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Short Communication

Response of nonlinear oscillators with random frequency of excitation, revisited

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Abstract

Periodically driven oscillators of low-frequency random excitations are analyzed. Computer simulation, which was carried out for the Duffing equation and forced vibrations of a pendulum, indicated that in these cases noise has a stabilizing effect. Computation of Lyapunov exponents showed that by adding noise to chaotic motion the largest Lyapunov exponent as a rule turns to negative and, consequently, the chaotic motion is annihilated.

1. Introduction

In [1] nonlinear periodically driven oscillators with a random frequency of excitation were investigated. There an unexpected result appeared—if the system has stable fixed points or a stable limit cycle then by adding noise the motion turns regular and is terminated in some of the fixed points or on the limit cycle. In Ref. [1] the equations of motion were integrated by the Runge–Kutta technique and noise was added at every time step. This brings to the case of high-frequency stochastic oscillations. In the present paper the effect of low-frequency oscillations is analyzed. A case for which the angular velocity is modeled with low-frequency stochastic oscillations is proposed in Section 3. This model is applied in the case of the Duffing equation and for the vibrations of a pendulum (Section 4).

An intricate question—can the noise suppress chaotic motions—arises. To give an answer to it in Section 5 Lyapunov exponents are calculated.

It is well-known that chaotic processes have the property that small numerical errors tend to grow exponentially fast. Therefore, the question if numerical orbits of chaotic processes represent true orbits was raised [2]. In the present paper the calculations obtained by the Runge–Kutta method were compared with the wavelet solution [3] and a good accordance was stated. Therefore, our numerical results can be considered trustworthy.

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2. Low-frequency excitations

Let us consider the following type of equations:

$$\ddot{x} + g(t, x, \dot{x}) = s(x)\cos[\omega(t)], \quad 0 \leqslant t \leqslant T, \tag{1}$$

where g and s are prescribed functions. We assume that the angular velocity ω has the form

$$\omega(t) = \omega_0[1 + \alpha\xi(t)], \tag{2}$$

where $\xi(t)$ is Gaussian noise with zero mean and standard deviation 1. The coefficient $0 \leqslant \alpha \leqslant 1$ characterizes the noise intensity (if $\alpha = 0$ the motion is deterministic). Due to the inertia of the driver stochastic oscillations cannot change abruptly and some smoothness of the angular velocity curve must take place. In other words, a model of low stochastic frequency is needed. It can be put together in the following way. We choose a number of time instants N_s in which the motion is disturbed. For simplicity sake we assume that these points are distributed uniformly over the interval $t \in [0, T]$. The instants t_j at which the stochastic excitation is applied are

$$t_j = j\frac{T}{N_s}, \quad j = 1, 2, \ldots, N_s. \tag{3}$$

By replacing Eq. (3) into Eq. (2) we obtain a set of points $P_0 = (0, \omega(0))$, $P_j = (t_j, \omega(t_j))$, $j = 1, \ldots, N_s$. Making use of the cubic spline interpolation for these points we get a stochastic realization for the modeled angular velocity $\tilde{\omega}(t)$. The corresponding modeled external force is

$$\tilde{f}(t) = s\cos[\tilde{\omega}(t)t]. \tag{4}$$

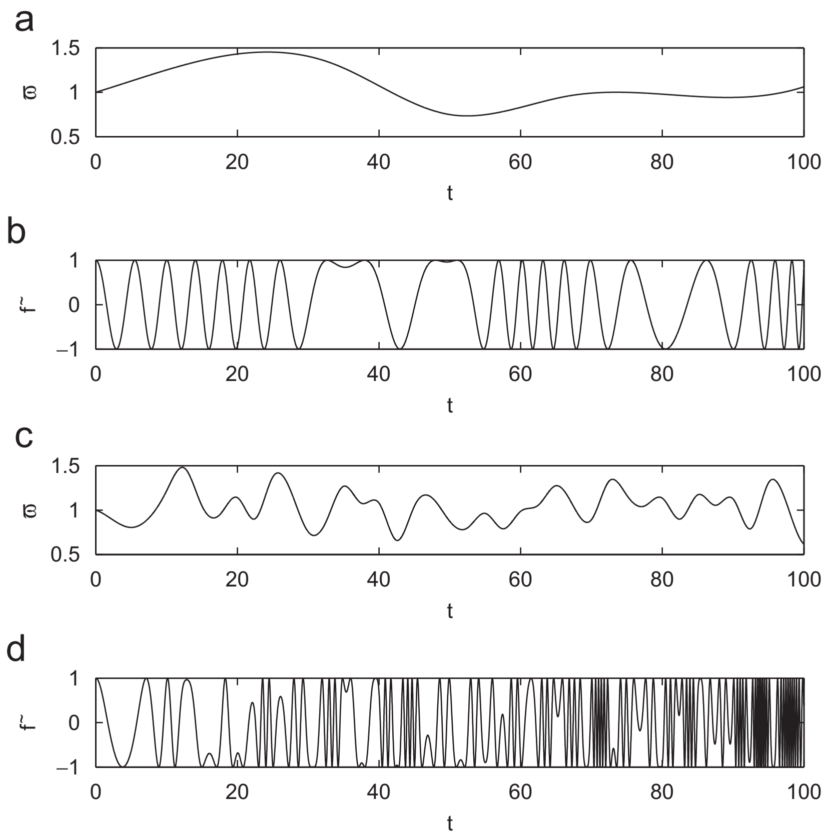

Fig. 1. Modeled angular velocity $\tilde{\omega}$ and external force $\tilde{f}$: (a)–(b) for $N_s = 3$, (c)–(d) for $N_s = 20$.

The approximation $\tilde{\omega}(t)$ consists of the stochastic excitations only at the finite number of time instants whereas the whole excitation is generated with the aid of cubic splines. The functions $\tilde{\omega}$ and $\tilde{f}$ for $\omega_0 = 1$, $s = 0.3$, $\alpha = 0.2$, $N_s = 6$, 40 are plotted in Fig. 1.

By integrating Eq. (1) with the aid of the Runge–Kutta method we find a stochastic realization for $x(t)$. By repeating this procedure v times we can calculate the mean $x_m(t)$ and standard deviation $\sigma(t)$.

3. Two examples

Let us consider the following examples:

(i) Duffing equation

The Duffing equation can be presented in the form of the following system:

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = -px_2 - qx_1 - rx_1^3 + s\cos\omega x_3,$$

$$\dot{x}_3 = 1. \tag{5}$$

Here p, q, r, s are prescribed constants, dots stand for time derivatives and $x_3 = t$. To Eq. (5) belong the boundary conditions $x_1(0) = x_0, x_2(0) = \dot{x}_0$. Computer simulation is carried out for $p = 0.25$, $q = -1$, $r = 1$, $s = 0.3$, $\omega_0 = 1$, $x_0 = 0$, $\dot{x}_0 = 1$ (this is the case of the two-well oscillator from Fig. 2 of Ref. [1]). For the number of points in which stochastic excitation is carried out are taken the values $N_s = 3$ or 20. The mean $x_m(t)$ and standard deviation $\sigma(t)$ are calculated from 20 stochastic realizations (as it was indicated in Ref. [1]) the motion can be terminated in one of the two foci $x_1 = 1$ and -1; for calculating x_m and σ only the

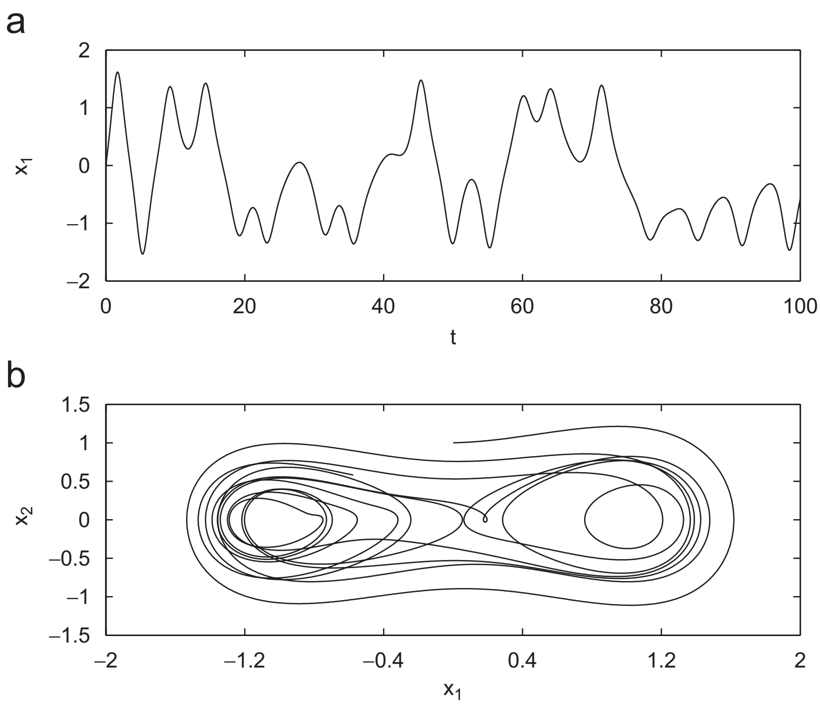

Fig. 2. Duffing equation in the deterministic case: (a) time history, (b) phase diagram.

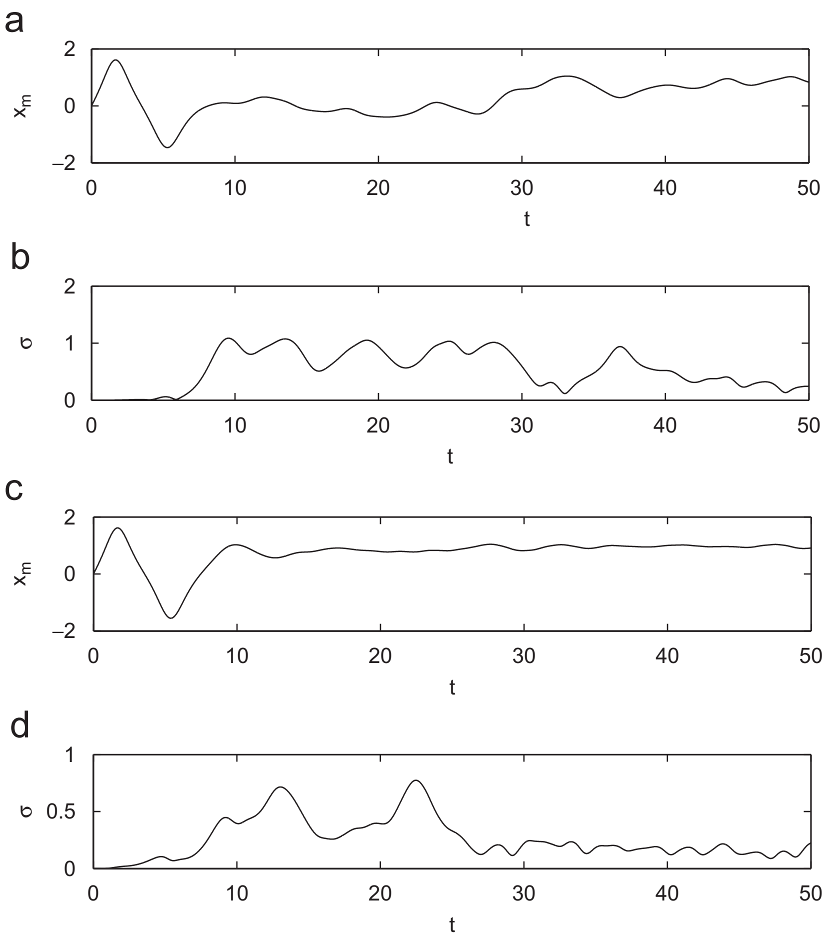

Fig. 3. Duffing equation, stochastic case: (a)–(b) for $N_s = 3$, (c)–(d) for $N_s = 20$.

realizations for which the motion terminates in $x_1 = 1$ are taken into account. Results of the computation are plotted in Figs. 2 (deterministic case) and 3 (stochastic case for $\alpha = 0.2$). In Fig. 4 the longer integration time is applied (time span is 1000 and the number of integration steps is 50,000); here $N_s = 60$ and 400.

(ii) Forced vibrations of the pendulum

The equations of a driven mathematical pendulum are [1]:

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = -\sin x_1[1 + a\cos(\omega x_3)] - bx_2,$$

$$\dot{x}_3 = 1. \tag{6}$$

Computer simulation was carried out for $a = 0.94$, $b = 0.15$, $\omega_0 = \pi/2$, $x_0 = 1$, $\dot{x}_0 = 1$. Time history and phase diagram for the deterministic case are plotted in Fig. 5. It follows from this diagram that deterministic motion is irregular, it consists of successive librations and rotations. Results for the stochastic case are

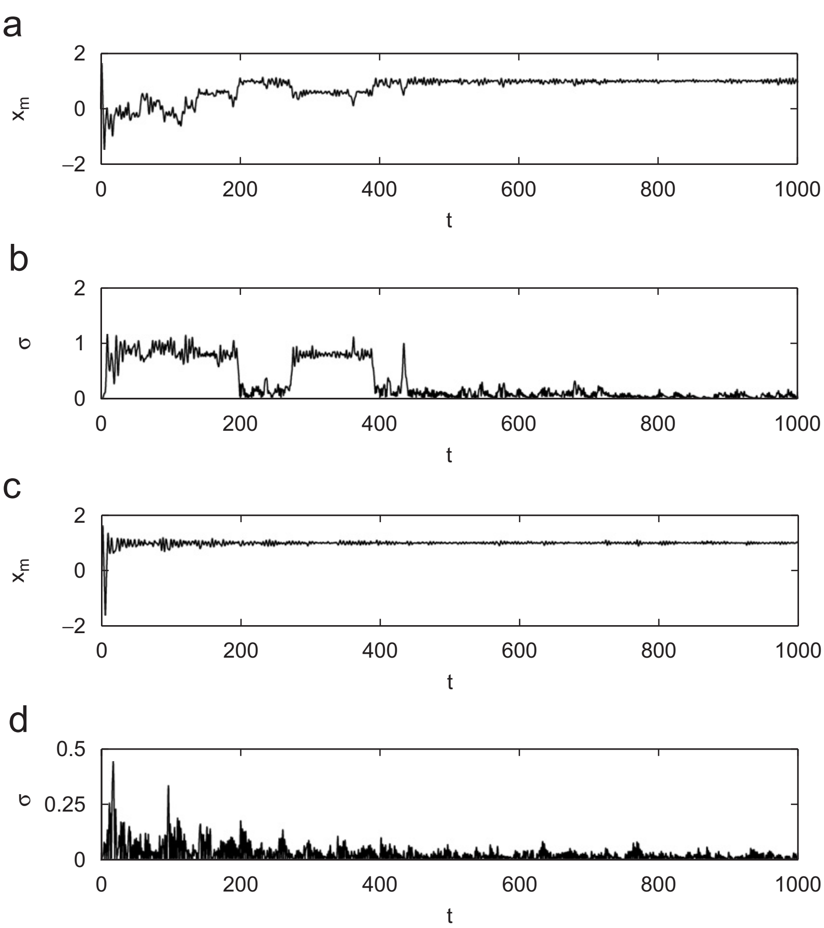

Fig. 4. Duffing equation, stochastic case: (a)–(b) for $N_s = 24$, (c)–(d) for $N_s = 400$.

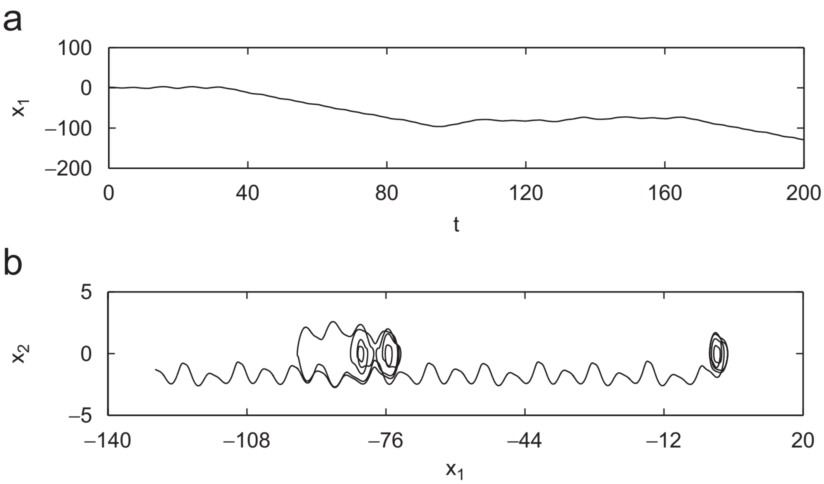

Fig. 5. Driven pendulum, deterministic case: (a) time history, (b) phase diagram.

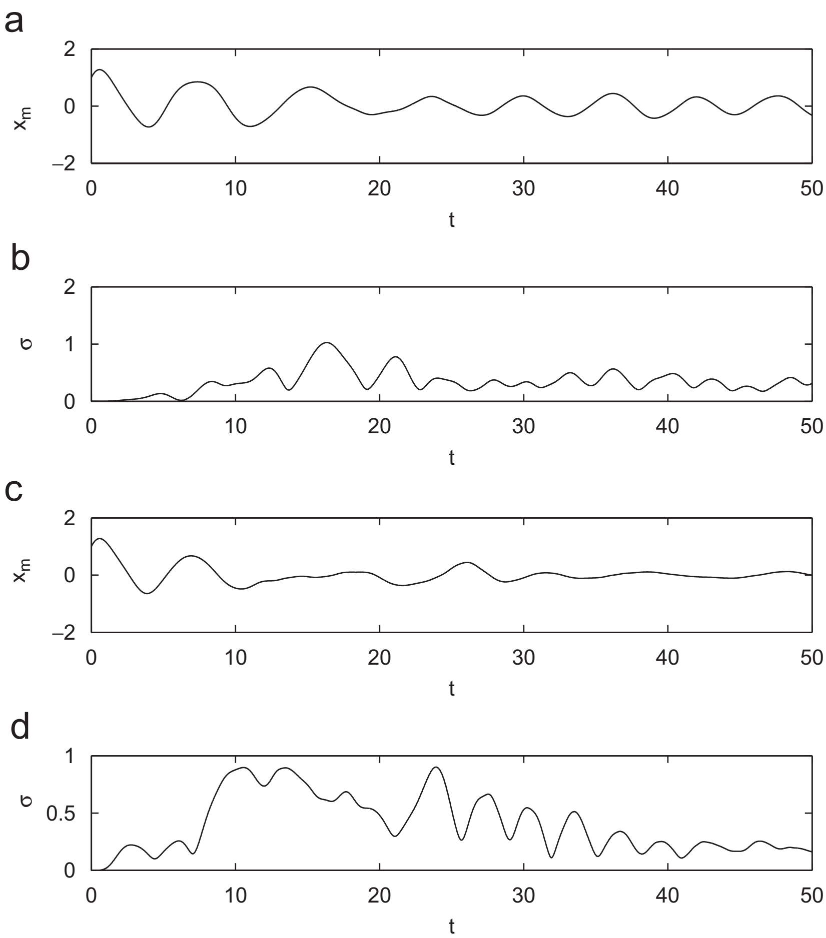

Fig. 6. Driven pendulum, stochastic case: (a)–(b) for $N_s = 3$, (c)–(d) for $N_s = 20$.

presented in Fig. 6; the mean x_m and standard deviation σ were calculated from 10 stochastic realizations. For the number of perturbation points again the values $N_s = 3$ and 20 were taken.

It follows from these calculations that noise regularizes irregular motion; this effect becomes evident already in the case of a small number of perturbation points ($N_s = 3$).

4. Can noise annihilate chaos?

In Ref. [1] and in Section 4 of the present paper it was shown that random frequency of excitation tends to stabilize the system. Here the question what happens with a chaotic system after adding noise—is the system still chaotic or is the chaos annihilated—arises. The answer can be given by applying the Lyapunov exponents.

Systems (5)–(6) have three Lyapunov exponents. Practically we need only the largest exponent: if it is negative the motion is regular; a positive value indicates chaos. By definition Lyapunov exponents are defined as the limits of infinite time history [4]:

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \frac{P(t)}{P(0)}. \tag{7}$$

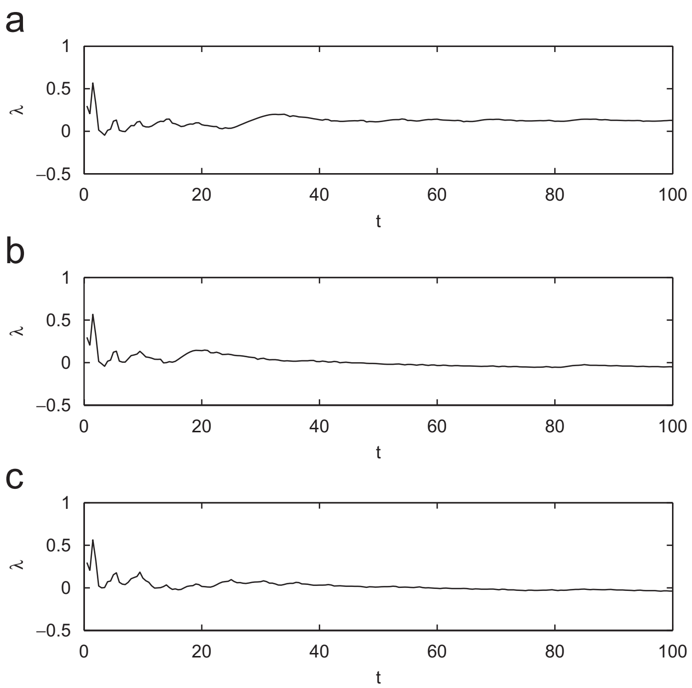

Fig. 7. The largest Lyapunov exponent versus time for the Duffing equation (2); (a) deterministic case; (b) stochastic case for $N_s = 3$; (c) stochastic case for $N_s = 20$.

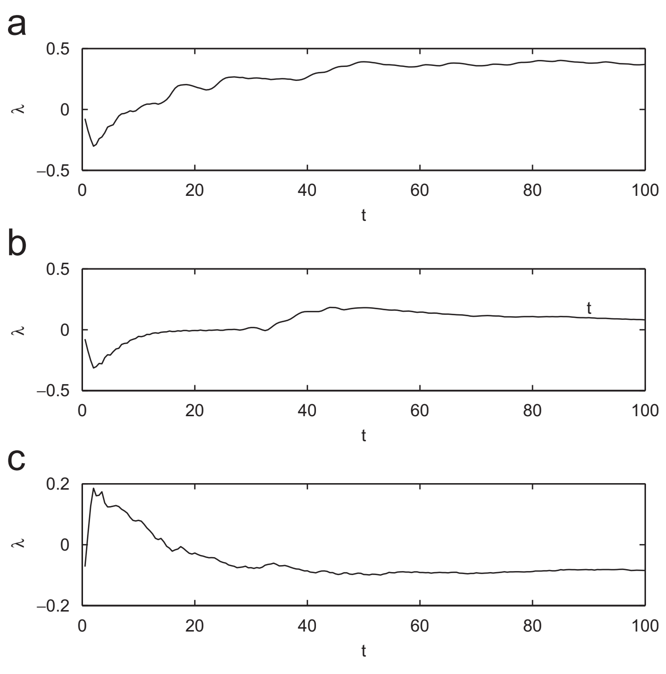

Fig. 8. The largest Lyapunov exponent versus time for the pendulum (a) deterministic case; (b) stochastic case for $N_s = 3$; (c) stochastic case for $N_s = 20$.

Here $P(t)$ is the length of the largest principal axis of the ellipsoid which is the deformed shape of the infinitesimal sphere of initial conditions. In computational practice time is always limited. Thus the Lyapunov exponent may be presented by an estimate which becomes a function of time

$$\lambda(t) = \frac{1}{t}\frac{P(t)}{P(0)}. \tag{8}$$

In Ref. [5] the conclusion has been made that it is necessary to use finite time Lyapunov exponents as a measure of complexity in order to examine the unpredictability of motion.

Let us return to the examples discussed in Section 4. Making use of Wolf's algorithm [6] the largest Lyapunov exponent versus time is computed. The results for the Duffing equation are plotted in Fig. 7 and for the pendulum in Fig. 8. In both cases the deterministic motion is chaotic (except a narrow region in Fig. 8a near $t = 0$, where the Lyapunov exponent is negative). As to stochastic motion then for twenty perturbation points ($N_s = 20$) the largest Lyapunov exponent is positive for small values of t; after that λ is permanently negative and the chaotic motion is suppressed. This tendency becomes evident also for $N_s = 3$ but not in so established form (in Fig. 8b addition of the noise reduces the Lyapunov exponent λ but it nevertheless obtains a small positive value).

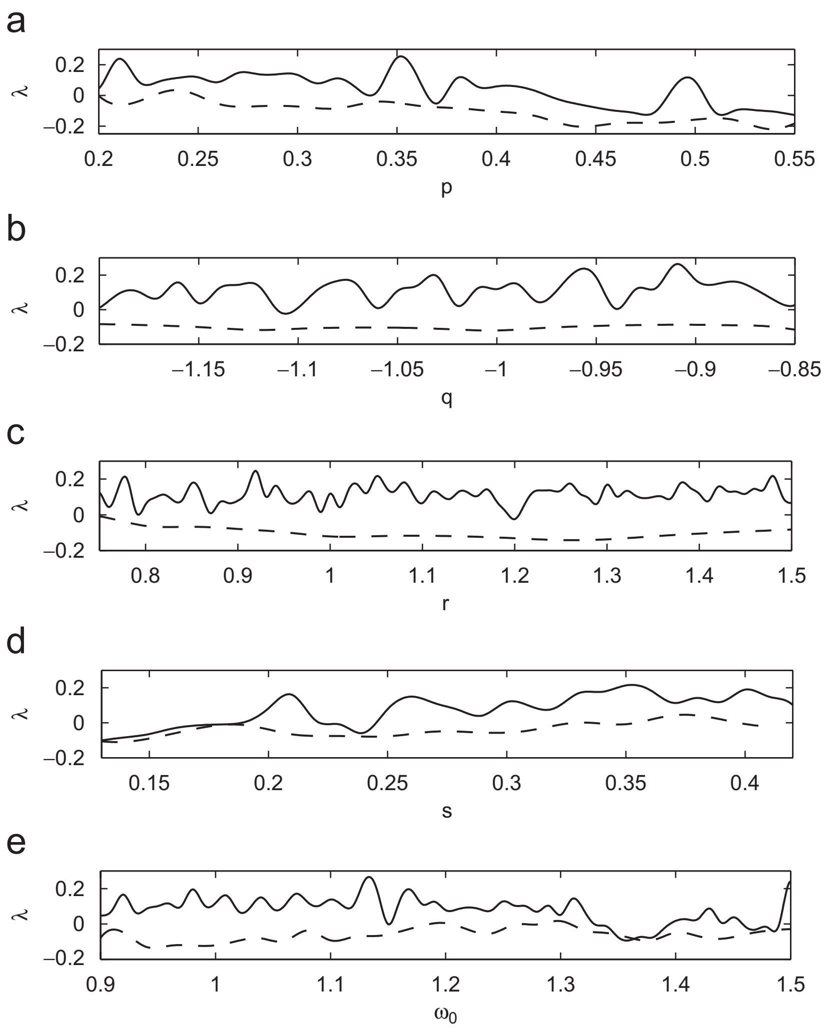

Fig. 9. Lyapunov exponents versus parameters for the Duffing equation, — deterministic case, - - - stochastic case.

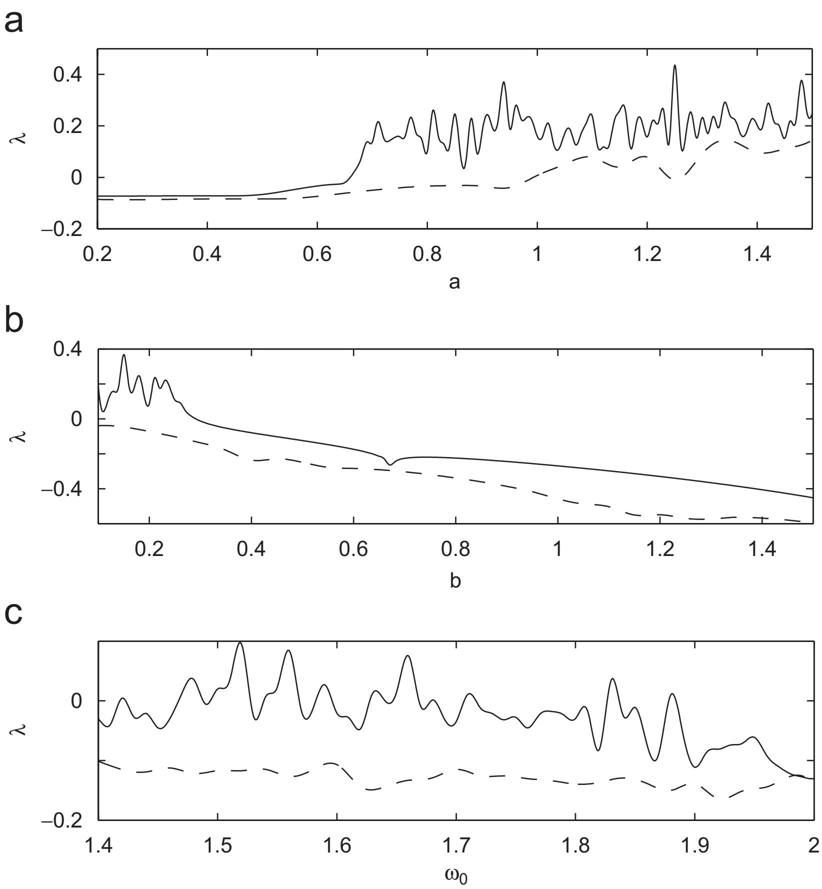

Fig. 10. Lyapunov exponents versus parameters for the driven pendulum, — deterministic case, - - - stochastic case.

Plots of Lyapunov exponents versus parameters are also of interest. For getting such diagrams in the case of the two examples of Section 4 all system parameters except one are fixed. Calculations were carried out for $N_s = 20, \alpha = 0.2$; the largest Lyapunov exponent was calculated for $t = 50$ (Duffing equation) or $t = 200$ (pendulum); a mean of 6 stochastic realizations is taken. The results are plotted in Figs. 9 and 10. It follows from these calculations that in all cases the values of the largest Lyapunov coefficient are considerably reduced. In most cases $\lambda < 0$ and the chaotic motion is annihilated (an exception is Fig. 10a where the motion of the pendulum remains chaotic for $a > 1$).

5. Conclusions

A model for investigating low-frequency stochastic excitations of dynamic systems is proposed. Computer experiments were carried out for the Duffing equation and pendulum. For the number of stochastic excitation points was taken ($N_s = 3$ or 20); by increasing this value we go over to the high frequency excitations, which were discussed in Ref. [1]. The calculations showed that noise has a stabilizing effect; if the noise level α is big enough then chaotic motions are suppressed. Calculations showed that similar results hold also for other cases discussed in Ref. [1].

Acknowledgments

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