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Short communication

Modal mass of clamped beams and clamped plates

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Abstract

The calculation of the forced vibration response of a clamped beam or clamped plate often involves the calculation of the *modal* (or generalized) mass. Calculation of this term for simply-supported structures is relatively easy and results in a value that is half the mass of the structure for beams, and a quarter of the mass of the structure for plates and cylinders. However, calculation of this term for clamped beams and clamped plates is algebraically more complicated and has led to the presentation of several formula in research literature that appear vastly different. This paper contains a derivation of the modal mass of a clamped beam and clamped plate and shows that several of the formula for the modal mass in the research literature are equivalent.

1. Introduction

The focus of this short communication is the derivation of the modal (or generalized) mass of a clamped beam or clamped plate, that is one aspect of the derivation of the forced vibration response of a structure using modal summation techniques. The reader should consult text books such as Soedel [1], Tse et al. [2] and Morse and Ingard [3] for a full description of modal methods.

The calculation of the modal mass of a structure involves integration of the square of the assumed mode shape function over the length of the structure in each of its dimension. For a one-dimensional beam, the integration is over the length of the beam. For a plate, the integration occurs across the length and width of the plate and for a cylinder, it occurs along the axial length and the circumference.

The equation for the assumed mode shape function depends on the boundary conditions of the structure. The boundary condition that is most commonly described in vibration handbooks is the simply-supported edge condition. For structures with these boundary conditions, the assumed mode shape ψ is a sinusoidal function such as

$$\psi(x) = \sin(n\pi x/L), \tag{1}$$

where $n = 1 \dots \infty$ is the mode number, $x = 0 \dots L$ is the position along the structure and L is the length of the structure. The modal mass for a simply-supported beam is derived by evaluating the square of the mode shape

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function given in Eq. (1) and is

$$\int_0^L \psi_m \psi_n \, \mathrm{d}x = L/2 \quad \text{for } m = n, \tag{2}$$

$$= 0 \quad \text{for } m \neq n. \tag{3}$$

This integral involving sine functions can be found in most mathematical handbook of integrals. The modal-mass is then $\rho S(L/2)$, which is half the mass of the beam, where ρ is the density, S is the cross-sectional area of the beam.

Warburton [4] presented the first comprehensive set of equations describing the vibration of plates for various boundary conditions. He described the out-of-plate deflection of plates $W(x, y)$ as the product of two mode shape functions for beams, that is,

$$W(x, y) = X(x)Y(y), \tag{4}$$

where $X(x)$ and $Y(y)$ are appropriate mode shape functions of beams that account for the boundary conditions along the length (x) and width (y) of the plate.

The derivation for the modal mass of a simply-supported plate utilizes these same results for a simply-supported beam. The modal mass is then $\rho h(L_x/2)(L_y/2)$, which is a quarter the mass of the plate, where ρ is the density, h is the thickness, L_x and L_y are the lengths of the plate in the x and y directions, respectively.

The derivation of the modal mass for a clamped–clamped beam or fully clamped plate is more complex. The evaluation of the integral has led to various formulas presented in the research literature. The expressions all appear different, but it is shown here that most are in fact equivalent.

2. Summary of formulas presented in the literature

Several authors have presented different formulas for the modal mass of a clamped beam or clamped plate. The following text is a brief review of some of these formulas. Although authors have used differing nomenclature in their respective papers, the equations presented here use consistent variables to enable comparisons of their equations.

The mode shape functions for a clamped–clamped beam used by all authors use the roots k_n of the following equation:

$$\cos(k_n)\cosh(k_n) - 1 = 0. \tag{5}$$

Table 1 lists the value of k_n for values of the modal indice n.

Other terms that are used in this paper are L the length of the beam, and the term

$$D_n(k_n) = \frac{\cosh(k_n) - \cos(k_n)}{\sinh(k_n) - \sin(k_n)}, \tag{6}$$

which is a factor in the expression for the mode shape function ψ.

Table 1
Values of k for modal indice n

n	k_n
1	4.7300408
2	7.8532046
3	10.9956078
4	14.1371655
5	17.2787596
6	20.4203522
$n>6$	$(2n+1)\pi/2$

2.1. Derivation by Young

Young [5] defined the mode shape function for a clamped–clamped beam as

$$\psi_n = \cosh\left(\frac{k_n x}{L}\right) - \cos\left(\frac{k_n x}{L}\right) - D_n\left[\sinh\left(\frac{k_n x}{L}\right) - \sin\left(\frac{k_n x}{L}\right)\right], \tag{7}$$

where D_n was not defined in the paper but listed as a table of values. The integral of the square of the mode shape function was defined in the paper as [5, Eq. (8)]

$$\int_0^L \psi_n^2 \,\mathrm{d}x = L \tag{8}$$

and hence the modal mass is then ρSL, which is the total mass of the beam.

2.2. Derivation by Carmichael

Carmichael defined the mode shape function for a clamped–clamped beam as [6, Eq. (10)]

$$\psi_n = E_n\left[\cosh\left(\frac{k_n x}{L}\right) - \cos\left(\frac{k_n x}{L}\right)\right] - \left[\sinh\left(\frac{k_n x}{L}\right) - \sin\left(\frac{k_n x}{L}\right)\right], \tag{9}$$

where $E_n = 1/D_n$. The integral of the square of the mode shape function was defined in the paper as [6, Eq. (11)]

$$\int_0^L \psi_n^2 \,\mathrm{d}x = LE_n^2 \tag{10}$$

and hence the modal mass is then ρSLE_n^2.

2.3. Derivation by Arenas

Other researchers have not explicitly defined the modal mass in their derivations, such as Sung and Jan [7,8]. The omission was noted by Arenas [9] who attempted to perform the integration of the beam mode shape functions that resulted in the following (after manipulating [9, Eqs. (6)–(9)] so that it is in a similar format to the other equations described here):

$$\int_0^L \psi_n^2 \,\mathrm{d}x = \frac{LA_n}{k_n}, \tag{11}$$

where

$$\begin{aligned} A_n = {} & \tfrac{1}{4}(1 + D_n^2)\sinh(2k_n) + \sinh(k_n)[2D_n \sin(k_n) - (1 - D_n^2)\cos(k_n)] \\ & - (1 + D_n^2)\sin(k_n)\cosh(k_n) + \tfrac{1}{2}(1 - D_n^2)\sin(k_n)\cos(k_n) + k_n \\ & - \tfrac{1}{2}D_n[1 + \cosh(2k_n)] + D_n\cos^2(k_n) \end{aligned} \tag{12}$$

and hence the modal mass is then $\rho SLA_n/k_n$.

2.4. Derivation by Barboni et al.

Barboni et al. [10] defined the mode shape of a clamped beam as

$$\psi_n(x) = -B_n\{[\sinh(k_n x/L) - \sin(k_n x/L)] - E_n[\cosh(k_n x/L) - \cos(k_n x/L)]\} \tag{13}$$

and

$$B_n = \left\{ E_n^2 + \frac{\sin 2k_n}{4k_n}(E_n^2 - 1) - \frac{E_n}{k_n}\sin^2 k_n + \frac{\sinh 2k_n}{4k_n}(E_n^2 + 1) + \frac{(1 - E_n^2)}{k_n}\cos k_n \sinh k_n \right.$$
$$\left. + \frac{2E_n}{k_n}\sin k_n \sinh k_n - \frac{(E_n^2 + 1)}{k_n}\sin k_n \cosh k_n - \frac{E_n}{2k_n}(\cosh(2k_n) - 1) \right\}^{-1/2}. \tag{14}$$

The integral of the square of the mode shape function is then [10, p. 113]

$$\int_0^L \psi_n^2 \, \mathrm{d}x = L, \tag{15}$$

which is the same result as derived by Young. However, the additional term B_n that precedes the mode shape function was perhaps included to normalize the mode shape function, so that the integral of the square of the mode shape function would evaluate to L.

It is clear from the above discussion that researchers have presented several formula for the integral that defines the modal mass of a clamped beam or clamped plate. The following section is a complete derivation of the modal mass of a fully clamped plate, where the mode shape for a clamped beam is assumed. It is shown through the derivation that several of the previous formula are equivalent, and that they reduce to the same expression.

3. Derivation of the modal mass

The mode shape for a clamped beam can be defined as Eq. (7). The modal mass is calculated as

$$\rho S \Lambda_n = \rho S \int_0^L \psi_n^2(x) \, \mathrm{d}x. \tag{16}$$

The evaluation of this integral involves several pages of algebraic manipulation. By making use of trigonometric circular functions and grouping appropriate terms, the equation can be simplified to

$$\Lambda_n = \frac{L}{k_n}\{2\sin(k_n)D_n \sinh(k_n) + 1/2 D_n^2 \sinh(k_n)\cosh(k_n) + 1/2\sinh(k_n)\cosh(k_n)$$
$$+ \sinh(k_n)\cos(k_n)D_n^2 - \sinh(k_n)\cos(k_n) - \sin(k_n)\cosh(k_n) - \sin(k_n)D_n^2\cosh(k_n)$$
$$- 1/2\cos(k_n)\sin(k_n)D_n^2 + 1/2\cos(k_n)\sin(k_n) - D_n(\cosh(k_n))^2 + D_n(\cos(k_n))^2 + k_n\}. \tag{17}$$

It can be shown that Eq. (14) by Barboni et al. is in fact equivalent to Eq. (17).

By shifting the L/k_n to the left-hand side, the equation can be simplified further into

$$(k_n/L) \times \Lambda_n = \tfrac{1}{4}(1 + D_n^2)\sinh(2k_n) + \sinh(k_n)[2D_n \sin(k_n) - (1 - D_n^2)\cos(k_n)] - (1 + D_n^2)\sin(k_n)\cosh(k_n)$$
$$+ \tfrac{1}{2}(1 - D_n^2)\sin(k_n)\cos(k_n) + k_n - \tfrac{1}{2}D_n[1 + \cosh(2k_n)] + D_n\cos^2(k_n), \tag{18}$$

which is identical to the expression Λ_n by Arenas shown here in Eq. (12) (see Ref. [9, Eq. (7)]). Eq. (18) can be further simplified by substituting Eq. (6) for D_n. After factoring the denominator, and returning the k_n/L on the left-hand side of Eq. (18) to the right-hand side, yields

$$\Lambda_n = \frac{-L}{k_n[\sinh(k_n) - \sin(k_n)]^2}\{-k_n[\sinh(k_n) - \sin(k_n)]^2 + \cosh(k_n)\sinh(k_n)(\cos(k_n))^2$$
$$- \cos(k_n)\sinh(k_n) - \sin(k_n)\cos(k_n)(\cosh(k_n))^2 + \sin(k_n)\cosh(k_n)\}. \tag{19}$$

The crucial step here, where some authors stop their derivation, is to factor the remaining terms after $-k_n[\sinh(k_n) - \sin(k_n)]^2$ so that Eq. (19) can be written as

$$\Lambda_n = \frac{-L}{k_n[\sinh(k_n) - \sin(k_n)]^2}\{-k_n[\sinh(k_n) - \sin(k_n)]^2$$
$$- (\cos k_n \cosh k_n - 1)(\sin k_n \cosh k_n - \cos k_n \sinh k_n)\}. \tag{20}$$

Noting the result in Eq. (5) where $\cos k_n \cosh k_n = 1$ and cancelling the remaining terms, the equation can be simplified to

$$\Lambda_n = L. \tag{21}$$

It has therefore been shown that the expressions derived by Young, Arenas, and Barboni et al. are equivalent.

4. Summary

The proof detailed here shows that the integral of the square of the mode shape function for a clamped beam is L, which was shown by Young's equation in Eq. (8). Carmichael's equation in Eq. (10) has the extra term E_n^2. For values of $n > 1$, $E_n^2 \approx 1$ and hence whilst not precisely correct, it does numerically calculate to the correct value. Arenas' expression for the modal mass in equation Eq. (11) is also correct, however the term for A_n could have been further simplified as shown in this paper. Similarly, Barboni's equation in Eq. (15) was also shown to be correct, however, the algebraic expression for B_n could have been further simplified. The equation derived by Barboni et al. is related to Arenas' expression as $B_n^2 = k_n/A_n$.

By using the result presented in this paper, it follows that the modal mass for a clamped beam is just the mass of the beam, and the modal mass of a clamped plate is simply the mass of the plate.

It is hoped that the derivation presented in this paper will provide a useful reference to future researchers attempting to find a formula for the modal mass of a clamped beam or clamped plate.

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