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Short Communication

Determination of periodic solution for a $u^{1/3}$ force by He's modified Lindstedt–Poincaré method

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Abstract

This paper applies He's modified Lindstedt–Poincaré method to determine the periodic solutions of oscillators in a $u^{1/3}$ force. The result obtained and comparison with analytical solution provides confirmation for the validity of He's Modified Lindstedt–Poincaré method.

1. Introduction

In nonlinear analysis, perturbation methods are well-established tools to study diverse aspects of nonlinear problems. Surveys of the early literature with numerous references, and useful bibliographies, have been given by Nayfeh [1], Mickens [2], Jordan and Smith [3] and Hagedorn [4]. However, the use of perturbation theory in many important practical problems is invalid, or it simply breaks down for parameters beyond a certain specified range. Therefore, new analytical techniques should be developed to overcome these shortcomings. Such a new technique should work over a large range of parameters and yield accurate analytical approximate solutions beyond the coverage and ability of the classical perturbation methods. For example, some extensions of the Lindstedt–Poincaré perturbation method to strongly nonlinear systems, so-called He's Modified Lindstedt–Poincaré method, have been proposed; see the comprehensive book by He [5] and the references therein. In He's Modified Lindstedt–Poincaré method, a constant, rather than the nonlinear frequency, is expanded in powers of the expanding parameter to avoid the occurrence of secular terms in the perturbation series solution. The results show that the obtained approximate solutions are uniformly valid on the whole solution domain and they are suitable not only for weakly nonlinear systems, but also for strongly nonlinear systems.

There also exists a wide range of literature dealing with the approximate determination of periodic solutions for nonlinear problems by using a mixture of methodologies [6–17].

The purpose of this paper is the determination of the periodic solutions to nonlinear oscillator equations for which the elastic restoring forces are non-polynomial functions of the displacement by applying He's Modified

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Lindstedt–Poincaré method. This class of equations represents a new class of nonlinear oscillating systems which were first studied in detail by Mickens [18].

2. He's modified Lindstedt–Poincaré method

Currently, we will study the properties of the periodic solutions to certain nonlinear oscillators by applying He's modified Lindstedt–Poincaré method for which the elastic restoring forces are non-polynomial functions of the displacement. In particular, this term is chosen to be

$$f(x) = -x^{1/3}.$$

As it is well known that the Lindstedt–Poincaré method [1–4] gives uniformly valid asymptotic expansions for the periodic solutions of weakly nonlinear oscillators, in general, the technique is not applicable in case of strongly nonlinear terms or elastic restoring forces are non-polinomial functions. Therefore, the work reported here applies He's modified Lindstedt–Poincaré method which also works for strongly nonlinear systems as well as the nonlinear systems with the elastic restoring forces are non-polynomial functions of displacement. The fundamental nature of the method is simply based on expanding both the solution and coefficient of u into series [19–21]. In the following examples, we will illustrate the usefulness and effectiveness of the proposed technique.

Example 1. Now, consider the following nonlinear oscillator which was first studied in detail by Mickens [18]:

$$u'' + u^{1/3} = 0, \quad u(0) = A, \quad u'(0) = 0. \tag{1}$$

By applying harmonic balance method and using the first-order approximate solution

$$u_0 \simeq A\cos\omega t \tag{2}$$

to Eq. (1), Mickens determined angular frequency, ω, as

$$\omega = \left(\frac{4}{3A^2}\right)^{1/6} \approx 1.04912A^{-1/3}. \tag{3}$$

Mickens [22] also used the second-order harmonic balance approximation to the periodic solution of Eq. (1) and determined ω as

$$\omega = \frac{1}{\left[\left(\frac{3}{4}\right) + \left(\frac{27}{4}\right)\bar{z} + \left(\frac{243}{2}\right)\bar{z}^2\right]^{1/6}}\left(\frac{1+\bar{z}}{A}\right)^{1/3}, \tag{4}$$

where $\bar{z}$ is the one of the solution having the smallest absolute magnitude of the polynomial equation

$$(1701)z^3 - (27)z^2 + (51)z + 1 = 0.$$

Comparing (3) with (4), it is clearly seen that the second harmonic balance approximation only provides small corrections to the periodic solution obtained in the first approximation and is negligible. This is the expected result.

More recently, He [5], and Xu [23], determined ω as in Eq. (3) by applying homotopy perturbation method and bookkeeping parameter method, respectively.

Now, to apply modified Lindstedt–Poincaré method, we re-write Eq. (1) in the form

$$u'' + 0u + pu^{1/3} = 0. \tag{5}$$

Supposing the constant zero in (5) can be expressed as

$$0 = \omega^2 + p\omega_1 + p^2\omega_2 + \cdots \tag{6}$$

and we can assume that the solution can be written in the form

$$u = u_0 + pu_1 + p^2u_2 + \cdots \tag{7}$$

Substituting Eqs. (6) and (7) into Eq. (5) and processing as the standard perturbation method, we have

$$u_0'' + \omega^2 u_0 = 0, \quad u_0(0) = A, \; u_0'(0) = 0, \tag{8}$$

$$u_0'' + \omega^2 u_1 + \omega_1 u_0 + u_0^{1/3} = 0. \tag{9}$$

Solving Eq. (8), we have

$$u_0 = A\cos\omega t. \tag{10}$$

Substituting Eq. (10) into Eq. (9) results into

$$u_1'' + \omega^2 u_1 + \omega_1 A\cos\omega t + A^{1/3}(\cos\omega t)^{1/3} = 0. \tag{11}$$

Fourier series representation is needed for $(\cos\omega t)^{1/3}$. It has been calculated [24] and is given by

$$(\cos\omega t)^{1/3} = \sum_{n=0}^{\infty} a_{2n+1}\cos(2n+1)\omega t, \tag{12}$$

$$a_{2n+1} = \frac{3\Gamma\left(\frac{7}{3}\right)}{2^{4/3}\Gamma\left(n+\frac{5}{3}\right)\Gamma\left(\frac{2}{3}-n\right)},$$

with $a_1 = 1.159595266963929$. Therefore, the first several terms are

$$(\cos\omega t)^{1/3} = a_1\left[\cos\omega t - \frac{\cos 3\omega t}{5} + \frac{\cos 5\omega t}{10} - \frac{7\cos 7\omega t}{110} + \frac{\cos 9\omega t}{22} - \frac{13\cos 11\omega t}{374} + \cdots\right]. \tag{13}$$

Substituting Eq. (13) into Eq. (11) yields

$$u_1'' + \omega^2 u_1 + \omega_1 A\cos\omega t + A^{1/3}a_1\left[\cos\omega t - \frac{\cos 3\omega t}{5} + \cdots\right] = 0. \tag{14}$$

The requirement of no secular term gives

$$\omega_1 A + A^{1/3}a_1 = 0 \tag{15}$$

and

$$\omega_1 = -\frac{a_1}{A^{2/3}} = -\frac{1.1596}{A^{2/3}}. \tag{16}$$

If the first-order approximation is enough, then from Eq. (6), we have

$$\omega^2 + \omega_1 = 0 \tag{17}$$

and the frequency can be obtained in the form of

$$\omega = \frac{1.0768}{A^{1/3}}. \tag{18}$$

We, therefore, obtain the following approximated period:

$$T = \frac{2\pi A^{1/3}}{1.0768} = 5.835A^{1/3}. \tag{19}$$

For the purpose of comparison, Mickens' first-order harmonic balance method [18], He's homotpy perturbation solution with first-order approximation [5] and Xu's bookkeeping method [23] give the frequency as $A^{1/3}\omega = 1.0491$. Mickens' second-order harmonic balance [25] gives the calculated value of the frequency as

$A^{1/3}\omega = 1.0704$. The exact value [26] of the frequency read $A^{1/3}\omega_{ex} = 1.070451$. Hence, the exact period is

$$T_{ex} = \frac{2\pi A^{1/3}}{1.070451} = 5.86966A^{1/3} \tag{20}$$

It can be easly shown that the maximal relative error is less than 0.59%.

Example 2. The second equation to be studied is a modified version of the van der Pol equation [25], i.e.,

$$u'' + u^{1/3} = \varepsilon(1 - u^2)u', \quad u(0) = A, \quad u'(0) = 0. \tag{21}$$

Again, to apply modified Lindstedt–Poincaré method, we rewrite Eq. (21) in the form:

$$u'' + 0u + p\left(u^{1/3} - \varepsilon(1 - u^2)u'\right) = 0. \tag{22}$$

Supposing the constant zero in Eq. (22) can be expressed as

$$0 = \omega^2 + p\omega_1 + p^2\omega_2 + \cdots \tag{23}$$

and we can assume that the solution can be written in the form

$$u = u_0 + pu_1 + p^2u_2 + \cdots. \tag{24}$$

Substituting Eqs. (23) and (24) into Eq. (22) and processing as the standard perturbation method, we have

$$u_0'' + \omega^2u_0 = 0, \quad u_0(0) = A, \quad u_0'(0) = 0, \tag{25}$$

$$u_1'' + \omega^2u_1 + \omega_1u_0 + u_0^{1/3} - \varepsilon(1 - u_0^2)u_0' = 0, \tag{26}$$

Solving Eq. (25), we have

$$u_0 = A\cos\omega t. \tag{27}$$

Substituting Eq. (27) into Eq. (26) results into

$$u_1'' + \omega^2u_1 + \omega_1A\cos\omega t + (\cos\omega t)^{1/3} - \varepsilon(1 - A^2\cos^2\omega t)(-A\omega\sin\omega t) = 0. \tag{28}$$

Replacing Fourier expansion of $(\cos\omega t)^{1/3}$ from Eq. (13), it follows that Eq. (28) becomes

$$u_1'' + \omega^2u_1 + \omega_1A\cos\omega t + A^{1/3}a_1\left(\cos\omega t - \frac{\cos 3\omega t}{5} + \cdots\right) + \varepsilon A\omega\left(1 - \frac{A^2}{4}\right)\sin\omega t - \frac{\varepsilon A^3\omega}{4}\sin 3\omega t = 0. \tag{29}$$

The requirement of no secular term gives

$$1 - \frac{A^2}{4} = 0 \quad \text{and} \quad \omega_1A\cos\omega t + A^{1/3}a_1 = 0 \tag{30}$$

and therefore, we obtain,

$$A = 2 \quad \text{and} \quad w_1 = -\frac{a_1}{A^{2/3}} = -\frac{1.1596}{2^{2/3}}. \tag{31}$$

If the first-order approximation is enough, then from Eq. (23), we have

$$\omega^2 + \omega_1 = 0 \tag{32}$$

and the frequency can be obtained in the form of

$$\omega = \frac{1.0768}{2^{1/3}} = 0.8547. \tag{33}$$

Therefore, we obtain the following approximated period:

$$T = \frac{2\pi}{0.8547}, \tag{34}$$

which agrees exactly with Mickens' solution [25].

3. Conclusion

In summary, we conclude that the method illustrated here is very accurate for entire solution domain. It is extremely simple and easy to use. We think that the method have great potential which still needs further development.

References

[1] A.H. Nayfeh, *Perturbation Methods*, Wiley-Interscience, New York, 1973.

[2] R.E. Mickens, *Oscillations in Planar Dynamics Systems*, World Scientific, Singapore, 1996.

[3] D.W. Jordan, P. Smith, *Nonlinear Ordinary Differential Equations*, Clarendon Press, Oxford, 1987.

[4] P. Hagedorn, *Nonlinear Oscillations (translated by WolframStadler)*, Clarendon Press, Oxford, 1981.

[5] J.H. He, *Non-Perturbative Methods for Strongly Nonlinear Problems Dissertion*, de-verlag im Internet GmbH, Berlin, 2006.

[6] J.H. He, Limit cycle and bifurcation of nonlinear problems, *Chaos Solitons and Fractals* 26 (3) (2005) 827–833.

[7] H.M. Liu, Approximate period of nonlinear oscillators with discontinuities by modified Lindstedt–Poincare method, *Chaos, Solitons and Fractals* 23 (2) (2005) 577–579.

[8] J.H. He, Determination of limit cycles for strongly nonlinear oscillators, *Physical Review Letters* 90 (17) (2003) art. no. 174301.

[9] T. Öziş, A. Yıldırım, A note on He's homotopy perturbation method for van der Pol oscillator with very strong nonlinearity, *Chaos Solitons and Fractals*, in press, doi:10.1016/j.chaos.2006.04.013.

[10] T. Öziş, A. Yıldırım, Determination of limit cycles by a modified straightforward expansion for nonlinear oscillators, *Chaos, Solitons and Fractals* 32 (2007) 445–448.

[11] T. Öziş, A. Yıldırım, Determination of frequency-amplitude relation for Duffing-Harmonic Oscillator by the Energy Balance Method, *Special Issue of Computers and Mathematics with Applications* (2006) (accepted).

[12] B.S. Wu, W.P. Sun, C.W. Lim, An analytical approximate technique for a class of strongly non-linear oscillators, *International Journal of Non-Linear Mechanics* 41 (6–7) (2006) 766–774.

[13] J.H. He, Analytical solution of a nonlinear oscillator by the linearized perturbation technique, *Communication in Nonlinear Science and Numerical Simulation* 4 (2) (1999) 109–113.

[14] J.H. He, Modified straightforward expansion, *Meccanica* 34 (4) (1999) 287–289.

[15] Y.K. Cheung, S.H. Mook, S.L. Lau, A modified Lindstedt–Poincaré method for certain strongly non-linear oscillators, *International Journal of Non-linear Mechanics* 26 (1991) 367–378.

[16] C.W. Lim, B.S. Wu, A modified Mickens procedure for certain non-linear oscillators, *Journal of Sound and Vibration* 257 (1) (2002) 202–206.

[17] Z.G. Xiong, H. Hu, Q. Tang, A modified eigenfunction expansion approximation for nonlinear oscillating equations, *Journal of Sound and Vibration* 290 (2006) 1315–1321.

[18] R.E. Mickens, Oscillations in an $x^{4/3}$ potential, *Journal of Sound and Vibration* 246 (2) (2001) 375–378.

[19] J.H. He, Modified Lindstedt–Poincaré methods for some strongly non-linear oscillations Part I: expansion of a constant, *International Journal of Non-linear Mechanics* 37 (2002) 309–314.

[20] J.H. He, Some asymptotic methods for strongly nonlinear equations, *International Journal of Modern Physics B* 20 (10) (2006) 1141–1199.

[21] J.H. He, Recent developments in asymptotic methods for nonlinear ordinary equations, *International Journal of Computational and Numerical analysis and Applications* 2 (2) (2002) 127–190.

[22] R.E. Mickens, Analysis of non-linear oscillators having non-polynomial elastic terms, *Journal of Sound and Vibration* 255 (4) (2002) 789–792.

[23] L. Xu, He's parameter-expanding methods for strongly nonlinear oscillators, *Journal of Computational and Applied Mathematics* (2006), doi:10.1016/j.cam.2006.07.020.

[24] R.E. Mickens, *Mathematical Methods for the Natural and Engineering Sciences*, World Scientific, Singapore, 2004.

[25] R.E. Mickens, Iteration method solutions for conservative and limit-cycle $x^{1/3}$ force oscillators, *Journal of Sound and Vibration* 292 (3–5) (2006) 964–968.

[26] H.P.W. Gottlieb, Frequencies of oscillators with fractional-power non-linearities, *Journal of Sound and Vibration* 261 (2003) 557–566.