1-v}{2} D \left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \right) \delta \phi_y \right]_0^a = 0, \\
&\left[\frac{1-v}{2} D \left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \right) \delta \phi_x \right]_0^b = 0.
\end{aligned} \tag{13}$$

It is well known that the special class of eigenvalue problems for rectangular plates with governing equations (i.e. Eq. (12)) admitting a closed-form solution is characterized by the fact that the two opposing edges are simply supported. The solution for this type of problem is generally obtained by the Lévy approach [7]. The present paper considers only such cases, i.e., a plate simply supported at $x = 0$ and a. Other assumptions are as follows:

$$\begin{aligned}
w &= W(y) \sin \frac{\alpha}{a} x \, e^{ipt}, \\
\phi_x &= \Phi_x(y) \cos \frac{\alpha}{a} x \, e^{ipt}, \\
\phi_y &= \Phi_y(y) \sin \frac{\alpha}{a} x \, e^{ipt},
\end{aligned} \tag{14}$$

where

$$\alpha = n_x \pi \quad (n_x = 1, 2, \ldots).$$

Further, we set

$$W = \bar{W} e^{(\gamma/b)y}, \quad \Phi_x = \bar{\Phi}_x e^{(\gamma/b)y}, \quad \Phi_y = \bar{\Phi}_y e^{(\gamma/b)y}. \tag{15}$$

From the condition that a solution of the form of Eq. (15) may exist, the following characteristic equation for γ can be obtained:

$$\left(\frac{\gamma}{b}\right)^6 + A_1\left(\frac{\gamma}{b}\right)^4 + A_2\left(\frac{\gamma}{b}\right)^2 + A_3 = 0, \tag{16}$$

where

$$\begin{aligned}
A_1 =& -3\left(\frac{\alpha}{a}\right)^2 + (3-v)\frac{\rho(1+v)}{E}p^2 - \frac{12}{h^2}k' + \frac{\rho}{k'G}p^2,\\
A_2 =& 3\left(\frac{\alpha}{a}\right)^4 + \left\{-2(3-v)\frac{\rho(1+v)}{E}p^2 + \frac{24}{h^2}k' - 2\frac{\rho}{k'G}p^2\right\}\left(\frac{\alpha}{a}\right)^2\\
&+ \left\{2\frac{\rho(1-v^2)}{E}p^2 + (3-v)\frac{\rho}{k'G}p^2\right\}\left\{\frac{\rho(1+v)}{E}p^2 - \frac{6}{h^2}k'\right\},\\
A_3 =& -\left(\frac{\alpha}{a}\right)^6 + \left\{(3-v)\frac{\rho(1+v)}{E}p^2 - \frac{12}{h^2}k' + \frac{\rho}{k'G}p^2\right\}\left(\frac{\alpha}{a}\right)^4\\
&- \left\{2\frac{\rho(1-v^2)}{E}p^2 + (3-v)\frac{\rho}{k'G}p^2\right\}\left\{\frac{\rho(1+v)}{E}p^2 - \frac{6}{h^2}k'\right\}\left(\frac{\alpha}{a}\right)^2\\
&+ 2(1-v)\frac{\rho}{k'G}p^2\left\{\frac{\rho(1+v)}{E}p^2 - \frac{6}{h^2}k'\right\}^2.
\end{aligned} \tag{17}$$

Solving Eq. (16), the roots γ_i ($i = 1$–3) are expressed as

$$\begin{aligned}
\left(\frac{\gamma_1}{b}\right)^2 &= \left(\frac{\alpha}{a}\right)^2 - \frac{1}{2}\rho\left(\frac{1-v^2}{E} + \frac{1}{k'G}\right)p^2 + \sqrt{\frac{1}{4}\rho^2\left(\frac{1-v^2}{E} - \frac{1}{k'G}\right)^2 p^4 + \frac{\rho h}{D}p^2},\\
\left(\frac{\gamma_2}{b}\right)^2 &= \left(\frac{\alpha}{a}\right)^2 - \frac{1}{2}\rho\left(\frac{1-v^2}{E} + \frac{1}{k'G}\right)p^2 - \sqrt{\frac{1}{4}\rho^2\left(\frac{1-v^2}{E} - \frac{1}{k'G}\right)^2 p^4 + \frac{\rho h}{D}p^2}.\\
\left(\frac{\gamma_3}{b}\right)^2 &= \left(\frac{\alpha}{a}\right)^2 - 2\left\{\frac{\rho(1+v)}{E}p^2 - \frac{6}{h^2}k'\right\}.
\end{aligned} \tag{18}$$

Here, if $\pm\gamma_3/b$ is substituted into the homogeneous equations with respect to $\bar{W}$, $\bar{\Phi}_x$ and $\bar{\Phi}_y$, it can be verified, with some effort, that $\bar{W}$ becomes zero identically. Therefore, $W(y)$, $\Phi_x(y)$ and $\Phi_y(y)$ can be expressed as

$$\begin{aligned}
W(y) =& C_1\,e^{(\gamma_1/b)y} + C_2\,e^{(-\gamma_1/b)y} + C_3\,e^{(\gamma_2/b)y} + C_4\,e^{(-\gamma_2/b)y},\\
\Phi_x(y) =& D_1C_1\,e^{(\gamma_1/b)y} + D_1C_2\,e^{(-\gamma_1/b)y} + D_2C_3\,e^{(\gamma_2/b)y} + D_2C_4\,e^{(-\gamma_2/b)y} + C_5\,e^{(\gamma_3/b)y} + C_6\,e^{(-\gamma_3/b)y},\\
\Phi_y(y) =& E_1C_1\,e^{(\gamma_1/b)y} - E_1C_2\,e^{(-\gamma_1/b)y} + E_2C_3\,e^{(\gamma_2/b)y} - E_2C_4 e^{(-\gamma_2/b)y}\\
&+ E_3C_5\,e^{(\gamma_3/b)y} - E_3C_6\,e^{(-\gamma_3/b)y},
\end{aligned} \tag{19}$$

where

$$D_1 = \frac{\frac{1+v}{2}\left\{\left(\frac{\gamma_1}{b}\right)^2 - \left(\frac{\alpha}{a}\right)^2 + \frac{\rho}{k'G}p^2\right\} + \frac{6(1-v)}{h^2}k'}{\frac{1-v}{2}\left(\frac{\alpha}{a}\right)^2 - \frac{1-v}{2}\left(\frac{\gamma_1}{b}\right)^2 - \frac{\rho(1-v^2)}{E} + \frac{6(1-v)}{h^2}k'}\frac{\alpha}{a},$$

$$D_2 = \frac{\frac{1+v}{2}\left\{\left(\frac{\gamma_2}{b}\right)^2 - \left(\frac{\alpha}{a}\right)^2 + \frac{\rho}{k'G}p^2\right\} + \frac{6(1-v)}{h^2}k'}{\frac{1-v}{2}\left(\frac{\alpha}{a}\right)^2 - \frac{1-v}{2}\left(\frac{\gamma_2}{b}\right)^2 - \frac{\rho(1-v^2)}{E} + \frac{6(1-v)}{h^2}k'}\frac{\alpha}{a},$$

$$
\begin{aligned}
E_1 &= \frac{\dfrac{1+v}{2}\left\{\left(\dfrac{\gamma_1}{b}\right)^2-\left(\dfrac{\alpha}{a}\right)^2+\dfrac{\rho}{k'G}p^2\right\}+\dfrac{6(1-v)}{h^2}k'}{\dfrac{1-v}{2}\left(\dfrac{\alpha}{a}\right)^2-\dfrac{1-v}{2}\left(\dfrac{\gamma_1}{b}\right)^2-\dfrac{\rho\left(1-v^2\right)}{E}+\dfrac{6(1-v)}{h^2}k'}\frac{\gamma_1}{a},\\
E_2 &= \frac{\dfrac{1+v}{2}\left\{\left(\dfrac{\gamma_2}{b}\right)^2-\left(\dfrac{\alpha}{a}\right)^2+\dfrac{\rho}{k'G}p^2\right\}+\dfrac{6(1-v)}{h^2}k'}{\dfrac{1-v}{2}\left(\dfrac{\alpha}{a}\right)^2-\dfrac{1-v}{2}\left(\dfrac{\gamma_2}{b}\right)^2-\dfrac{\rho\left(1-v^2\right)}{E}+\dfrac{6(1-v)}{h^2}k'}\frac{\gamma_2}{a},\\
E_3 &= \frac{\alpha b}{a\gamma_3}.
\end{aligned}
\tag{20}
$$

Here, C_i ($i = 1$–6) are the integral constants determined from the six boundary conditions with respect to the two edges parallel to the y-axis.

2.3. Problems in traditional theories

In Timoshenko beam theory, under the premise that the bending deflection is a physical entity that can be recognized, the bending deflection amplitude W_b is obtained by integration of the rotation angle Φ because the relation $\Phi = \mathrm{d}W_b/\mathrm{d}x$ holds. Integrating the second part of Eq. (9) gives

$$
\begin{aligned}
W_b(x) = {} & \frac{\ell}{\lambda_1}B_1C_1\,\mathrm{e}^{(\lambda_1/\ell)x}+\frac{\ell}{\lambda_1}B_1C_2\,\mathrm{e}^{(-\lambda_1/\ell)x}+\frac{\ell}{\lambda_2}B_2C_3\,\mathrm{e}^{(\lambda_2/\ell)x}\\
&+\frac{\ell}{\lambda_2}B_2C_4\,\mathrm{e}^{(-\lambda_2/\ell)x}+C,
\end{aligned}
\tag{21}
$$

where C_i ($i = 1$–4) are determined in advance from the boundary conditions. However, the constant C cannot be determined because no other boundary conditions exist. Therefore, the bending deflection amplitude $W_b(x)$ is indeterminate by a constant value and thus cannot be obtained uniquely, despite suggestions to the contrary [16]. This result is therefore unacceptable from a physical point of view.

In traditional Mindlin plate theory, if the bending deflection w_b is premised to be recognizable physically and we assume

$$
w_\mathrm{b}(x,y,t) = W'_\mathrm{b}(x,y)\,\mathrm{e}^{\mathrm{i}pt},
\tag{22}
$$

then the following relations must hold:

$$
\Phi'_x = \frac{\partial W'_\mathrm{b}}{\partial x},\quad \Phi'_y = \frac{\partial W'_\mathrm{b}}{\partial y},\quad W'_\mathrm{b} = \int_x \Phi'_x\,\mathrm{d}x = \int_y \Phi'_y\,\mathrm{d}y.
\tag{23}
$$

Integrating the first and second parts of Eq. (23) with respect to x and y leads to the following expressions of the Lévy solutions for the plate:

$$
\begin{aligned}
\int_x \Phi'_x\,\mathrm{d}x = {} & \Phi_x(y)\int_x \cos\frac{\alpha}{a}x\,\mathrm{d}x = \Big(\frac{a}{\alpha}D_1C_1\mathrm{e}^{(\gamma_1/b)y}+\frac{a}{\alpha}D_1C_2\,\mathrm{e}^{(-\gamma_1/b)y}+\frac{a}{\alpha}D_2C_3\,\mathrm{e}^{(\gamma_2/b)y}+\frac{a}{\alpha}D_2C_4\,\mathrm{e}^{(-\gamma_2/b)y}\\
&+\frac{a}{\alpha}C_5\mathrm{e}^{(\gamma_3/b)y}+\frac{a}{\alpha}C_6\mathrm{e}^{(-\gamma_3/b)y}\Big)\sin\frac{\alpha}{a}x+F(y),\\
\int_y \Phi'_y\,\mathrm{d}y = {} & \int_y \Phi_y\,\mathrm{d}y\,\sin\frac{\alpha}{a}x\\
= {} & \left(\frac{b}{\gamma_1}E_1C_1\,\mathrm{e}^{(\gamma_1/b)y}+\frac{b}{\gamma_1}E_1C_2\,\mathrm{e}^{(-\gamma_1/b)y}+\frac{b}{\gamma_2}E_2C_3\,\mathrm{e}^{(\gamma_2/b)y}+\frac{b}{\gamma_2}E_2C_4\,\mathrm{e}^{(-\gamma_2/b)y}\right.\\
&\left.+\frac{b}{\gamma_3}E_3C_5\,\mathrm{e}^{(\gamma_3/b)y}+\frac{b}{\gamma_3}E_3C_6\,\mathrm{e}^{(-\gamma_3/b)y}\right)\sin\frac{\alpha}{a}x+F'(x).
\end{aligned}
\tag{24}
$$

For Eq. (24) to satisfy the third relation of Eq. (23), the relation $F(y) = F'(x) = C' = 0$ is necessary, and the relations

$$\frac{a}{\alpha} D_1 = \frac{b}{\gamma_1} E_1, \quad \frac{a}{\alpha} D_2 = \frac{b}{\gamma_2} E_2, \quad \frac{a}{\alpha} = \frac{b}{\gamma_3} E_3, \tag{25}$$

must hold. Here, from Eq. (20), the first and second relations of Eq. (25) can be satisfied, but the third relation cannot hold. Thus, the bending deflection w_b cannot be defined, which is not physically realistic.

These arguments point out certain inexpediencies in the boundary value problem of traditional Timoshenko beam and Mindlin plate theories.

3. An alternative formulation

An alternative formulation of the boundary value problem is therefore proposed for the Timoshenko beam and Mindlin plate. The governing equations and boundary conditions are derived using Hamilton's principle, which is considered to be the most certain procedure for specifying the boundary conditions. The approach essentially involves treating the total transverse deflection w and the bending deflection w_b as fundamental variables for both the Timoshenko beam and the Mindlin plate, and the shearing deflection w_s is then obtained by the relation

$$w = w_b + w_s. \tag{26}$$

3.1. Timoshenko beam

Fig. 2 shows a schematic of the alternative concept of deformation for the Timoshenko beam. The relations for the bending rotation ϕ, shear angle β, bending deflection w_b, shearing deflection w_s and total deflection w are assumed to be given by

$$\phi = \frac{\partial w_b}{\partial x}, \quad \beta = \frac{\partial w_s}{\partial x}, \quad \frac{\partial w}{\partial x} = \frac{\partial w_b}{\partial x} + \frac{\partial w_s}{\partial x}. \tag{27}$$

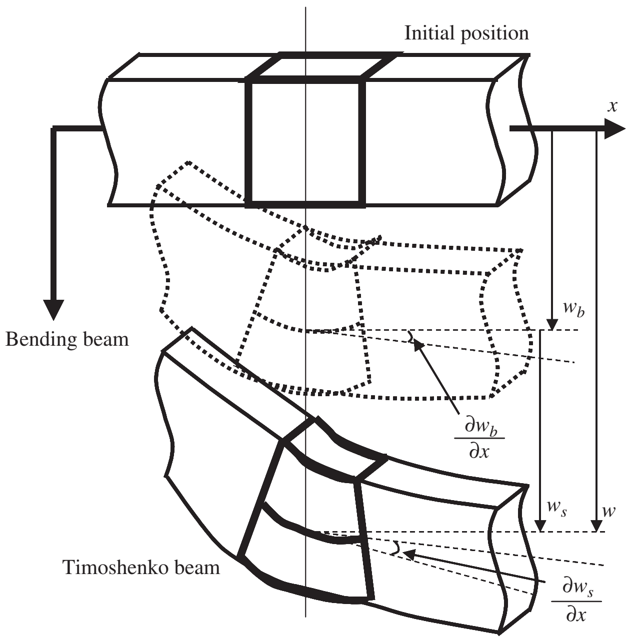

Fig. 2. Schematic of alternative concept of deformation for Timoshenko's beam.

The kinetic and potential energies T and U can then be expressed using w and w_b, and then substituted into Hamilton's principle. After the usual procedures of partial integration according to the calculus of variations [19], the governing equations and boundary conditions expressed by w and w_b are finally obtained as follows:

$$\rho A \frac{\partial^2 w}{\partial t^2} - k' GA\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w_b}{\partial x^2}\right) = 0,$$
$$\rho I \frac{\partial^4 w_b}{\partial x^2 \partial t^2} - EI\frac{\partial^4 w_b}{\partial x^4} - k' GA\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w_b}{\partial x^2}\right) = 0, \tag{28}$$

$$\left[EI\frac{\partial^2 w_b}{\partial x^2}\delta\left(\frac{\partial w_b}{\partial x}\right)\right]_0^\ell = 0,$$
$$\left[k' GA\left(\frac{\partial w}{\partial x} - \frac{\partial w_b}{\partial x}\right)\delta w\right]_0^\ell = 0,$$
$$\left[\left\{\rho I\frac{\partial^3 w_b}{\partial x\,\partial t^2} - EI\frac{\partial^3 w_b}{\partial x^3}\right\}\delta w_b\right]_0^\ell = 0. \tag{29}$$

The first part of Eq. (28) is the same as the first part of Eq. (3) in the traditional formulation if the first expression of Eq. (27) is used. The second relation of Eq. (28) may be obtained by differentiating the second part of Eq. (3) with respect to x and substituting in a similar manner. However, the order of the modeled system increases from 4 to 6 in this case, and the third relation of Eq. (29) is consequently added as a new boundary condition. This means that the degree of freedom of deformation for the present model is more restrictive than in the traditional model. On the other hand, the first and second boundary conditions have the same form if the expression $\phi = \partial w_b/\partial x$ is substituted. Generally speaking, it is supposed that the left-hand side of the boundary condition $[A\delta B]_0^\ell = 0$ indicates the virtual work done by a virtual displacement δB of the boundary, and A is considered to be a generalized work-conjugate load corresponding to its virtual displacement. Therefore, the term expressed in the braces $\{\cdot\}$ of the third boundary condition of Eq. (29) is more precisely defined as the work-conjugate internal transverse force that performs work via the bending virtual displacement δw_b. With the boundary condition of $[A\delta B]_0^\ell = 0$, the condition $A = 0$ is generally referred to as the dynamic boundary condition, while the condition $B = 0$ or other constant represents the geometric boundary condition in the calculus of variations. Thus, in the third expression of Eq. (29), $\{\cdot\} = 0$ is the dynamic boundary condition, and $w_b = 0$ is the geometric boundary condition. These definitions can also be applied to the fifth and sixth expressions in Eq. (40) related to δw_b, as will be shown later.

For the separation of variables with respect to time t, we assume

$$w = W(x)\mathrm{e}^{\mathrm{i}pt}, \quad w_b = W_b(x)\mathrm{e}^{\mathrm{i}pt}, \tag{30}$$

Then, put W and W_b as

$$W = \bar{W}\mathrm{e}^{(\lambda/l)x}, \quad W_\mathrm{b} = \bar{W}_\mathrm{b}\mathrm{e}^{(\lambda/\ell)x}. \tag{31}$$

From the condition that a solution of the form of Eq. (31) may exist, the characteristic equation for λ is obtained as follows:

$$\left(\frac{\lambda}{\ell}\right)^6 + \rho\left(\frac{1}{E} + \frac{1}{k'G}\right)p^2\left(\frac{\lambda}{\ell}\right)^4 + \rho^2\left(\frac{1}{E}\frac{1}{k'G}\right)p^2\left(p^2 - \frac{k'GA}{\rho I}\right)\left(\frac{\lambda}{\ell}\right)^2 = 0. \tag{32}$$

Solving Eq. (32) yields the roots λ_i (i = 1–3), as given by

$$\left(\frac{\lambda_1}{\ell}\right)^2 = -\frac{1}{2}\rho\left(\frac{1}{E} + \frac{1}{k'G}\right)p^2 + \sqrt{\frac{1}{4}\rho^2\left(\frac{1}{E} - \frac{1}{k'G}\right)^2 p^4 + \frac{\rho A}{EI}p^2},$$

$$\left(\frac{\lambda_2}{\ell}\right)^2 = -\frac{1}{2}\rho\left(\frac{1}{E}+\frac{1}{k'G}\right)p^2 - \sqrt{\frac{1}{4}\rho^2\left(\frac{1}{E}-\frac{1}{k'G}\right)^2 p^4 + \frac{\rho A}{EI}p^2},$$
$$\left(\frac{\lambda_3}{\ell}\right)^2 = 0. \tag{33}$$

Here, substituting $\pm\lambda_3/\ell$ into the homogeneous equations with respect to $\bar{W}$ and $\bar{W}_b$ leads to $\bar{W}=0$ under the assumption of $p\neq 0$. Hence, the general solutions for W and W_b are given by

$$W(x) = C_1\,\mathrm{e}^{(\lambda_1/\ell)x} + C_2\,\mathrm{e}^{(-\lambda_1/\ell)x} + C_3\,\mathrm{e}^{(\lambda_2/\ell)x} + C_4\,\mathrm{e}^{(-\lambda_2/\ell)x},$$
$$W_b(x) = B_1'C_1\,\mathrm{e}^{(\lambda_1/\ell)x} + B_1'C_2\,\mathrm{e}^{(-\lambda_1/\ell)x} + B_2'C_3\,\mathrm{e}^{(\lambda_2/\ell)x} + B_2'C_4\,\mathrm{e}^{(-\lambda_2/\ell)x} + C_5\frac{x}{\ell} + C_6, \tag{34}$$

where

$$B_1' = 1 + \frac{\rho}{k'G}p^2\left(\frac{\ell}{\lambda_1}\right)^2, \quad B_2' = 1 + \frac{\rho}{k'G}p^2\left(\frac{\ell}{\lambda_2}\right)^2.$$

In Eq. (34), C_i (i = 1–6) are the integral constants, which are determined from the boundary conditions at both ends of a beam ($x = 0$ and ℓ), as follows.

(1) *Simply supported end*:

$$W=0, \quad W_b=0, \quad EI\frac{\mathrm{d}^2W_b}{\mathrm{d}x^2}=0. \tag{35}$$

(2) *Clamped end*:

$$W=0, \quad W_b=0, \quad \frac{\mathrm{d}W_b}{\mathrm{d}x}=0. \tag{36}$$

(3) *Free end*:

$$EI\frac{\mathrm{d}^2W_b}{\mathrm{d}x^2}=0, \quad k'GA\left(\frac{\mathrm{d}W}{\mathrm{d}x}-\frac{\mathrm{d}W_b}{\mathrm{d}x}\right)=0,$$
$$\rho I p^2\frac{\mathrm{d}W_b}{\mathrm{d}x} + EI\frac{\mathrm{d}^3W_b}{\mathrm{d}x^3}=0. \tag{37}$$

3.2. Mindlin plate

In the alternative formulation for the Mindlin plate, the total deflection $w(x, y, t)$ of the mid-plane and the bending deflection $w_b(x, y, t)$ itself are regarded as the fundamental variables. The shearing deflection $w_s(x, y, t)$ is obtained from the relation $w_s = w - w_b$.

Under the premise of the existence of the physical entity of $w_b(x,y,t)$, the following relations hold:

$$\phi_x = \frac{\partial w_b}{\partial x}, \quad \beta_x = \frac{\partial w_s}{\partial x}, \quad \phi_y = \frac{\partial w_b}{\partial y}, \quad \beta_y = \frac{\partial w_s}{\partial y},$$
$$\frac{\partial w}{\partial x} = \frac{\partial w_b}{\partial x} + \frac{\partial w_s}{\partial x}, \quad \frac{\partial w}{\partial y} = \frac{\partial w_b}{\partial y} + \frac{\partial w_s}{\partial y}, \tag{38}$$

where ϕ_x and ϕ_y are the rotation angles of a transverse normal due to bending about the x- and y-axis, and β_x and β_y are the angles of distortion due to shear with respect to the x and y directions,. Rewriting the kinetic and potential energies T and U of Eqs. (10) and (11) with respect to w and w_b on the basis of Eq. (38), and substituting the result into Hamilton's principle, the usual procedure of the calculus of variations leads to the following two governing equations and seven boundary conditions:

$$\rho h\frac{\partial^2 w}{\partial t^2}-k'Gh\left(\frac{\partial^2 w}{\partial x^2}-\frac{\partial^2 w_b}{\partial x^2}\right)-k'Gh\left(\frac{\partial^2 w}{\partial y^2}-\frac{\partial^2 w_b}{\partial y^2}\right)=0,$$
$$\frac{\rho h^3}{12}\left(\frac{\partial^4 w_b}{\partial x^2\,\partial t^2}+\frac{\partial^4 w_b}{\partial y^2\,\partial t^2}\right)-D\left(\frac{\partial^4 w_b}{\partial x^4}+2\frac{\partial^4 w_b}{\partial x^2\,\partial y^2}+\frac{\partial^4 w_b}{\partial y^4}\right)$$
$$-\left\{k'Gh\left(\frac{\partial^2 w}{\partial x^2}-\frac{\partial^2 w_b}{\partial x^2}\right)+k'Gh\left(\frac{\partial^2 w}{\partial y^2}-\frac{\partial^2 w_b}{\partial y^2}\right)\right\}=0, \tag{39}$$

$$\left[k'Gh\left(\frac{\partial w}{\partial x}-\frac{\partial w_b}{\partial x}\right)\delta w\right]_0^a=0,$$
$$\left[k'Gh\left(\frac{\partial w}{\partial y}-\frac{\partial w_b}{\partial y}\right)\delta w\right]_0^b=0,$$
$$\left[D\left(\frac{\partial^2 w_b}{\partial x^2}+v\frac{\partial^2 w_b}{\partial y^2}\right)\delta\left(\frac{\partial w_b}{\partial x}\right)\right]_0^a=0,$$
$$\left[D\left(\frac{\partial^2 w_b}{\partial y^2}+v\frac{\partial^2 w_b}{\partial x^2}\right)\delta\left(\frac{\partial w_b}{\partial y}\right)\right]_0^b=0,$$
$$\left[\left\langle\frac{\rho h^3}{12}\frac{\partial^3 w_b}{\partial x\,\partial t^2}-D\left\{\frac{\partial^3 w_b}{\partial x^3}+(2-v)\frac{\partial^3 w_b}{\partial x\,\partial y^2}\right\}\right\rangle\delta w_b\right]_0^a=0,$$
$$\left[\left\langle\frac{\rho h^3}{12}\frac{\partial^3 w_b}{\partial y\,\partial t^2}-D\left\{\frac{\partial^3 w_b}{\partial y^3}+(2-v)\frac{\partial^3 w_b}{\partial x^2\partial y}\right\}\right\rangle\delta w_b\right]_0^b=0,$$
$$\left[\left[2(1-v)D\frac{\partial^2 w_b}{\partial x\,\partial y}\delta w_b\right]_0^a\right]_0^b=0. \tag{40}$$

The number of governing equations is thus reduced from three in the traditional formulation to two, but the order of the modeled system remains unchanged. The expressions for the boundary conditions, however, change according to the variation in the physical representation of deformation for a flat plate; i.e., the fundamental variables w and w_b are used in place of the traditional w, ϕ_x and ϕ_y.

The Lévy solution approach [7] is adopted in the following to obtain the closed-form solution for a rectangular plate. The two edges of the plate ($x=0$ and a) are considered to be simply supported. Hence, put

$$w=W(y)\sin\frac{\alpha}{a}x\,\mathrm{e}^{ipt},\quad w_b=W_b(y)\sin\frac{\alpha}{a}x\,\mathrm{e}^{ipt}, \tag{41}$$

where

$$\alpha=n_x\pi\quad(n_x=1,2,\ldots)$$

and further assume

$$W=\bar{W}\,\mathrm{e}^{(\gamma/b)y},\quad W_b=\bar{W}_b\,\mathrm{e}^{(\gamma/b)y}. \tag{42}$$

From the condition that a solution of the form of Eq. (42) may exist for the homogeneous equations with respect to $\bar{W}$ and $\bar{W}_b$, the following characteristic equation for γ is obtained:

$$\left\{\left(\frac{\gamma}{b}\right)^2-\left(\frac{\alpha}{a}\right)^2\right\}^3+\rho\left(\frac{1-v^2}{E}+\frac{\rho}{k'G}\right)p^2\left\{\left(\frac{\gamma}{b}\right)^2-\left(\frac{\alpha}{a}\right)^2\right\}^2$$
$$+\rho^2\left(\frac{1-v^2}{E}\frac{\rho}{k'G}\right)p^2\left(p^2-\frac{12k'G}{\rho h^2}\right)\left\{\left(\frac{\gamma}{b}\right)^2-\left(\frac{\alpha}{a}\right)^2\right\}=0. \tag{43}$$

Solving Eq. (43) yields the roots γ_i ($i = 1$–3), which are expressed as

$$\left(\frac{\gamma_1}{b}\right)^2 = \left(\frac{\alpha}{a}\right)^2 - \frac{1}{2}\rho\left(\frac{1-v^2}{E} + \frac{1}{k'G}\right)p^2 + \sqrt{\frac{1}{4}\rho^2\left(\frac{1-v^2}{E} - \frac{1}{k'G}\right)^2 p^4 + \frac{\rho h}{D}p^2},$$
$$\left(\frac{\gamma_2}{b}\right)^2 = \left(\frac{\alpha}{a}\right)^2 - \frac{1}{2}\rho\left(\frac{1-v^2}{E} + \frac{1}{k'G}\right)p^2 - \sqrt{\frac{1}{4}\rho^2\left(\frac{1-v^2}{E} - \frac{1}{k'G}\right)^2 p^4 + \frac{\rho h}{D}p^2},$$
$$\left(\frac{\gamma_3}{b}\right)^2 = \left(\frac{\alpha}{a}\right)^2. \tag{44}$$

If $\pm\gamma_3/b$ is substituted into the homogeneous equations with respect to $\bar{W}$ and $\bar{W}_b$, $\bar{W}$ is zero under the condition of $p \neq 0$. The general solutions for $W(y)$ and $W_b(y)$ can then be given by

$$W(y) = C_1\,\mathrm{e}^{(\gamma_1/b)y} + C_2\,\mathrm{e}^{(-\gamma_1/b)y} + C_3\,\mathrm{e}^{(\gamma_2/b)y} + C_4\,\mathrm{e}^{(-\gamma_2/b)y},$$
$$W_b(y) = D'_1C_1\,\mathrm{e}^{(\gamma_1/b)y} + D'_1C_2\,\mathrm{e}^{(-\gamma_1/b)y} + D'_2C_3\,\mathrm{e}^{(\gamma_2/b)y} + D'_2C_4\,\mathrm{e}^{(-\gamma_2/b)y} + C_5\,\mathrm{e}^{(\gamma_3/b)y} + C_6\,\mathrm{e}^{(-\gamma_3/b)y}, \tag{45}$$

where

$$D'_1 = 1 + \frac{\rho}{k'G}p^2\left\{\left(\frac{\gamma_1}{b}\right)^2 - \left(\frac{\alpha}{a}\right)^2\right\}^{-1}, \quad D'_2 = 1 + \frac{\rho}{k'G}p^2\left\{\left(\frac{\gamma_2}{b}\right)^2 - \left(\frac{\alpha}{a}\right)^2\right\}^{-1}.$$

Here, C_i ($i = 1$–6) are the integral constants, which are determined from the following boundary conditions at the two edges $y = 0$ and b:

(1) *Simply supported edge*:

$$W = 0, \quad W_b = 0, \quad D\left\{\frac{\mathrm{d}^2W_b}{\mathrm{d}y^2} - v\left(\frac{\alpha}{a}\right)^2 W_b\right\} = 0. \tag{46}$$

(2) *Clamped edge*:

$$W = 0, \quad W_b = 0, \quad \frac{\mathrm{d}W_b}{\mathrm{d}y} = 0. \tag{47}$$

(3) *Free edge*:

$$D\left\{\frac{\mathrm{d}^2W_b}{\mathrm{d}y^2} - v\left(\frac{\alpha}{a}\right)^2 W_b\right\} = 0, \quad k'Gh\left(\frac{\mathrm{d}W}{\mathrm{d}y} - \frac{\mathrm{d}W_b}{\mathrm{d}y}\right) = 0,$$
$$\frac{\rho h^3}{12}p^2\frac{dW_b}{dy} + D\left\{\frac{\mathrm{d}^3W_b}{\mathrm{d}y^3} - (2-v)\left(\frac{\alpha}{a}\right)^2\frac{\mathrm{d}W_b}{\mathrm{d}y}\right\} = 0. \tag{48}$$

4. Comparisons of alternative and traditional formulations

The natural frequencies and mode shapes determined for a Timoshenko beam by the traditional and alternative formulations are compared below, along with the natural frequencies determined for a Mindlin plate. In the following, the simply supported, clamped and free boundary conditions are abbreviated by S, C and F.

Consider a rectangular cross-sectional beam with length of $\ell = 0.5$ m and thickness h as a Timoshenko beam, and a flat rectangular plate with two edges of lengths $a = 0.8$ m and $b = 0.5$ m and thickness h as a Mindlin Plate. The material is assumed to be aluminum with the parameters: Young's modulus $E = 68.6$ GPa, density $\rho = 2700\,\text{kg/m}^3$, Poisson's ratio $v = 0.33$, and shear modulus $G = E/2(1+v)$. The shear coefficient k' is assumed to be 5/6 [15].

For the beam, the characteristic equations used to calculate the natural frequencies in the proposed alternative formulation can be verified for boundary conditions FF, SF, CF and SS to be exactly coincident with those in the traditional case. However, the characteristic equations for the SC and CC boundary conditions given by the alternative approach differ from those in the traditional cases. In the proposed formulation, both bending and shearing deflections are recognized as physical entities that are assigned zero values at supported ends (S or C) of the beam. The degree of system deformation is thus more restrictive in the proposed case when both ends of the beam are supported. In contrast, if either or both of the two ends are free (i.e., FF, SF or CF), the proposed formulation will not exhibit such restrictiveness and the characteristic equations used to calculate the natural frequencies will be the same as those of the traditional approach. This can also be understood by comparing the mode shapes of the two cases (see Figs. 7–14). In the case of the SS boundary conditions, both the bending and shearing deflections (w_b and w_s) have exactly sinusoidal curves as solutions. The characteristic equations for both formulations then become coincident, since the fundamental variables w and ϕ in the traditional theory also have sinusoidal curve solutions. The results for the SC and CC boundary conditions are compared in more detail below as cases in which the present formulation is of most interest.

Fig. 3 shows a comparison of the natural frequencies for the first to fourth modes of the Timoshenko beam with one end simply supported and the other end clamped (i.e., SC). Fig. 4 shows the differences expressed as a percentage with respect to the traditional results. It is observed that the differences between the traditional and alternative methods become larger as the non-dimensional thickness h/ℓ increases, and the natural frequencies for the alternative method are always higher than for the traditional method. As mentioned above, this difference in natural frequency is related to the more restrictive degree of freedom of deformation in the alternative formulation compared to the traditional model. Fig. 5 shows a comparison of the natural frequencies for the CC boundary condition, and Fig. 6 shows the differences expressed as a percentage. For the symmetrical CC boundary condition, the differences in the natural frequencies for the odd-order modes are very small (but not exactly zero) and cannot be distinguished on the graph, whereas those for the even modes exhibit relatively large differences. This feature will be discussed later in relation to the mode shape behavior. It should be noted that the relative difference in natural frequency increases dramatically with the non-dimensional thickness h/ℓ for the third and fourth modes in Fig. 4 and for the fourth mode in Fig. 6. These features depend on the particular behavior of the natural frequency curves in the large non-dimensional thickness range of the traditional results, i.e., in Figs. 3(a) and 5(a). Although the calculated range of beam size up to $h/\ell = 0.5$ may not strictly be sufficiently small for first-order shear deformation theory, it has been shown through comparison of higher-order shear deformation theory with Mindlin plate theory (i.e., first-order shear deformation theory) that the maximum difference between the two theories in terms of the first

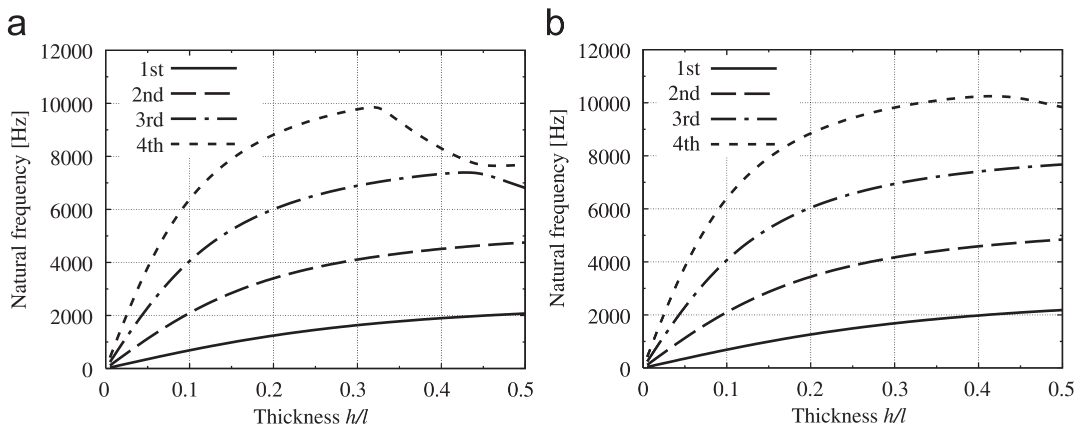

Fig. 3. Comparison of natural frequencies (SC beam) between (a) traditional and (b) alternative formulations.

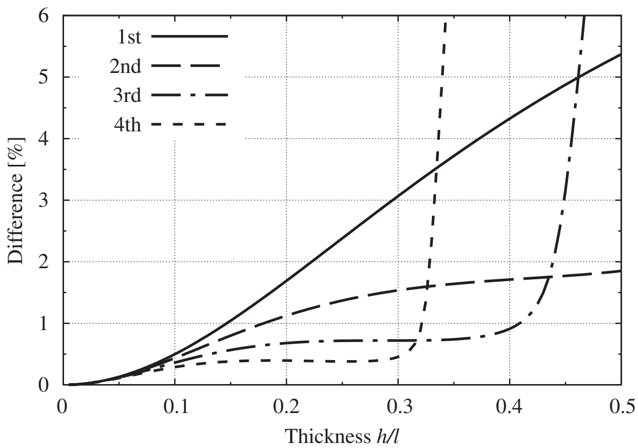

Fig. 4. Difference between alternative and traditional frequencies (SC beam).

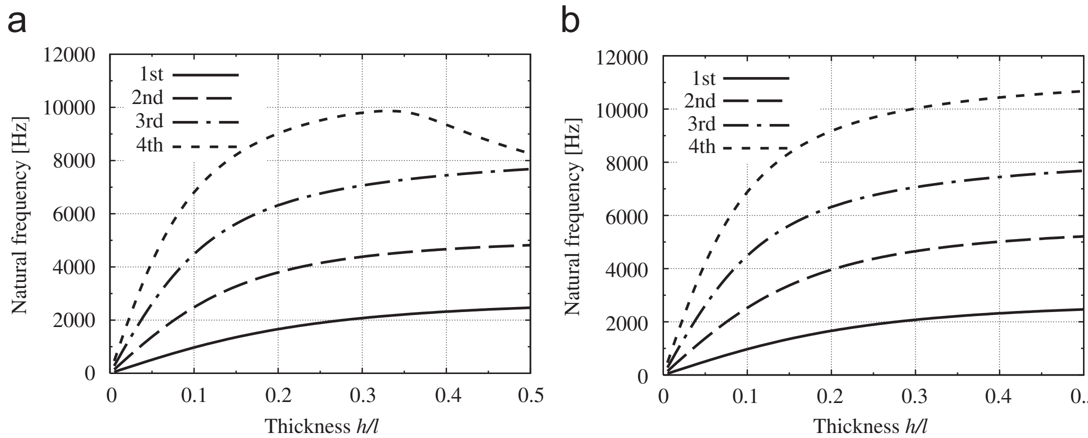

Fig. 5. Comparison of natural frequencies (CC beam) between (a) traditional and (b) alternative formulations.

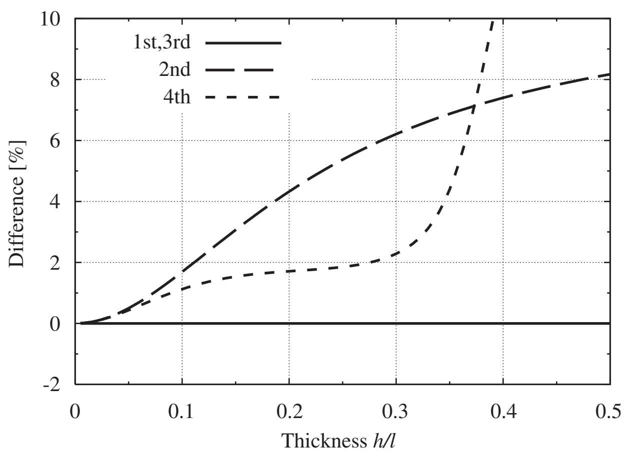

Fig. 6. Difference between alternative and traditional frequencies (CC beam).

five natural frequencies is 2.71%, even for the case of a square plate of size $a \times a$ and a non-dimensional thickness h/a of 0.5 [20]. A comparison of the natural frequencies of 3D theory using Ritz's method and Mindlin plate theory with $\kappa' = 5/6$ has also shown that the difference between the two theories for the first five natural frequencies is at most 3.5% even for a cube of $h/a = 1$ with boundary condition SSSS [21]. The present results for $h/\ell = 0.5$ are therefore considered useful for characterizing the behavior of the natural frequency curves in the traditional formulation (see Figs. 3(a) and 5(a)).

Figs. 7–10 show comparisons of the mode shapes of the first to fourth modes for the SC Timoshenko beam of $h/\ell = 0.5$. The traditional results are shown with an integral constant C (Eq. (21)) of zero, although the bending deflection amplitude $W_b(x)$ and shearing deflection amplitude $W_s(x)$ are in fact indeterminate due to this constant. Similarly, Figs. 11–14 show comparisons of the mode shapes of the first to fourth modes for the CC Timoshenko beam. These results indicate that the bending and shearing deflections cannot be determined uniquely using the traditional formulation, whereas almost physically normal deflection curves are obtained using the alternative formulation of Timoshenko's beam. For the odd-order modes (first and third) in the case of boundary condition CC, if the constant C in Eq. (21) is adjusted such that W_b becomes zero at one end, W_b becomes zero at the other end, resulting in a shearing deflection W_s of zero concurrently at both ends. Such deformation behavior is consistent with the fundamental premise of the proposed formulation (see Figs. 11(b) and 13(b)). In contrast, for the even-order modes (second and fourth), even if the constant C is adjusted such that W_b becomes zero at one end, W_b takes a very large value at the other end. Such deformation behavior is

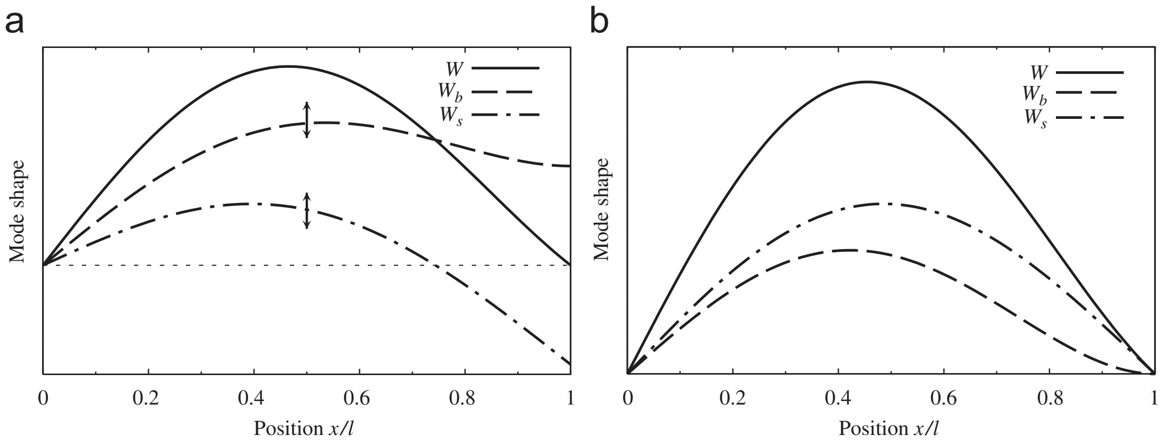

Fig. 7. Comparison of mode shapes (SC beam, $h/\ell = 0.5$, first mode): (a) traditional, and (b) alternative.

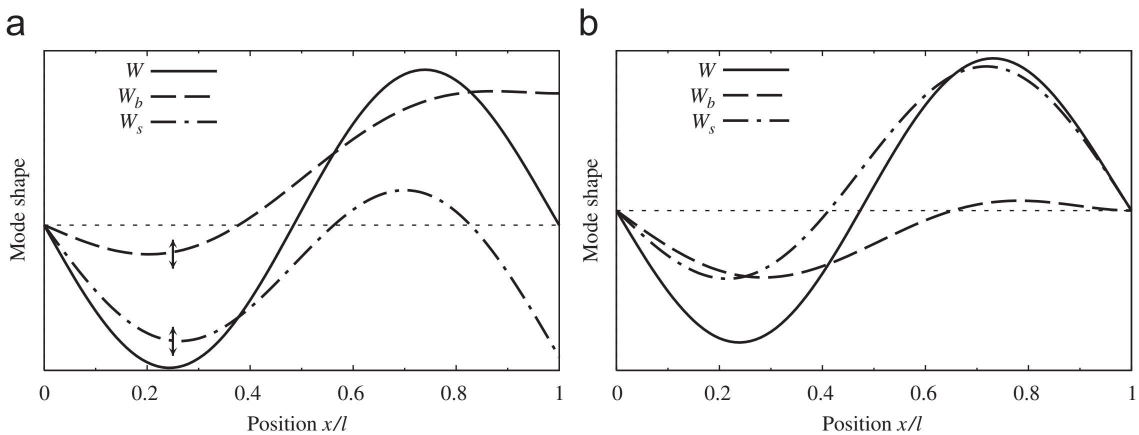

Fig. 8. Comparison of mode shapes (SC beam, $h/\ell = 0.5$, 2nd mode): (a) traditional, and (b) alternative.

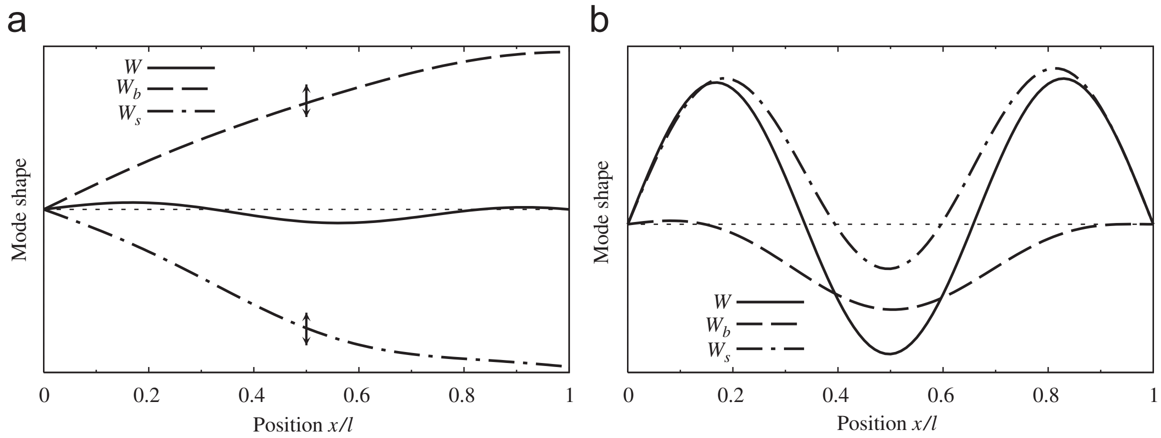

Fig. 9. Comparison of mode shapes (SC beam, $h/\ell = 0.5$, 3rd mode): (a) traditional, and (b) alternative.

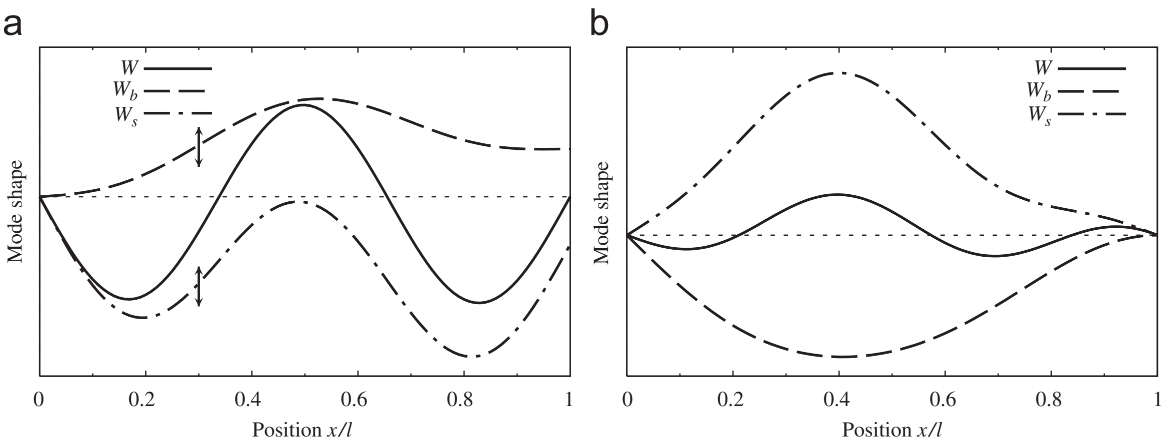

Fig. 10. Comparison of mode shapes (SC beam, $h/\ell = 0.5$, 4th mode): (a) traditional, and (b) alternative.

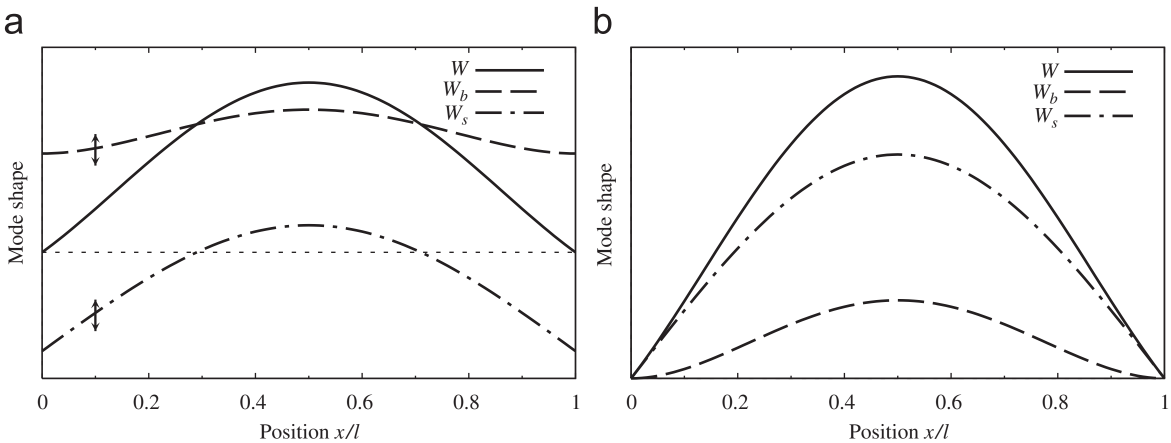

Fig. 11. Comparison of mode shapes (CC beam, $h/\ell = 0.5$, 1st mode): (a) traditional, and (b) alternative.

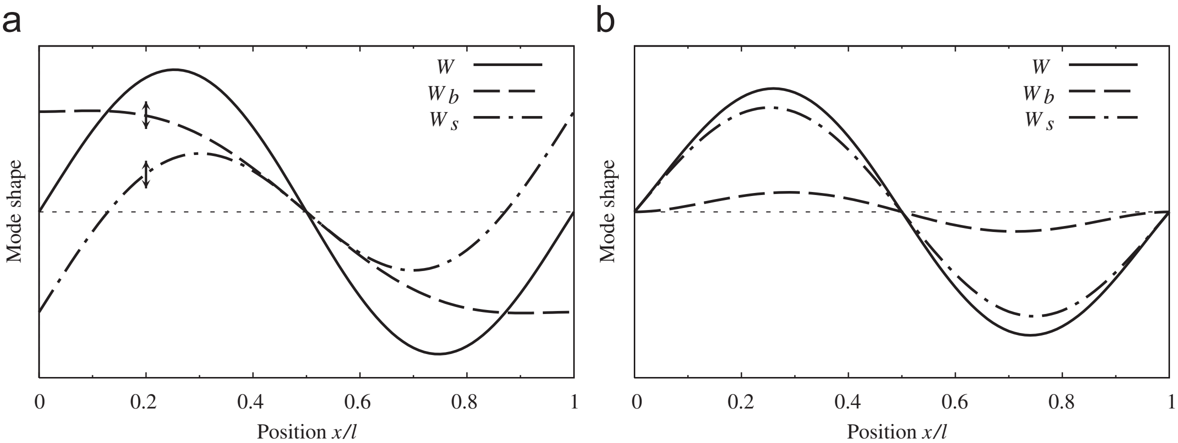

Fig. 12. Comparison of mode shapes (CC beam, $h/\ell = 0.5$, 2nd mode): (a) traditional, and (b) alternative.

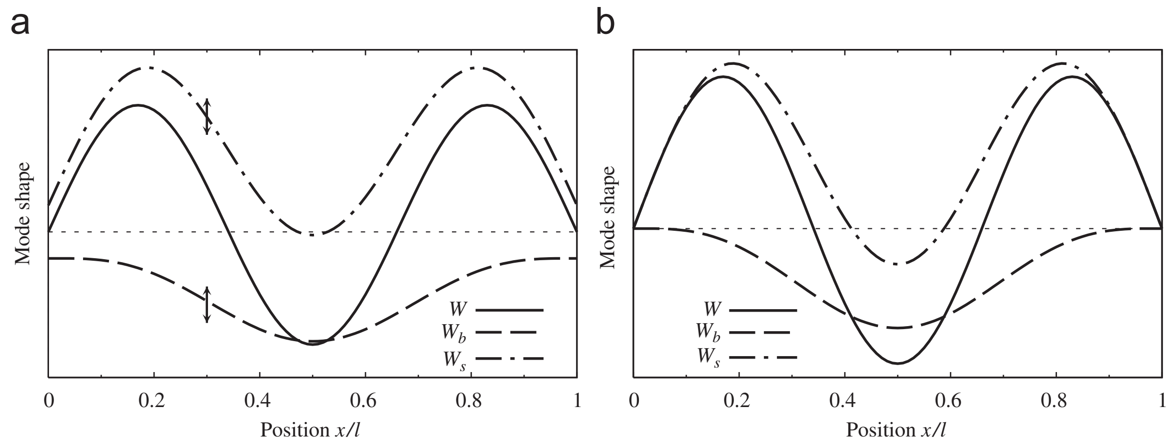

Fig. 13. Comparison of mode shapes (CC beam, $h/\ell = 0.5$, 3rd mode): (a) traditional, and (b) alternative.

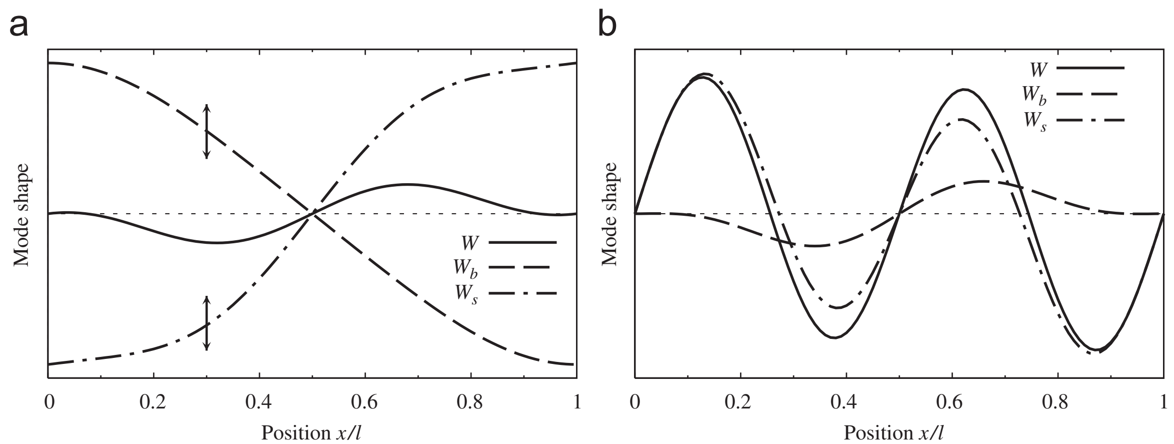

Fig. 14. Comparison of mode shapes (CC beam, $h/\ell = 0.5$, 4th mode): (a) traditional, and (b) alternative.

not consistent with the proposed concept of deformation. This feature may be the reason why the first and third frequencies given by the traditional and alternative formulations are almost coincident (but not exactly the same) even in the case of the CC boundary condition, despite the two approaches giving dissimilar second and fourth frequencies.

The natural frequencies of a flat rectangular plate simply supported at edges $x = 0$ and a ($= 0.8$ m), i.e., an SCSC boundary condition, are shown in Fig. 15 for the fundamental (first) mode and the second mode with respect to the x direction, including two anti-nodes. As in the case for the beam, the alternative method produces higher frequencies than the traditional method. Fig. 16(a) shows the differences in the fundamental natural frequencies as a percentage with respect to the traditional results for various boundary conditions, and Fig. 16(b) shows the differences for the second mode with respect to the x direction. Again, the natural frequencies obtained by the alternative formulation are higher.

Thus, the alternative formulation affords natural frequencies that are equal to or higher than those obtained by the traditional calculation under a given boundary condition, and more importantly, allows the bending and shearing deflections to be obtained concurrently and uniquely using a deductive methodology. This alternative approach, although derived primarily for dynamic analyses of the Timoshenko beam and Mindlin plate, is valid also for static analyses in cases where the bending and shearing deflections are recognizable as

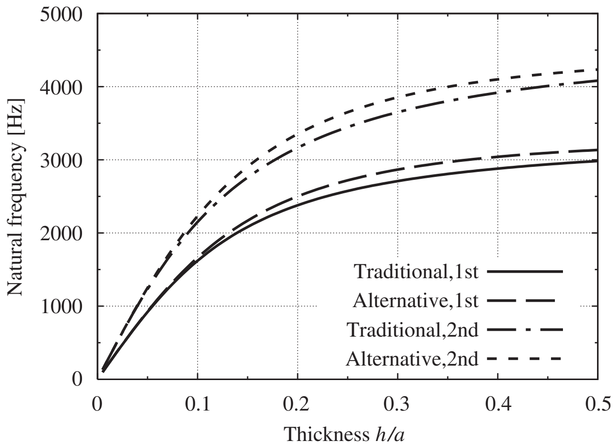

Fig. 15. Comparison of natural frequencies for the first and second modes (SCSC flat plate, $a = 0.8$ m, $b = 0.5$ m).

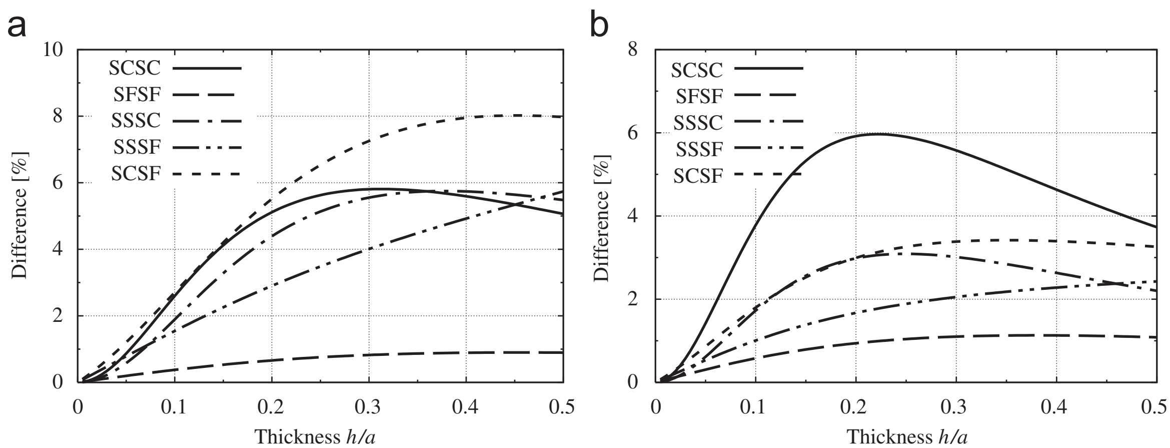

Fig. 16. Difference between alternative and traditional frequencies (flat plate, $a = 0.8$ m, $b = 0.5$ m) for the (a) fundamental (first) mode and (b) second mode with respect to the x direction.

physical entities and are to be obtained independently. This is the first such proposal of a consistent procedure for static analyses. The alternative formulation presented here also provides adjustability with respect to the physical recognition of deformation for beams and flat plates.

5. Conclusions

The conventional concept of deformation in the Timoshenko beam and Mindlin Plate theories involves some inexpediency in that the bending and shearing deflections cannot be determined independently and uniquely as physical entities. The notion that the bending and shearing deflections are distinguishable in beams and flat plates has gained widespread acceptance for static analyses involving shearing and bending. The alternative formulations of Timoshenko's beam and Mindlin's plate proposed in this study regard the bending deflection and total deflection as two fundamental variables, and the natural conditions (governing equations and boundary conditions) are derived on the basis of Hamilton's principle. These alternative formulations afford natural frequencies equal to or higher than those obtained by the traditional methods for certain boundary conditions owing to a more restrictive degree of system deformation under the proposed formulation. If the alternative formulation is applied to static problems, the total deflection can be obtained by a deductive methodology. The proposed formulation thus represents the first consistent procedure proposed for static analysis. This feature of consistency is of particular importance for both dynamic and static analyses, and should be of interest as an alternative to first-order shear deformation theory for beams and flat plates.

References

[1] S. Timoshenko, On the correction for shear of the differential equation for transverse vibrations of prismatic bars, *Philosophical Magazine (series 6)* 41 (1921) 744–746.

[2] S. Timoshenko, D.H. Young, W. Weaver Jr., *Vibration Problems in Engineering*, Wiley, New York, 1928, pp. 432–435 (Chapter 5).

[3] R.D. Mindlin, Influence of rotatory inertia and shear on flexural motions of isotropic, elastic plates, *Journal of Applied Mechanics* 18 (1) (1951) 31–38.

[4] A.K. Noor, W.S. Burton, Assessment of shear deformation theories for multilayered composite plates, *Applied Mechanics Review* 42 (1) (1989) 1–12.

[5] M. Levinson, An accurate, simple theory of the statistics and dynamics of elastic plates, *Mechanics Research Communications* 7 (6) (1980) 343–350.

[6] Y.K. Cheung, D. Zhou, Vibration of tapered Mindlin plates in terms of static Timoshenko beam functions, *Journal of Sound and Vibration* 260 (2003) 693–709.

[7] Y. Xiang, J.N. Reddy, Natural vibration of rectangular plates with an internal line hinge using the first order shear deformation plate theory, *Journal of Sound and Vibration* 263 (2003) 285–297.

[8] M. El-Raheb, Exact solution of the asymmetric Mindlin's plate equations applied to a disk, *Journal of Sound and Vibration* 261 (2003) 153–168.

[9] M. El-Mously, A timoshenko-beam-on-pasternak-foundation analogy for cylindrical shells, *Journal of Sound and Vibration* 261 (2003) 635–652.

[10] S. Timoshenko, *Strength of Materials; Part I Elementary*, Van Nostrand Reinhold, New York, 1930, pp. 170–175 (Chapter 5).

[11] L.H. Donnell, Discussion of the literature by E. Reissner, the effect of transverse shear deformation on the bending of elastic plates, *Journal of Applied Mechanics, Transactions of the ASME* 67 (1946) A-69–A-77.

[12] L.H. Donnell, *Beams, Plates and Shells*, McGraw-Hill, New York, 1976, pp. 298–301 (Chapter 5).

[13] I. Nakahara, *Strength of Materials, Part I*, Yoken-do, Tokyo, 1965, pp. 192–194 (Chapter 7) (in Japanese).

[14] A. Okumura, *Strength of Materials*, Korona Company, Tokyo, 1973, pp. 286–289 (Chapter 4) (in Japanese).

[15] R.P. Shimpi, Refined plate theory and its variants, *AIAA Journal* 40 (1) (2002) 137–146.

[16] L.S. Jacobsen, R.S. Ayre, *Engineering Vibrations with Applications to Structures and Machinery*, McGraw-Hill, New York, 1958, pp. 496–513 (Chapter 10).

[17] R.A. Anderson, Flexural vibrations in uniform beams according to the Timoshenko theory, *Journal of Applied Mechanics* 20 (4) (1953) 504–510.

[18] R.A. Anderson, Transient Response of Uniform Beams, Doctoral Thesis submitted to the California Institute of Technology, Pasadena, CA, USA, 1953.

[19] L. Meirovitch, *Principles and Techniques of Vibrations*, Prentice-Hall, Englewood Cliffs, NJ, 1997, pp. 423–430 (Chapter 7).

[20] N.F. Hanna, A.W. Leissa, A higher order shear deformation theory for the vibration of thick plates, *Journal of Sound and Vibration* 170 (1994) 545–555.

[21] Y. Narita, Y. Ohta, S. Morika, Three-dimensional vibration analysis of symmetrically laminated, cross-ply thick rectangular plates, *Transactions of the JSME* 61-584 (1995) 842–848 (in Japanese).