Available online at www.sciencedirect.com
ScienceDirect
Journal of Sound and Vibration 301 (2007) 420–426
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

On the eigencharacteristics of a cantilevered visco-elastic beam carrying a tip mass and its representation by a spring-damper-mass system

M. Gürgöze[a,*], A.N. Doğruoğlu[b], S. Zeren[a]

[a]Faculty of Mechanical Engineering, Technical University of Istanbul, 34439 Gümüşsuyu, Istanbul, Turkey
[b]Faculty of Civil Engineering, Technical University of Istanbul, 34469 Maslak, Istanbul, Turkey

Received 31 May 2006; received in revised form 15 September 2006; accepted 5 October 2006
Available online 6 December 2006

Abstract

The present note is concerned with the derivation of the characteristic equation of a cantilevered, visco-elastic bending beam (Kelvin–Voigt model), carrying a tip mass. Further, it is attempted to represent this continuous system by an "equivalent" spring-damper-mass system. Then, the "first" eigenvalues of these systems are calculated and tabulated for a wide range of the non-dimensional mass parameter.

1. Introduction

A very recent study of two of the present authors published in this journal was concerned with the eigencharacteristics of an axially vibrating visco-elastic rod carrying a tip mass and its representation by a spring-damper-mass system [1]. The present work is to some extent, the counterpart of the previous work for the bending vibrations. After having derived and solved the characteristic equation of a visco-elastic (Kelvin–Voigt model) cantilever, carrying a tip mass, it is attempted to represent the system mentioned above, by an "equivalent" spring-damper-mass system.

It is hoped that especially the characteristic equation established and the numerical results collected in tables can be helpful to design engineers working in this field.

2. Theory

The mechanical system under consideration is shown in Fig. 1. It consists of a cantilevered visco-elastic bending beam carrying a tip mass M. It is assumed that its visco-elastic properties fit the Kelvin–Voigt model. The bending rigidity, length, mass per unit length and visco-elastic constant of the beam material are EI, L, m and α, respectively.

*Corresponding author. Fax: +90 212 2450795.
E-mail address: gurgozem@itu.edu.tr (M. Gürgöze).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2006.10.002

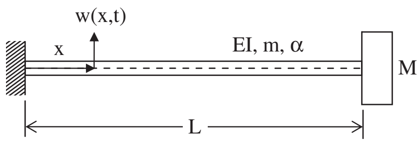

Fig. 1. Visco-elastic bending beam with a tip mass.

Equation of motion of the visco-elastic beam can be found in the literature [2] as

$$EIw^{\mathrm{IV}}(x,t) + \alpha I w^{\mathrm{IV}\cdot}(x,t) + m\ddot{w}(x,t) = 0, \tag{1}$$

where I is the moment of inertia of the beam section, $w(x,t)$ represents the lateral displacement of the beam at the location x and time t. Primes and dots denote partial derivatives with respect to x and t, as usual. The corresponding boundary conditions are obtained from Ref. [2] as special case in the following form:

$$w(0,t) = 0, \tag{2}$$

$$w'(0,t) = 0, \tag{3}$$

$$EIw''(L,t) + \alpha I w''^{\cdot}(L,t) = 0, \tag{4}$$

$$EIw'''(L,t) + \alpha I w'''^{\cdot}(L,t) - M\ddot{w}(L,t) = 0, \tag{5}$$

where the first two are obvious and the third and fourth conditions express the moment and force balances at the tip.

Assuming a solution in the form

$$w(x,t) = W(x)\mathrm{e}^{\lambda t}, \tag{6}$$

where $W(x)$ and λ denote the amplitude function and an eigenvalue, both being complex in general, and substitution into Eqs (1)–(5) leads to an ordinary differential equation of order four for $W(x)$ with corresponding four boundary conditions.

The substitution of the general solution of the differential equation mentioned above:

$$W(x) = C_1\mathrm{e}^{\beta x} + C_2\mathrm{e}^{-\beta x} + C_3\mathrm{e}^{\mathrm{i}\beta x} + C_4\mathrm{e}^{-\mathrm{i}\beta x}, \tag{7}$$

in which C_1–C_4 are unknown coefficients, into the corresponding boundary conditions leads to a set of four linear homogeneous equations for the unknowns C_1–C_4 . Setting the determinant of the coefficients matrix to zero, gives after lengthy rearrangements:

$$\begin{aligned} 4 &+ \mathrm{e}^{-\bar{\beta}}\left\{\left[1-(1+\mathrm{i})\alpha_M\bar{\beta}\right]\mathrm{e}^{-\mathrm{i}\bar{\beta}} + \left[1-(1-\mathrm{i})\alpha_M\bar{\beta}\right]\mathrm{e}^{\mathrm{i}\bar{\beta}}\right\} \\ &+ \mathrm{e}^{\bar{\beta}}\left\{\left[1+(1-\mathrm{i})\alpha_M\bar{\beta}\right]\mathrm{e}^{-\mathrm{i}\bar{\beta}} + \left[1+(1+\mathrm{i})\alpha_M\bar{\beta}\right]\mathrm{e}^{\mathrm{i}\bar{\beta}}\right\} = 0, \end{aligned} \tag{8}$$

where the following abbreviations are used:

$$\beta^4 = -\frac{m\lambda^2}{EI + \alpha I\lambda}, \quad \bar{\beta} = \beta L, \quad \alpha_M = \frac{M}{mL} \tag{9}$$

and i denotes the imaginary unit, as usual.

Anticipating that $\bar{\beta}$ should be a complex number in general, due to the damped character of the system, it is reasonable to rearrange further in order to obtain a more compact form of Eq. (8).

It can be shown after some rearrangements, that the above equation can be brought into the following form:

$$1 + \cos\bar{\beta}\cosh\bar{\beta} + \alpha_M\bar{\beta}(\cos\bar{\beta}\sinh\bar{\beta} - \sin\bar{\beta}\cosh\bar{\beta}) = 0. \tag{10}$$

A look at the frequency equation of the cantilevered elastic beam with a tip mass in Ref. [3] reveals the surprising fact that the characteristic equation of the visco-elastic beam with the tip mass is formally the same. However, one must be aware of the fact that the definition of $\bar{\beta}$ in the elastic and visco-elastic cases are:

$$\bar{\beta}^2 = \omega\sqrt{\frac{mL^4}{EI}} \tag{11}$$

and

$$\bar{\beta}^2 = \pm i\sqrt{\frac{mL^4\lambda^2}{EI + \alpha I\lambda}}, \tag{12}$$

respectively, where ω denotes the eigenfrequency of the elastic system, whereas λ is the eigenvalue of the visco-elastic system.

As in Refs. [1,3] one can think of representing the vibrational system in Fig. 1 by an "equivalent" single degree-of-freedom system. It is reasonable to make use of the simplified model in Fig. 2 for this purpose, where

$$k = 3EI/L^3, \quad c = 3\alpha I/L^3. \tag{13}$$

δ is the ratio of the mass to be added to the tip, to the mass of the beam itself.

The constant δ is not yet known and will be determined requiring that the "first" eigenvalue of the continuous system in Fig. 1 is equal to the eigenvalue of the model in Fig. 2.

Before proceeding further, it is quite in order to represent the system in Fig. 2 in terms of the non-dimensional parameters as in Fig. 3, where a non-dimensional damping parameter

$$\bar{d} = \frac{\alpha I}{mL^4\omega_0} \tag{14}$$

is introduced, with $\omega_0^2 = EI/mL^4$.

The characteristic equation of the model in Fig. 3 is simply:

$$(\alpha_M + \delta)\bar{\lambda}^{*2} + 3\bar{d}\bar{\lambda}^* + 3 = 0, \tag{15}$$

where $\bar{\lambda}^* = \lambda^*/\omega_0$, $\bar{\lambda}^*$ being the non-dimensional eigenvalue. On the other side, after having obtained the parameter $\bar{\beta}$ by solving numerically the characteristic Eq. (10), in order to obtain the eigenvalue λ of the continuous system, relationship (12) has to be used, which leads to the following quadratic equation:

$$\bar{\lambda}^2 + \bar{d}\bar{\beta}^4\bar{\lambda} + \bar{\beta}^4 = 0 \tag{16}$$

with the non-dimensional eigenvalue: $\bar{\lambda} = \lambda/\omega_0$.

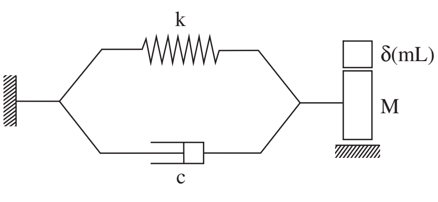

Fig. 2. Equivalent spring-damper-mass system for obtaining the "first" eigenvalue of the continuous system in Fig. 1.

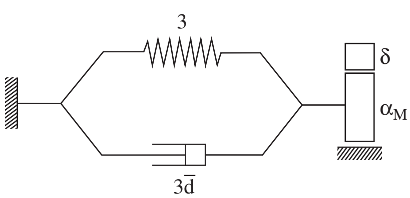

Fig. 3. Non-dimensionalized version of the model in Fig. 2.

The solutions of the characteristic equations in Eqs. (15) and (16) yield

$$\bar{\lambda}^* = \frac{-3\bar{d} \pm \sqrt{9\bar{d}^2 - 12(\alpha_M + \delta)}}{2(\alpha_M + \delta)}, \tag{17}$$

$$\bar{\lambda} = \frac{-\bar{d}\bar{\beta}^4 \pm \bar{\beta}^2\sqrt{\bar{d}^2\bar{\beta}^4 - 4}}{2}. \tag{18}$$

Requiring $\bar{\lambda}^* = \bar{\lambda}$ leads to:

$$\alpha_M + \delta = \frac{3\bar{d}\bar{\lambda} + 3}{\bar{d}\bar{\beta}^4\bar{\lambda} + \bar{\beta}^4}. \tag{19}$$

Numerical evalutions reveal that the right side of Eq. (19) is independent of $\bar{d}$ such that it can be set $\bar{d} = 0$, corresponding to the undamped case. Hence, Eq. (19) results in

$$\delta = \frac{3}{\bar{\beta}^4} - \alpha_M. \tag{20}$$

It is in order to note that this is the same formula as in Ref. [3], written here in different notations.

In conjunction with Fig. 3, expression (20) can be interpreted in the manner that the factor δ by which the own mass of the bending beam must be multiplied in order to be taken into account as is the same as in the undamped case. $\bar{\beta}$ denotes the first root of Eq. (10) with respect to $\bar{\beta}$, for the corresponding α_M value.

Let us return to the eigenvalue pair given in Eq. (17) which can be written as

$$\bar{\lambda}^* = -\frac{3\bar{d}}{2(\alpha_M + \delta)} \pm i\frac{\sqrt{12(\alpha_M + \delta) - 9\bar{d}^2}}{2(\alpha_M + \delta)}. \tag{21}$$

Recalling the fact noticed previously, that δ does not depend on $\bar{d}$, it is seen from above that the real part of the eigenvalues of the damped system depends upon $\bar{d}$ linearly, whereas the imaginary part decreases, as $\bar{d}$ increases. All these trends are similar to those observed in Ref. [1] for the axially vibrating visco-elastic rod with a tip mass.

3. Numerical evaluations

In Table 1, for a wide range of the non-dimensional mass parameter α_M, the corresponding $\bar{\beta}_1$ values are listed which are the "first" roots of the transcendental equation (10). Although the roots of this equation can be found in Ref. [4] for several values of the mass parameter, they are given here, for the sake of completeness, for the same range of α_M as used in Ref. [1].

The eigenvalues of the continuous system in Fig. 1 and the discrete model in Fig. 3 are given in Table 2 for $\bar{d} = 0.1$ and 0.5, respectively, which are complex, due to the presence of the internal damping.

The complex numbers $\bar{\lambda}_{0.1}$ in the second column are eigenvalues of the continuous system in Fig. 1 for $\bar{d} = 0.1$, determined from Eq. (18). The complex numbers $\bar{\lambda}^*_{0.1}$ which are the roots of the characteristic equation (15), given in Eq. (17), i.e., eigenvalues of the discrete model in Fig. 3, are exactly the same as $\bar{\lambda}_{0.1}$. Therefore, they are not repeated in a separate column. The corresponding eigenvalues for $\bar{d} = 0.5$ are denoted as $\bar{\lambda}_{0.5}$ and $\bar{\lambda}^*_{0.5}$, respectively, and given in the third column of Table 2. The non-dimensional factor δ in the last column is calculated from Eq. (20).

The exact agreement of $\bar{\lambda}$ and $\bar{\lambda}^*$ values in each column justifies the fact that the spring-damper-mass system in Fig. 3 yields the "first" eigenvalue of the continuous system in Fig. 1 exactly.

As stated in the previous section, the δ value is the same, irrespective of $\bar{d}$. It is clearly seen from Table 2 that δ approaches $\delta = 33/140 = 0.235714$, if α_M tends to infinity.

An inspection of the second and third columns reveals further that the absolute values of the real parts of the eigenvalues in the third column are five times of those in the second column, as expected. Further, the

Table 1
$\bar{\beta}_1$ values which are the “first” roots of the transcendental Eq. (10) for a wide range of the non-dimensional mass parameter α_M

α_M	$\bar{\beta}_1$
0	1.875104
0.001	1.873233
0.002	1.871372
0.003	1.869519
0.004	1.867674
0.005	1.865838
0.006	1.864011
0.007	1.862193
0.008	1.860382
0.009	1.858581
0.01	1.856787
0.02	1.839294
0.03	1.822562
0.04	1.806538
0.05	1.791171
0.06	1.776415
0.07	1.762230
0.08	1.748578
0.09	1.735426
0.1	1.722742
0.2	1.616400
0.3	1.536143
0.4	1.472408
0.5	1.419964
0.6	1.375669
0.7	1.337499
0.8	1.304087
0.9	1.274462
1	1.247917
2	1.076196
3	0.981231
4	0.917358
5	0.870021
6	0.832826
7	0.802429
8	0.776877
9	0.754937
10	0.735782
15	0.666137
20	0.620512
25	0.587187
30	0.561242
35	0.540175
40	0.522549
45	0.507469
50	0.494342
55	0.482753
60	0.472407
65	0.463083
70	0.454612
75	0.446863
80	0.439732
85	0.433137
90	0.427008
95	0.421289
100	0.415934
200	0.349861

Table 1 (*continued*)

α_M	$\bar{\beta}_1$
300	0.316166
400	0.294240
500	0.278283
600	0.265889
700	0.255840
800	0.247443
900	0.240265
1000	0.234021

Table 2
Collection of the "first" eigenvalues of the system in Figs. 1 and 3 and δ values for the same range of α_M as in Table 1; $\bar{d} = 0.1$ and 0.5

α_M	$\bar{\lambda}_{0.1} = \bar{\lambda}^*_{0.1}$	$\bar{\lambda}_{0.5} = \bar{\lambda}^*_{0.5}$	δ
0	−0.618118 ± 3.461256i	−3.090591 ± 1.676488i	0.242672
0.001	−0.615655 ± 3.454573i	−3.078276 ± 1.684435i	0.242643
0.002	−0.613211 ± 3.447927i	−3.066056 ± 1.692195i	0.242614
0.003	−0.610786 ± 3.441317i	−3.053930 ± 1.699774i	0.242585
0.004	−0.608379 ± 3.434743i	−3.041897 ± 1.707176i	0.242557
0.005	−0.605991 ± 3.428206i	−3.029955 ± 1.714407i	0.242528
0.006	−0.603621 ± 3.421704i	−3.018104 ± 1.721472i	0.242500
0.007	−0.601269 ± 3.415237i	−3.006343 ± 1.728373i	0.242473
0.008	−0.598934 ± 3.408806i	−2.994670 ± 1.735117i	0.242445
0.009	−0.596617 ± 3.402409i	−2.983084 ± 1.741707i	0.242418
0.01	−0.594317 ± 3.396046i	−2.971586 ± 1.748148i	0.242390
0.02	−0.572235 ± 3.334253i	−2.861173 ± 1.805098i	0.242130
0.03	−0.551696 ± 3.275598i	−2.758478 ± 1.850598i	0.241889
0.04	−0.532548 ± 3.219836i	−2.662738 ± 1.887003i	0.241665
0.05	−0.514657 ± 3.166744i	−2.573284 ± 1.916075i	0.241456
0.06	−0.497906 ± 3.116122i	−2.489531 ± 1.939165i	0.241262
0.07	−0.482192 ± 3.067790i	−2.410961 ± 1.957322i	0.241079
0.08	−0.467423 ± 3.021585i	−2.337115 ± 1.971384i	0.240908
0.09	−0.453518 ± 2.977360i	−2.267588 ± 1.982018i	0.240748
0.1	−0.440403 ± 2.934980i	−2.202016 ± 1.989771i	0.240597
0.2	−0.341323 ± 2.590357i	−1.706613 ± 1.978364i	0.239467
0.3	−0.278417 ± 2.343253i	−1.392087 ± 1.905372i	0.238759
0.4	−0.235008 ± 2.155212i	−1.175042 ± 1.821934i	0.238275
0.5	−0.203273 ± 2.006026i	−1.016365 ± 1.741397i	0.237924
0.6	−0.179071 ± 1.883973i	−0.895355 ± 1.667261i	0.237657
0.7	−0.160009 ± 1.781734i	−0.800045 ± 1.600034i	0.237447
0.8	−0.144609 ± 1.694483i	−0.723046 ± 1.539282i	0.237278
0.9	−0.131910 ± 1.618888i	−0.659550 ± 1.484316i	0.237140
1	−0.121259 ± 1.552570i	−0.606294 ± 1.434428i	0.237023
2	−0.067071 ± 1.156253i	−0.335355 ± 1.108584i	0.236435
3	−0.046350 ± 0.961697i	−0.231752 ± 0.934506i	0.236212
4	−0.035410 ± 0.840801i	−0.177050 ± 0.822711i	0.236094
5	−0.028648 ± 0.756395i	−0.143239 ± 0.743261i	0.236021
6	−0.024054 ± 0.693182i	−0.120270 ± 0.683092i	0.235972
7	−0.020730 ± 0.643559i	−0.103649 ± 0.635495i	0.235936
8	−0.018213 ± 0.603263i	−0.091065 ± 0.596629i	0.235909
9	−0.016241 ± 0.569698i	−0.081205 ± 0.564115i	0.235888
10	−0.014654 ± 0.541177i	−0.073272 ± 0.536394i	0.235871
15	−0.009845 ± 0.443630i	−0.049226 ± 0.441000i	0.235820
20	−0.007413 ± 0.384964i	−0.037063 ± 0.383247i	0.235793
25	−0.005944 ± 0.344737i	−0.029720 ± 0.343505i	0.235778
30	−0.004961 ± 0.314953i	−0.024805 ± 0.314014i	0.235766

Table 2 (*continued*)

α_M	$\bar{\lambda}_{0.1}=\bar{\lambda}^*_{0.1}$	$\bar{\lambda}_{0.5}=\bar{\lambda}^*_{0.5}$	δ
35	−0.004257±0.291758i	−0.021285±0.291012i	0.235760
40	−0.003728±0.273032i	−0.018640±0.272421i	0.235754
45	−0.003316±0.257504i	−0.016580±0.256991i	0.235748
50	−0.002986±0.244355i	−0.014930±0.243917i	0.235746
55	−0.002716±0.233035i	−0.013578±0.232655i	0.235743
60	−0.002490±0.223155i	−0.012451±0.222821i	0.235741
65	−0.002299±0.214434i	−0.011497±0.214138i	0.235739
70	−0.002136±0.206661i	−0.010678±0.206396i	0.235737
75	−0.001994±0.199676i	−0.009969±0.199437i	0.235736
80	−0.001869±0.193355i	−0.009347±0.193138i	0.235734
85	−0.001760±0.187599i	−0.008799±0.187401i	0.235733
90	−0.001662±0.182328i	−0.008312±0.182146i	0.235732
95	−0.001575±0.177478i	−0.007875±0.177310i	0.235731
100	−0.001496±0.172995i	−0.007482±0.172839i	0.235730
200	−0.000749±0.122400i	−0.003746±0.122345i	0.235722
300	−0.000500±0.099959i	−0.002498±0.099930i	0.235720
400	−0.000375±0.086576i	−0.001874±0.086557i	0.235718
500	−0.000300±0.077441i	−0.001499±0.077427i	0.235717
600	−0.000250±0.070696i	−0.001250±0.070686i	0.235717
700	−0.000214±0.065454i	−0.001071±0.065446i	0.235717
800	−0.000187±0.061228i	−0.000937±0.061221i	0.235716
900	−0.000167±0.057727i	−0.000833±0.057721i	0.235716
1000	−0.000150±0.054766i	−0.000750±0.054761i	0.235716

imaginary parts of the corresponding eigenvalues in case for $\bar{d}=0.5$ are less than those for $\bar{d}=0.1$, this trend being more apparent for smaller α_M values.

4. Conclusion

The present note is concerned first with the derivation of the characteristic equation of a laterally vibrating visco-elastic beam obeying the Kelvin–Voigt model, carrying a tip mass. The interesting fact to be noted is that the characteristic equation of the visco-elastic beam with a tip mass is found to be formally the same as of the elastic beam with the tip mass. Further, it is attempted to represent the original continuous system by an ''equivalent'' spring-damper-mass system. It is hoped that especially the characteristic equation established and the tables given, might be helpful to design engineers working in this field.

References

[1] M. Gürgöze, S. Zeren, On the eigencharacteristics of an axially vibrating visco-elastic rod carrying a tip mass and its representation by a single degree-of-freedom system, *Journal of Sound and Vibration* 294 (2006) 388–396.

[2] H.T. Banks, D.J. Inman, On damping mechanisms in beams, *Journal of Applied Mechanics* 58 (1991) 716–723.

[3] M. Gürgöze, On the representation of a cantilevered beam carrying a tip mass by an equivalent spring-mass system, *Journal of Sound and Vibration* 282 (2005) 538–542.

[4] P.A.A. Laura, J.L. Pombo, E.A. Susemihl, A note on the vibrations of a clamped-free beam with a mass at the free end, *Journal of Sound and Vibration* 37 (1974) 161–168.