

# Axisymmetric forced vibrations of an elastic free circular plate on a tensionless two parameter foundation

Z. Celep\*, K. Güler

*Department of Civil Engineering, Faculty of Civil Engineering, Istanbul Technical University, TR-34469 Maslak, Istanbul, Turkey*

Received 26 July 2004; received in revised form 17 August 2006; accepted 4 September 2006

Available online 22 December 2006

## Abstract

The static and dynamic responses of a circular elastic plate on a two-parameter tensionless foundation are investigated by assuming that the external load is rotationally symmetric and the plate experiences axially symmetric displacements. In the solution procedure, the vertical displacement of the foundation is determined by the corresponding governing equation, whereas the vertical displacement of the plate is expressed in series in terms of the mode shapes of the completely free circular plate. For the case of complete contact, the corresponding governing equation of the plate incorporated with the edge reaction from the foundation is satisfied through the Galerkin's approximation technique. The contact radius is obtained by requiring that the surface of the foundation satisfies the corresponding continuity equations. It is shown that the problem displays a highly nonlinear character due to the lift-off of the plate from the foundation and the numerical treatment of the governing equation is accomplished by adopting iterative processes in terms of the contact radius. The governing equation of the problem is solved numerically for the static and dynamic cases and the results are presented in figures to demonstrate the nonlinear behavior of the plate for various values of the parameters of the problem comparatively.

© 2006 Elsevier Ltd. All rights reserved.

## 1. Introduction

The response of elastic plates, such as slabs and pavements, on an elastic foundation of earthquake response of footings is structural engineering problem of theoretical and practical interest. It is therefore natural that a large number of studies has been devoted to the subject. Since the soil exhibits a very complex behavior, a number of different foundation models with various degrees of sophistication have been proposed. In the analysis of structures on soil the simplest model is the Winkler model. The Winkler model represents the soil medium as a system of identical but mutually independent elastic springs. The model has various shortcomings. The most serious deficiency of the model is the one pertaining to the independence of the springs. On the other hand, elastic continuum model is a conceptual approach of the infinite soil media. The second model provides much more information on the stresses and deformations within the soil mass than Winkler model. However, this modeling of the soil by semi-infinite elastic continuum model leads to

\*Corresponding author.

*E-mail address:* [celep@itu.edu.tr](mailto:celep@itu.edu.tr) (Z. Celep).

many-fold intricacies from mathematical viewpoint. This severely limits the application of this model in practice. In order to take care of shortcoming of both approaches, some modified approaches have been proposed. In the Winkler foundation model, the foundation reaction is assumed to be proportional to the displacement of the soil at the same point. The Winkler model can be considered as a system of closely spaced linear springs and is referred as a one-parameter model, the elastic constant of spring being the parameter. Because the springs are assumed to be independent and unconnected, no interaction exists between the springs. The foundation outside the loading area does not contribute to the foundation response. Various attempts have been made to improve the Winkler model and to obtain more realistic results by postulating a two-parameter model. The shortcomings of the Winkler model are eliminated partly by adopting two-parameter models with the simplest being the Pasternak model. This two-parameter model can be seen as a liquid with a surface tension in addition to its buoyancy effect or a system of closely spaced linear springs coupled to each other with elements which transmit a shear force proportional to the slope of the foundation surface or a membrane having a surface tension laid on a system of elastic springs. The analytical aspects of the continuously supported structures on various foundation models have been discussed by Kerr [1,2] and it is pointed out that the intuitive approach may lead to the incorrect formulation of the boundary conditions for the case of a two-parameter foundation model.

The response of structural elements resting on the one- and two-parameter foundation is usually analyzed by assuming that the foundation supports compressive as well as tensile stresses, which simplifies the analysis considerably. However, this assumption is questionable or not valid for many supporting media including the soil. For structural elements on soil capable of supporting compressive reactions only, the tensionless foundation model should be adopted for realistic results. It is difficult to determine the region of contact in advance. When separation occurs, analysis involving the tensionless Winkler foundation displays a nonlinear character gets complicated and its numerical treatment needs to be performed iteratively. As a result, only a limited number of studies dealing with the tensionless Winkler foundation is published. There are even fewer investigations exist on dynamic problems. When a tensionless two-parameter foundation model is considered, the solution gets more complicated due to the contact conditions even in the static problems.

When the static behavior of plates on a tensionless Winkler foundation is considered, there are various papers dealing with the subject (Weisman [3], Villaggio [4], Dempsey et al. [5], Celep et al. [6–8], Hong et al. [9]). Some of these problems are generalized to include the dynamic loads including oscillations of the plate as well (Celep and Güler [10–12] and Celep and Gençoğlu [13]). Generally, solutions are only given for a circular and rectangular plate by applying approximate numerical techniques to the nonlinear governing equations of the problem. On the other hand, for the dynamic problems, i.e., for oscillations of a plate on a tensionless foundation, the contact region of the plate depends on time. The numerical analysis is carried out by adopting step-wise integration in the time domain by updating the contact region continuously.

Various studies are carried out to improve the results by using two-parameter model for plates on an elastic foundation. There is a variety of studies dealing with the problem of plates on the conventional two-parameter foundation which provides continuous contact irrespective the type of the contact stress whether tensile or compression (Vallabhan et al. [14,15], Ayvaz et al. [16,17], Çelik and Saygun [18,19], Buczkowski and Torbacki [20]). The boundary conditions of beams on a two-parameter foundation is discussed in detail by Kerr [21] utilizing the variational approach.

In spite of the mathematical elegance, research and the number of publications on a two-parameter foundation model that reacts in compression only are very limited. The authors presume that the major difficulties in the implementation of the model are the definition of the contact region, the formulation of the corresponding boundary conditions and the continuous variation of the contact region in the case of time-dependent loading. There is only one study in the literature that deals with the problem of a circular plate subjected to a uniformly distributed load and a central concentrated load (Güler [22]). The aim of the present paper is to extend this study to include the dynamic behavior of the elastic circular plate on the two-parameter tensionless foundation under rotationally symmetric loading and displacement. Special attention is paid on the boundary conditions and the force equilibrium in static and dynamic cases. The study is carried out by assuming that an axially symmetric lift-off takes place, consequently the highly nonlinear governing equation of the problem is discretized by using Galerkin's technique. Numerical results are obtained by adopting

iterations and presented to illustrate the validity of the solution procedure as well as the response of the plate subjected to static and dynamic loadings.

### 2. Statement of the problem

Consider a completely free circular elastic thin plate of radius  $a$  and stiffness  $D$ , on a two-parameter foundation having stiffnesses  $K$  and  $G$  as Fig. 1 shows. The plate is assumed to be subjected to a concentrated load  $P(t)$  applied at the center and a rotationally symmetric distributed load  $Q(R, t)$ ,  $R$  is the coordinate in radial direction. Since the geometry and the loading of the problem are axially symmetrical, all the parameters of the problem reflect the same symmetry including the vertical displacement of the plate and that of the elastic foundation. The loads are assumed to be time dependent, consequently the parameters of the system depends on the radius  $R$  and on the time  $t$ .

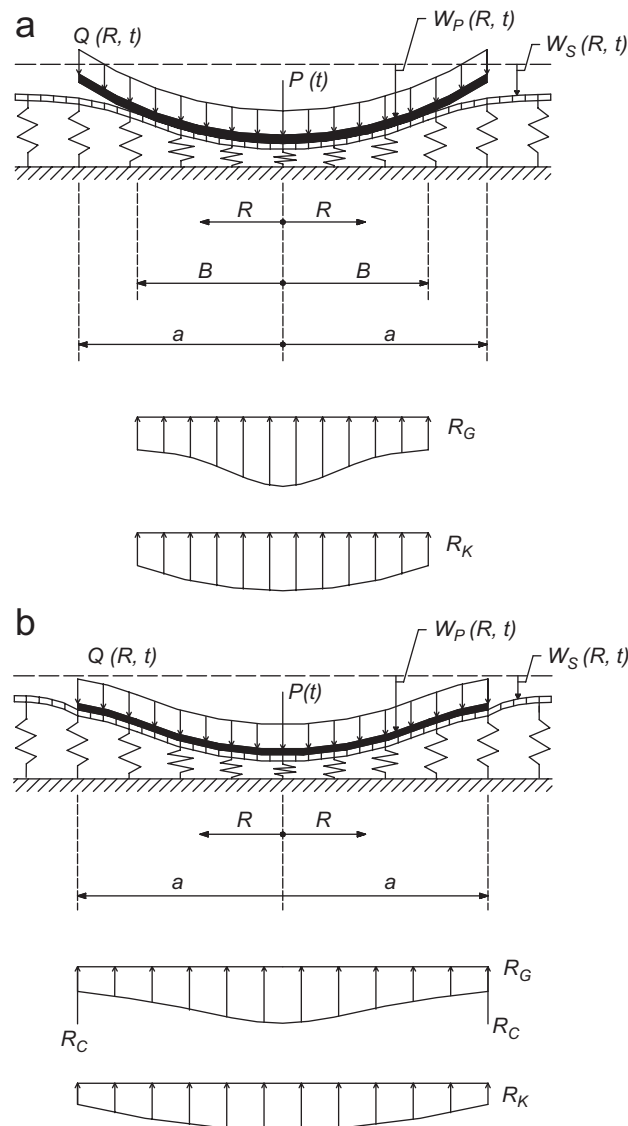


Fig. 1. Circular plate on the two-parameter foundation subjected to axially symmetric loads: (a) partial contact and (b) complete contact.

In the two parameter foundation, it is assumed that foundation pressure  $P_f(R, t)$  and the foundation displacement  $W_s(R, t)$  are related to each other according to

$$P_f = KW_s - G\Delta W_s. \quad (1)$$

As the inspection of Eq. (1) reveals, the two-parameter foundation consists of essentially can be seen as closely spaced linear springs of stiffness  $K$  connected to each other by a membrane having a surface tension stiffness  $G$ . However, it is also possible to give a more sophisticated interpretation as well. The surface of the foundation is divided into two regions. The first one is the free surface of the foundation and its displacement  $W_s(R, t)$  is controlled by the equation

$$G\Delta W_s - KW_s = 0. \quad (2)$$

Evaluation of the parameters  $K$  and  $G$  is discussed in various papers and recently in Ref. [17]. No pressure is exerted on the foundation surface, i.e., there is no interaction between the plate and the surface where the region beyond the plate area, i.e., for  $R > a$  [1]. However, this equation is also valid for the surface part of the foundation under the plate where separation takes place, i.e., for  $B > R > a$ . For the present axially symmetric case the solution of Eq. (2) is expressed as

$$W_s(R, t) = aw_s(r, \tau) = aC(\tau)K_o(\lambda r) \quad \text{for } b \geq r = R/a, \quad (3)$$

where  $b = B/a$ ,  $\lambda = \sqrt{Ka^2/G}$  and  $K_o$  is the modified Bessel function of second kind.

The displacement of the circular plate is governed by the equation

$$D\Delta\Delta W_p + [KW_p - G\Delta W_p]H(R, t) - \frac{P}{\pi R}\delta(R) - Q + G\left[\frac{\partial W_p}{\partial R} - \frac{\partial W_s}{\partial R}\right]_{R=a} \frac{\delta(R-a)}{R-a}H(R=a, t) = -m\frac{\partial^2 W_p}{\partial t^2}, \quad (4)$$

where  $W_p(R, t) = aw_p(r, t)$  is the displacement of the plate and  $m$  is the mass of the plate per unit area and  $\delta(R)$  is the Dirac's delta function [3,11]. In addition to the regular terms, the equation includes  $P(t)$  the concentrated load at the center of the plate, the foundation reaction distributed on the contact region ( $0 \leq R \leq B$ ) and the edge foundation reaction when the complete contact is established, i.e., for  $R = a$ . When a partial contact between the plate and the foundation takes place, then the reaction of the foundation is defined as a vertically distributed force within the region  $0 \leq R = a \leq r \leq B = ab$  as shown in Fig. 1a, where  $B(\tau)$  denotes the contact radius. When the complete contact is the case, then a circumferentially distributed edge reaction is exerted by the foundation at  $R = a$  as shown in Fig. 1b and it is included in the governing equation (4). However, when separation takes place, this edge reaction vanishes as stated by Kerr [2].  $H(R, t)$  is an auxiliary function known as the contact function and is defined as

$$\begin{aligned} H(R, t) &= 1 \quad \text{for } 0 \leq R \leq B, \\ H(R, t) &= 0 \quad \text{for } B \leq R \leq a, \end{aligned} \quad (5)$$

which reflects the axial symmetry of the problem as well. The presented formulation does not include any type of damping, such as material and radiation damping. More realistic results can be obtained by including these damping. Damping can be defined parallel to the definition of the foundation stiffnesses  $K$  and  $G$ . Furthermore, mass and damping can be included into the formulation to represent the foundation participation to the response. However, the present paper aims to give a comprehensive formulation by generalizing the static case and stressing the boundary conditions which have not been treated properly in various papers. Due to the tensionless character of the foundation, the problem is highly nonlinear; therefore Galerkin's method is adopted for the solution. The displacement function of the plate is assumed to be a linear combination of the axially symmetric free vibration mode shapes of the completely free circular plate including a rigid vertical translation as follows:

$$W_p(R, t) = aw_p(r, \tau) = a\left[\tilde{T}_o(\tau) + \sum_{n=0}^{\infty} T_n(\tau)w_n(r)\right], \quad (6)$$

where  $\bar{T}_o(\tau)$  and  $T_n(\tau)$  are time dependent parameters of the series and

$$w_n(r) = J_o(\lambda_n r) + A_n I_o(\lambda_n r), \tag{7}$$

where  $J_o$  and  $I_o$  are the regular and modified Bessel functions of first kind, respectively, and  $\lambda_n$  are the roots of the frequency equation (7)

$$\frac{J_1(\lambda_n)}{I_1(\lambda_n)} = \frac{(1 - \nu)J_1(\lambda_n) - \lambda_n J_o(\lambda_n)}{(1 - \nu)I_1(\lambda_n) - \lambda_n I_o(\lambda_n)} \quad A_n = -\frac{J_1(\lambda_n)}{I_1(\lambda_n)} \tag{8}$$

where  $\nu$  is the Poisson’s ratio. Eq. (8) is obtained by requiring that the corresponding boundary conditions of the completely free plate are satisfied. By substituting the displacement functions (3) and (6) into the governing equation of problem (4), using the identities

$$\Delta \Delta w_n(r) = \lambda_n^4 w_n \quad \Delta J_o(\lambda_n r) = -\lambda_n^2 J_o(\lambda_n r) \quad \Delta I_o(\lambda_n r) = \lambda_n^2 I_o(\lambda_n r),$$

the following non-dimensional equation is obtained for the unknown parameters  $\bar{T}_o(\tau)$  and  $T_n(\tau)$ ,

$$\begin{aligned} & \sum_{n=0}^{\infty} \lambda_n^4 w_n(r) T_n + H(r, \tau) k \left[ \bar{T}_o + \sum_{n=0}^{\infty} w_n(r) T_n \right] \\ & - H(r, \tau) g \sum_{n=0}^{\infty} \lambda_n^2 \bar{w}_n(r) T_n - \frac{p}{\pi r} \delta(r) - q \\ & + g \left[ \frac{\partial w_p}{\partial r} - \frac{\partial w_s}{\partial r} \right]_{r=1} \frac{\delta(r-1)}{r-1} H(r=1, \tau) \\ & = - \left[ \frac{d^2 \bar{T}_o}{d\tau^2} + \sum_{n=0}^{\infty} w_n(r) \frac{d^2 T_n}{d\tau^2} \right], \end{aligned} \tag{9}$$

where the non-dimensional parameters related to the geometry, the load and the foundation are defined so that a wide range of numerical values of the parameters of the problem can be covered:

$$\begin{aligned} k &= \frac{Ka^4}{D} \quad g = \frac{Ga^2}{D} \quad p = \frac{Pa}{D} \quad q = \frac{Qa^3}{D}, \\ t^2 &= \frac{ma^4}{D} \tau^2 \quad \lambda^2 = \frac{k}{g} = \frac{Ka^2}{G} \quad \bar{w}_n(r) = -J_o(\lambda_n r) + A_n I_o(\lambda_n r). \end{aligned} \tag{10}$$

By employing Galerkin’s procedure, i.e., by requiring the error in the governing equation (9) to be orthogonal to each mode shape within the definition region of the equation, the following system of differential equations is obtained:

$$\mathbf{M}\ddot{\mathbf{T}} + \mathbf{K}\mathbf{T} = \mathbf{F}, \tag{11}$$

where the dots denote the differentiation with respect to the non-dimensional time  $\tau$  and

$$\begin{aligned} \mathbf{T}(\tau) &= [\bar{T}_o \ T_o \ T_1 \ T_2, \dots]^T \quad \text{diag } \mathbf{M} = [0.5 \ m_o \ m_1 \ m_2, \dots], \\ \mathbf{K}(\tau) &= \begin{bmatrix} k_o & \bar{k}_{oo} & \bar{k}_{o1} & \bar{k}_{o2} & \dots \\ \bar{k}_{oo} & k_{oo} & k_{o1} & k_{o2} & \dots \\ \bar{k}_{1o} & k_{1o} & k_{11} & k_{12} & \dots \\ \bar{k}_{2o} & k_{2o} & k_{21} & k_{22} & \dots \\ \cdot & \cdot & \cdot & \cdot & \dots \end{bmatrix} \quad \mathbf{F}(\tau) = [\bar{f}_o \ f_o \ f_1 \ f_2, \dots]^T, \\ m_n &= \int_0^1 w_n(r) w_n(r) r \, dr, \\ k_o(\tau) &= k \int_0^1 H(r, \tau) r \, dr + g \lambda H(r=1, \tau) \frac{K_1(\lambda)}{K_o(\lambda)}, \end{aligned}$$

$$\begin{aligned}
 \bar{k}_{on}(\tau) &= k \int_0^1 H(r, \tau) w_n(r) r \, dr - g \lambda_n^2 \int_0^1 H(r, \tau) \bar{w}_n(r) r \, dr \\
 &\quad + g H(r = 1, \tau) \left[ \lambda_n \bar{w}_n(r = 1) + \lambda \frac{K_1(\lambda)}{K_o(\lambda)} w_n(r = 1) \right], \\
 \bar{k}_{no}(\tau) &= k \int_0^1 H(r, \tau) w_n(r) r \, dr + g \lambda H(r = 1, \tau) \frac{K_1(\lambda)}{K_o(\lambda)} w_n(r = 1), \\
 k_{mm}(\tau) &= \lambda_m^4 \int_0^1 w_m(r) w_n(r) r \, dr + k \int_0^1 H(r, \tau) w_m(r) w_n(r) r \, dr \\
 &\quad - g \lambda_m^2 \int_0^1 H(r, \tau) \bar{w}_m(r) w_n(r) r \, dr \\
 &\quad + g H(r = 1, \tau) \left[ \lambda_m \bar{w}_m(r = 1) w_n(r = 1) + \lambda \frac{K_1(\lambda)}{K_o(\lambda)} w_m(r = 1) w_n(r = 1) \right], \\
 \bar{f}_o(\tau) &= \frac{p(\tau)}{2\pi} + \frac{q(\tau)}{2} \quad f_n(\tau) = \frac{p(\tau)}{2\pi} (1 + A_n) \quad \bar{w}_n(r) = -J_1(\lambda_n r) + A_n I_1(\lambda_n r).
 \end{aligned} \tag{12}$$

The vertical force equilibrium of the plate can be written as follows:

$$P(t) + 2\pi \int_0^a Q(R, t) R \, dR = R_K(t) + R_G(t) + R_C(t) + R_I(t), \tag{13}$$

where

$$\begin{aligned}
 R_K(t) &= 2\pi \int_0^a K W_p H(R, t) R \, dR \quad R_G(t) = -2\pi \int_0^a G \Delta W_p H(R, t) R \, dR, \\
 R_C(t) &= 2\pi G a \left[ \frac{\partial W_p}{\partial R} - \frac{\partial W_s}{\partial R} \right] H(R = a, t) \quad R_I(t) = 2\pi \int_0^a m \frac{\partial^2 W_p}{\partial t^2} R \, dR,
 \end{aligned} \tag{14}$$

where  $P(t)$  and  $Q(R, t)$  are the external loads, and  $R_K(t)$  and  $R_G(t)$  denote the distributed foundation reactions exerted by the foundation proportional to the two stiffness parameters of the foundation  $K$  and  $G$ , respectively.  $R_C(t)$  corresponds to the foundation reaction which is proportional to the difference of the slopes of the foundation surfaces at the two sides of the edge of the plate. This force is distributed along the edge of the plate and it develops, when there is complete contact between the soil and the foundation. All these reactions have to be non-negative, i.e., compression due to the tensionless character of the two-parameter foundation model. Finally,  $R_I(t)$  corresponds to the resultant of the inertia forces of the plate. In fact this is the only resultant in the force equilibrium in Eq. (13) which can change its sign. Using the assumption for the vertical displacement functions, the force equilibrium in Eq. (13) can be expressed as follows in the following non-dimensional form:

$$\begin{aligned}
 p(\tau) + 2\pi \int_0^1 q(r, \tau) r \, dr &= r_k + r_g + r_c + r_i, \\
 r_k(\tau) &= \frac{aR_K}{D} = 2\pi k \int_0^1 \left[ \bar{T}_o + \sum_{n=0}^{\infty} T_n w_n(r) \right] H(r, \tau) r \, dr, \\
 r_g(\tau) &= \frac{aR_G}{D} = -2\pi g \int_0^1 \left[ \sum_{n=0}^{\infty} T_n \lambda_n^2 \bar{w}_n(r) \right] H(r, \tau) r \, dr, \\
 r_c(\tau) &= \frac{aR_C}{D} = 2\pi g \left[ \sum_{n=0}^{\infty} T_n \lambda_n \bar{w}_n(r = 1) + \lambda \frac{K_1(\lambda)}{K_o(\lambda)} \left[ \bar{T}_o + \sum_{n=0}^{\infty} T_n w_n(r = 1) \right] \right] H(r = 1, \tau), \\
 r_i(\tau) &= \frac{aR_I}{D} = 2\pi \left[ 0.5 \frac{d^2 \bar{T}_o}{d\tau^2} + \sum_{n=0}^{\infty} \frac{d^2 T_n}{d\tau^2} \int_0^1 w_n(r) r \, dr \right],
 \end{aligned} \tag{15}$$

where substitution of the displacement functions into the integrals leads to the following relations:

$$\int_0^1 H(r, \tau) r \, dr = 0.5b^2$$

$$\int_0^1 H(r, \tau) w_n(r) r \, dr = \frac{b}{\lambda_n} [J_1(\lambda_n b) + A_n I_1(\lambda_n b)]$$

$$\int_0^1 H(r, \tau) \bar{w}_n(r) r \, dr = -\frac{b}{\lambda_n} [J_1(\lambda_n b) - A_n I_1(\lambda_n b)],$$

$$\int_0^1 H(r, \tau) w_m(r) w_n(r) r \, dr = \frac{b}{\lambda_m^2 - \lambda_n^2} [\lambda_m J_o(\lambda_n b) J_1(\lambda_m b) - \lambda_n J_o(\lambda_m b) J_1(\lambda_n b)]$$

$$+ \frac{A_m b}{\lambda_m^2 + \lambda_n^2} [\lambda_m J_o(\lambda_n b) I_1(\lambda_m b) + \lambda_n J_o(\lambda_m b) J_1(\lambda_n b)]$$

$$+ \frac{A_n b}{\lambda_m^2 + \lambda_n^2} [\lambda_n J_o(\lambda_m b) I_1(\lambda_n b) + \lambda_m I_o(\lambda_n b) J_1(\lambda_m b)]$$

$$- \frac{A_m A_n b}{\lambda_m^2 - \lambda_n^2} [\lambda_m I_o(\lambda_n b) I_1(\lambda_m b) - \lambda_n I_o(\lambda_m b) I_1(\lambda_n b)] \quad \text{for } m \neq n,$$

$$\int_0^1 H(r, \tau) w_n(r) w_n(r) r \, dr = 0.5b^2 [J_o^2(\lambda_n b) + J_1^2(\lambda_n b)]$$

$$+ \frac{A_n}{\lambda_n} b [J_o(\lambda_n b) I_1(\lambda_n b) + I_o(\lambda_n b) J_1(\lambda_n b)]$$

$$+ 0.5A_n^2 b^2 [I_o^2(\lambda_n b) - I_1^2(\lambda_n b)]$$

$$\int_0^1 H(r, \tau) \bar{w}_m(r) w_n(r) r \, dr = -\frac{b}{\lambda_m^2 - \lambda_n^2} [\lambda_m J_o(\lambda_n b) J_1(\lambda_m b) - \lambda_n J_o(\lambda_m b) J_1(\lambda_n b)]$$

$$+ \frac{A_m b}{\lambda_m^2 + \lambda_n^2} [\lambda_m J_o(\lambda_n b) I_1(\lambda_m b) + \lambda_n J_o(\lambda_m b) J_1(\lambda_n b)]$$

$$- \frac{A_n b}{\lambda_m^2 + \lambda_n^2} [\lambda_n J_o(\lambda_m b) I_1(\lambda_n b) + \lambda_m I_o(\lambda_n b) J_1(\lambda_m b)]$$

$$+ \frac{A_m A_n b}{\lambda_m^2 - \lambda_n^2} [\lambda_m I_o(\lambda_n b) I_1(\lambda_m b) - \lambda_n I_o(\lambda_m b) I_1(\lambda_n b)] \quad \text{for } m \neq n,$$

$$\int_0^1 H(r, \tau) \bar{w}_n(r) w_n(r) r \, dr = -0.5b^2 [J_o^2(\lambda_n b) + J_1^2(\lambda_n b)] - 0.5A_n^2 b^2 [I_o^2(\lambda_n b) - I_1^2(\lambda_n b)],$$

$$\int_0^1 w_m(r) w_n(r) r \, dr = 0 \quad \text{for } m \neq n,$$

$$\int_0^1 w_n(r) w_n(r) r \, dr = 0.5 [J_o^2(\lambda_n) + J_1^2(\lambda_n)]$$

$$+ \frac{A_n}{\lambda_n} [J_o(\lambda_n) I_1(\lambda_n) + I_o(\lambda_n) J_1(\lambda_n)]$$

$$+ 0.5A_n^2 [I_o^2(\lambda_n) - I_1^2(\lambda_n)],$$

$$\int_0^1 w_n(r) r \, dr = 0, \tag{16}$$

where  $b(\tau)$  denotes the non-dimensional contact radius. For the case of partial contact  $b(\tau) < 1$ , the above expressions depend on time. In the present formulation, it is assumed that the foundation cannot support tensile reactions and the interaction between the foundation and the plate is only possible when the reaction under the plate is compressive. In general, a separation takes place to avoid the tensile reactions. In the case of the complete contact, one of the boundary conditions of the problem is the continuity of the displacement of the foundation surface at the edge of the plate. Considering the symmetry of the problem, this can be expressed as

$$W_p(R = a, t) = W_s(R = a, t) \quad (17)$$

or by using the non-dimensional parameters of the problem

$$w_p(r = 1, \tau) = w_s(r = 1, \tau),$$

$$\bar{T}_o + \sum_{n=0}^{\infty} T_n w_n(r = 1) = CK_o(\lambda). \quad (18)$$

The second boundary condition is the inclusion of the edge reaction into the analysis of the problem. In fact, this is done already in the governing equation of the problem (3) by using the Dirac's delta function and it is also included in the global checking of the vertical force equilibrium (13).

When a separation takes place, in the Winkler foundation model, the foundation displacement is continuous at the point that separates the contact and lift-off regions, whereas no continuity for the slope of the displacement is demanded. An excellent discussion about the boundary conditions involving the two-parameter foundation is given by Kerr [2]. Generally, in the case of partial contact, the three anticipated conditions can be stated at the point of separation; they are the continuity of the displacement of foundation, its slope and zero pressure, as it is the case for the elastic continuum. However, Kerr [21] pointed out that because of the reduced order of the governing differential equation of the two-parameter foundation model, only two of the three anticipated conditions can be satisfied which can be obtained by a variational analysis. For the problem under consideration, they are the continuity of the displacement and its slope. By taking into account the axial symmetry of the problem, the contact curve which separates the contact and the lift-off region will be a circle of the radius  $B = ba$ . The boundary conditions of the partial contact will be the continuity of the foundation surface and its slope at the contact radius:

$$W_p(R = B, t) = W_s(R = B, t) \quad \frac{\partial W_p}{\partial R}(R = B, t) = \frac{\partial W_s}{\partial R}(R = B, t),$$

$$w_p(r = b, \tau) = w_s(r = b, \tau) \quad \frac{\partial w_p}{\partial r}(r = b, \tau) = \frac{\partial w_s}{\partial r}(r = b, \tau), \quad (19)$$

$$\bar{T}_o + \sum_{n=0}^{\infty} T_n w_n(r = b) = CK_o(\lambda b) \quad \sum_{n=0}^{\infty} T_n \lambda_n \bar{w}_n(r = b) = -C\lambda K_1(\lambda b).$$

Since a partial contact is in question, no edge reaction exists and due to the properties of the Dirac's delta function, the corresponding term in the governing Eq. (3) is excluded. With Eq. (19), the formulation of the problem is completed. It is worth to note that at least in the present axially symmetric case a negative value of the plate displacement, i.e., upward displacement guarantees a separation between the plate and the foundation. However, contrary to the Winkler model, a positive value of the plate displacement does not always indicate that there is contact at that point, as shown in Fig. 1a. Inspection of the boundary conditions together with the governing equation of the problem justifies once more the use of the Galerkin's approximation, since it is very difficult, if not impossible to find a close solution for the displacement functions for the plate as well as for the foundation surface, which satisfy all these equations.

The static configuration of the plate on the foundation subjected to the concentrated load  $P$  and the distributed load  $Q$  can be studied easily by using the static version of the governing Eq. (11)

$$\mathbf{KT} = \mathbf{F} \quad (20)$$



for both the conventional and the tensionless foundation models. Since the mode shapes of the completely free plate is used in the expansion of the plate displacement function, the stiffness matrix  $\mathbf{K}$  will be diagonal one for  $g = 0$ . Although the Winkler foundation model is a special case of the two-parameter model for  $g = 0$ , the formulation as well as the numerical treatment of the problem cannot yield the corresponding result directly, due to the definition of  $\lambda = k/g$ . However, the numerical result which corresponds to the Winkler model can be obtained in a acceptable degree of approximation for  $g \rightarrow 0$  [7].

The evaluation of the system of the nonlinear equations requires an iterative solution by the fact that after lift-off, these equations depend continuously on the varying degrees of the contact between the plate and the foundation. On the other hand, they are relatively simple for a conventional foundation model for which the coefficients of the stiffness matrix can be evaluated in a straightforward manner on the assumption that the complete contact is established regardless of the foundation pressure. In this case the governing equation (11) will be a linear one.

### 3. Numerical results and discussion

The linear versions of the governing equations (11) and (20) are valid for the conventional foundation model and they can be obtained by assuming that the full contact is established between the plate and the foundation, or in other words by assuming that the foundation supports compression as well as tension, i.e., by assuming  $H(r, \tau) = 1$ . Although in the present problem small displacements for the plate and the foundation are assumed, the governing equation of problem (11) is highly nonlinear due to the tensionless character of the foundation and it requires several iterative procedures for the evaluation of the numerical results for the static case. Usually, numerical iterations require initially estimated values of the unknown contact radius. After having evaluated the elements of the coefficients of the stiffness matrix, Eq. (20) can be solved, the displacement functions are obtained. The contact radius is evaluated and checked. Iterative process is continued until an acceptable approximation is attained. In order to decide on the rate of convergence of the series in the solution, numerical treatment of the problem is carried out by considering various numbers of terms in the series. It is found that an acceptable accuracy for the graphical representation can be obtained by taking into account three mode shapes in addition to the rigid vertical translation. In the dynamic solution of the problem, inclusion of more mode shapes does not increase the overall accuracy of the graphical representation of oscillations. However, oscillations with small amplitudes but higher frequencies appear in the solution. In all numerical calculations it is assumed that  $\nu = 0.33$  and  $q$  is uniformly distributed. The force equilibrium (15) including the inertia force of the plate is checked in the static case as well as in the dynamic case for each time step.

Assuming that the plate is subjected to static loads  $p$  and  $q$ , the numerical results are given in Figs. 2 and 3. Figs. 2a–c show the contact radius  $b$ , the displacement at the center  $w_o = w_p(r = 0)$  and the edge  $w_a = w_p(r = 1)$  of the plate, respectively, for  $q/p = 0$ . As it is well known, the contact radius depends on the foundation stiffnesses  $g$  and  $k$ . It is independent of the level of the loading, provided that there is only one type of loading. In the present case, however, it depends on the loading ratio  $q/p$ . For a specific foundation model, the contact radius  $b$  does not change, as the loads  $p$  and  $q$  increase proportionally. The vertical equilibrium is maintained due to increase in vertical displacements proportionally. The plate subjected to a vertical central load on the tensionless Winkler foundation has been investigated by Celep [7]. The present results agree well for  $g = 0$  with those given in this study. Fig. 2a shows that the complete contact  $b = 1$  develops for rather low values of the foundation stiffnesses  $k$  and  $g$ . As the stiffnesses increase, the plate starts lift off and the contact region decreases. Figs. 2b and c show that the displacements of the center and that of the edge of the plate decrease with increasing stiffnesses as expected. As the stiffness  $k$  increases, the effect of the stiffness  $g$  on the displacements gets smaller. Although no result is presented, the same is valid for other way around, i.e., the effect of  $k$  gets significant for smaller values of  $g$ . Since  $q = 0$ , the displacements are proportional to the load  $p$ . Fig. 2d shows the edge reaction  $r_c$  which is generated when the complete contact develops. The edge reaction is proportional to the difference of the slopes at the two sides of the plate edge and to the foundation parameter  $g$ . The variation of  $r_c$  starts from zero, where the difference of the slope is very large for  $g = 0$ . It increases very rapidly to a maximum value, and then it decreases and becomes zero when the complete contact develops. Comparison of Figs. 2a and d shows that no edge reaction exists, when a partial contact is in question. The

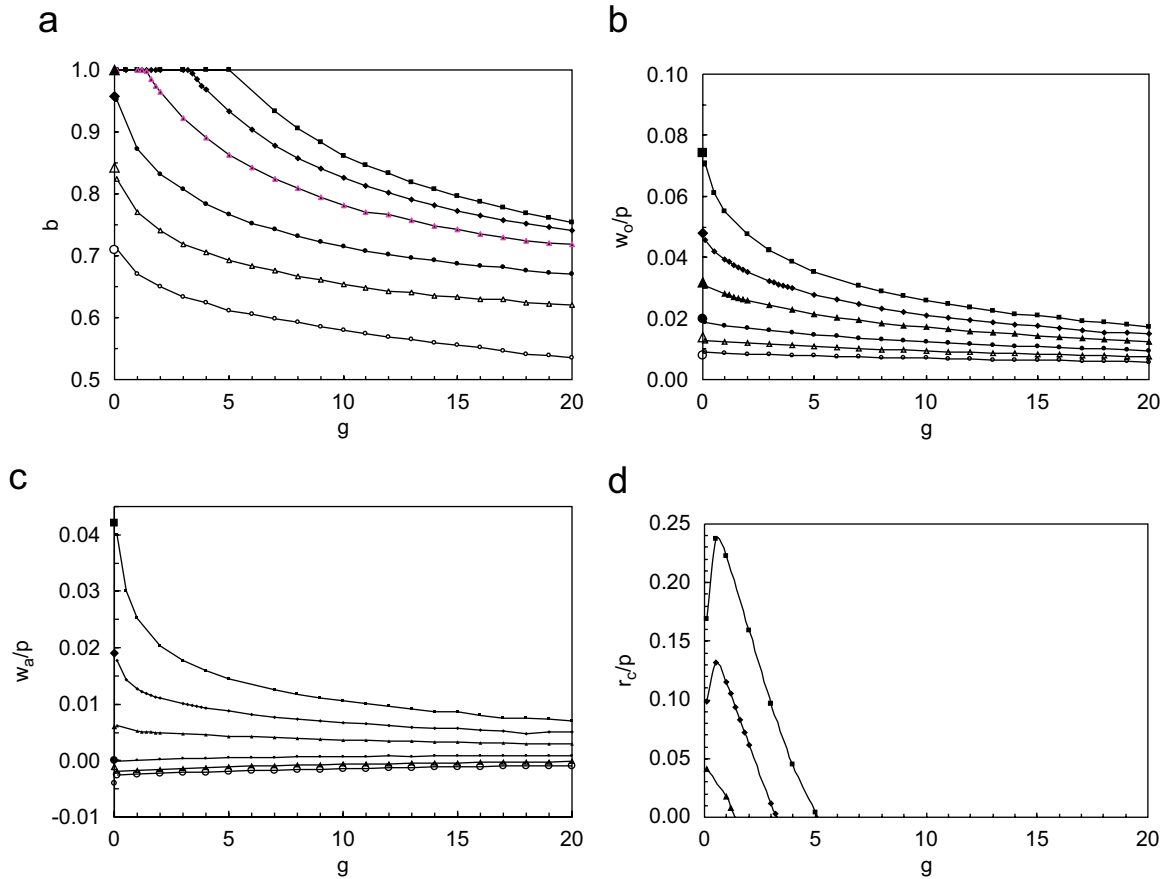


Fig. 2. Variation of (a) the contact radius  $b$ , (b) the displacement of the center of the plate  $w_o$ , (c) the displacement of the edge of the plate  $w_a$ , (d) the foundation reaction  $r_c$  with the foundation stiffness  $g$  for  $q/p = 0$  and for  $k = 5, \blacksquare; 10, \blacklozenge; 20, \blacktriangle; 50, \bullet; 100, \triangle; 200, \circ$ ; (for  $k = 5-20, \blacktriangle; 50, \blacklozenge; 100, \blacktriangle; 200, \bullet$  Celep [7] for  $g = 0$ ).

foundation reaction increases, when the corresponding foundation stiffness gets larger. Since the sum of the reactions has to be equal to the external load, an increase in one of the foundation reactions results in a decrease in the other foundation reactions.

Fig. 3 shows the similar variations for various values of the loading ratio  $q/p$  for  $k = 50$ . Comparison of Figs. 2 and 3 yields that the presence of the uniformly distributed load extends the complete contact region for the same combinations of the foundation stiffnesses, delays the onset of the partial contact and decreases the difference between the displacements of the plate at the center and at the edge. As  $q/p$  increases, the deformed shape of the plate gets flatter and the number of the numerical iterations for evaluations of the contact radius  $b$  increases rapidly. This is due to the fact that not only equality of the displacements, also the equivalence of the radial slopes are required at the contact circle.

For the dynamic problems, on the other hand, i.e. for oscillations of the plate on the foundation, the contact region of the plate depends on time. Numerical solution of the governing equation (20) is carried out for the forced vibrations by assuming that the plate is in static equilibrium under the loading  $p$  and  $q$ . Oscillations of the plate starts by changing the loading level to  $\beta p$  and  $\beta q$  by the dynamic load factor  $\beta$ . The time variation of the loading can be written as  $p + (\beta - 1)pH(\tau)$ , where  $H(\tau)$  denotes Heaviside step function. The governing equation (11) is a system of nonlinear differential equations, because the coefficients of the stiffness matrix have time-dependent terms. When the plate is partially uplifted, the coefficients depend continuously on the vertical displacements of the plate on the contact area. When a partial contact develops, the solution of the

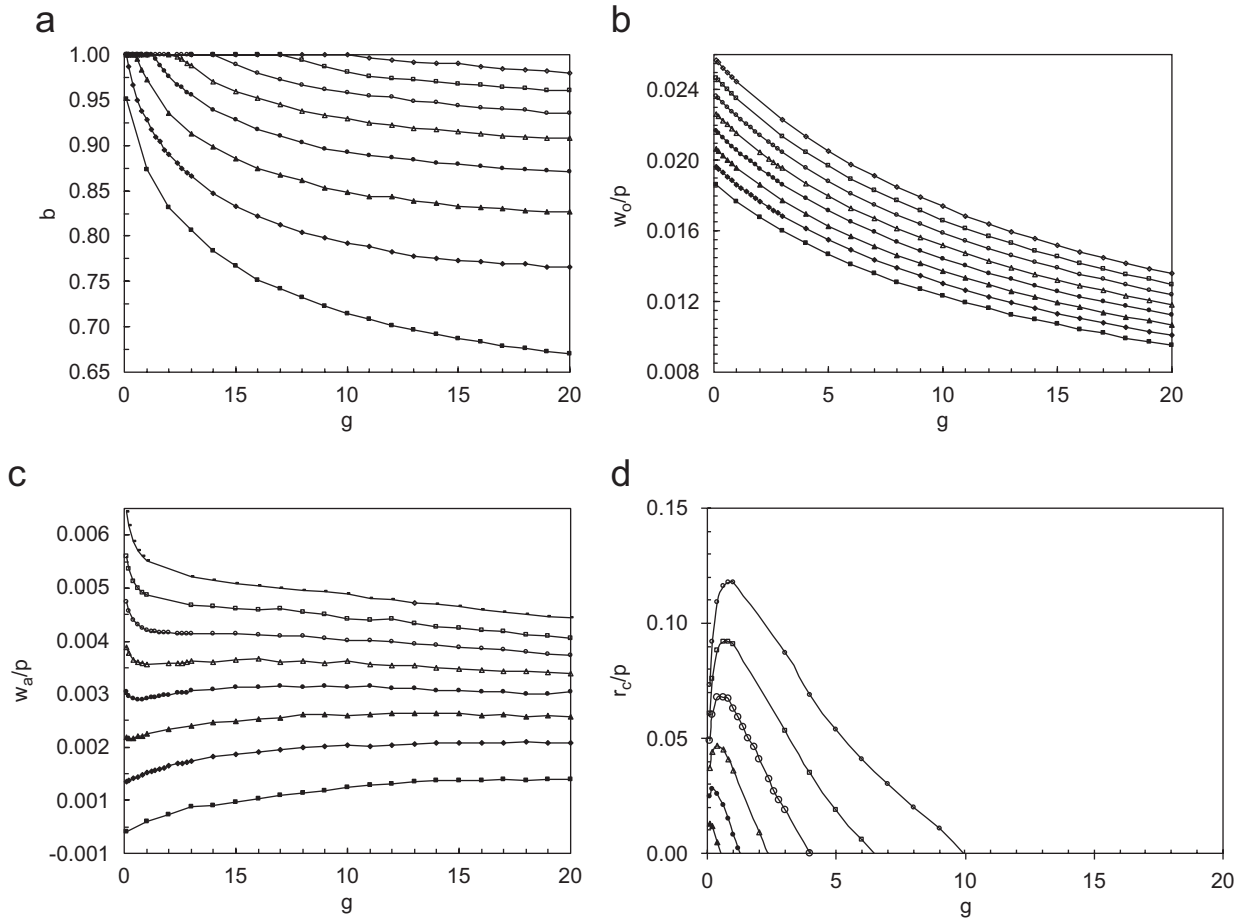


Fig. 3. Variation of (a) the contact radius  $b$ , (b) the displacement of the center of the plate  $w_o$ , (c) the displacement of the edge of the plate  $w_e$ , (d) the foundation reaction  $r_c$  with the foundation stiffness  $g$  for  $k = 50$  and for  $q/p = 0$ , ■; 0.05, ◆; 0.10, ▲; 0.15, ●; 0.20, △; 0.25, ○; 0.30, □; 0.35, ◇.

static case or the initial configuration of the plate requires an iterative solution. Similarly, the initial configuration of the dynamic behavior of the system is obtained by iteration and the governing differential equation (11) is solved along the time axis by employing a step-wise numerical integration. At each time step the contact function, the contact radius and the parameters of the problem including the coefficients of the stiffness matrix are evaluated numerically and updated by considering the displacement configuration of the plate at the previous time step. For identification of the response of the plate, numerous results are produced for static and dynamic cases and selected ones are presented in figures.

Fig. 4 shows oscillations of the parameters of problem for  $g = 2$  and  $k = 5$  by assuming  $q/p = 0$ , i.e., only a concentrated load is present. At the onset of the uplift, the system starts oscillating and the oscillations resemble to a harmonic variation. At this stage, the periods of the free vibrations can be evaluated approximately. For the given combination of the foundation stiffnesses, the first two non-dimensional period of the plate are calculated as 1.817 and 0.527, provided that complete contact is established. They correspond to rotationally symmetric free vibrations of the plate on the two-parameter conventional elastic foundation. In Fig. 4, oscillations of the corresponding parameters are not of harmonic nature and they display complex variations. However, these two approximate periods of the plate can be detected by a close inspection of these curves. It shows that the first vibration mode shape appears to be much more effective in oscillations of the plate.

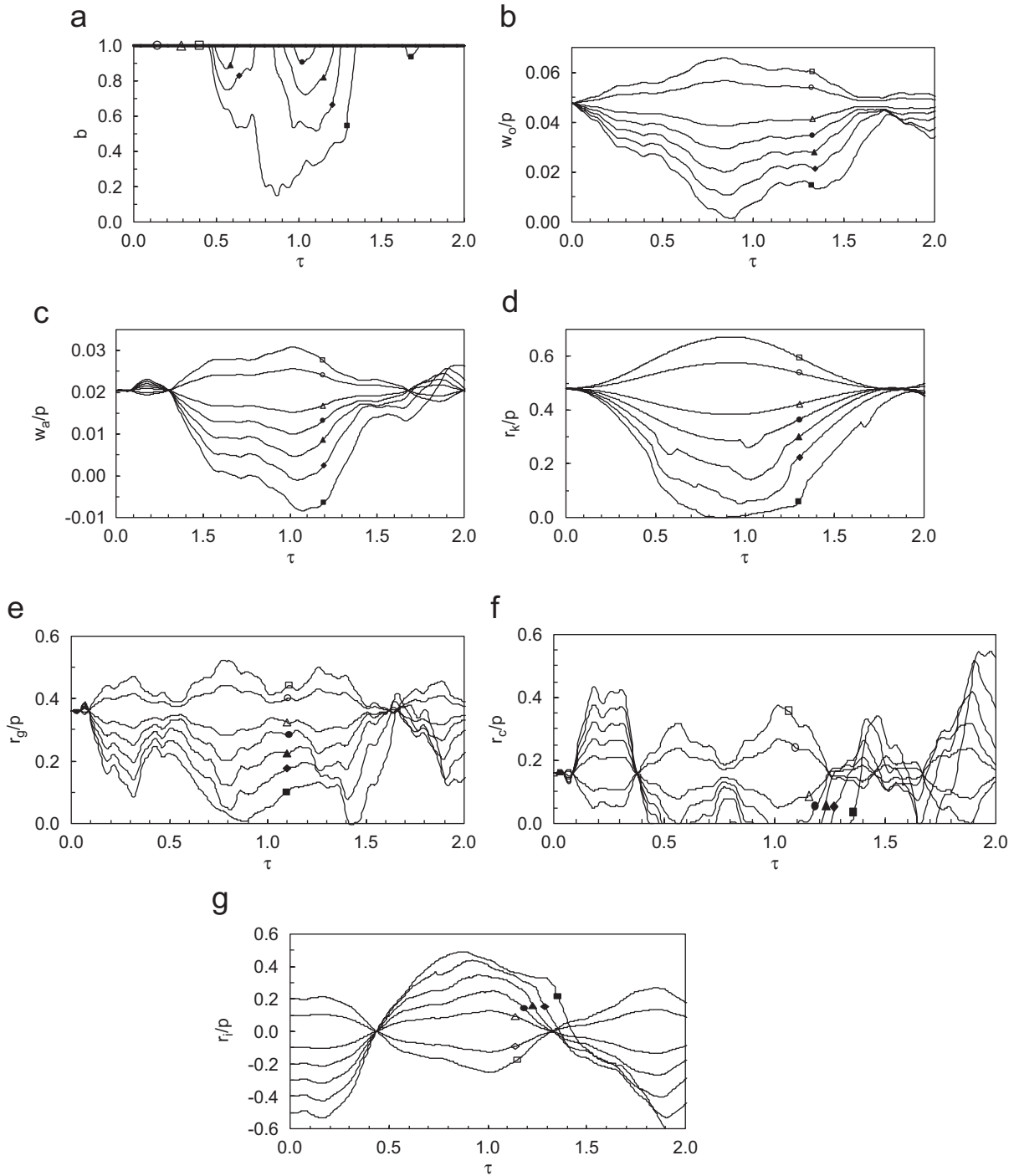


Fig. 4. Time variation of (a) the contact radius  $b$ , (b) the displacement of the center of the plate  $w_o$ , (c) the displacement of the edge of the plate  $w_a$ , (d) the foundation reaction  $r_k$ , (e) the foundation reaction  $r_g$ , (f) the foundation reaction  $r_c$ , (g) the inertia force  $r_i$  for  $g = 2, k = 5$  and  $q/p = 0$  and for  $\beta = 0.5, \blacksquare; 0.6, \blacklozenge; 0.7, \blacktriangle; 0.8, \bullet; 0.9, \triangle; 1.1, \circ; 1.2, \square$ .

As oscillations take place the contact region changes and the stiffness matrix has to be updated. Consequently, the corresponding periods of the plate changes as well. When the lift-off region increases, i.e., when the contact radius decreases, the approximate period of the oscillation elongates.

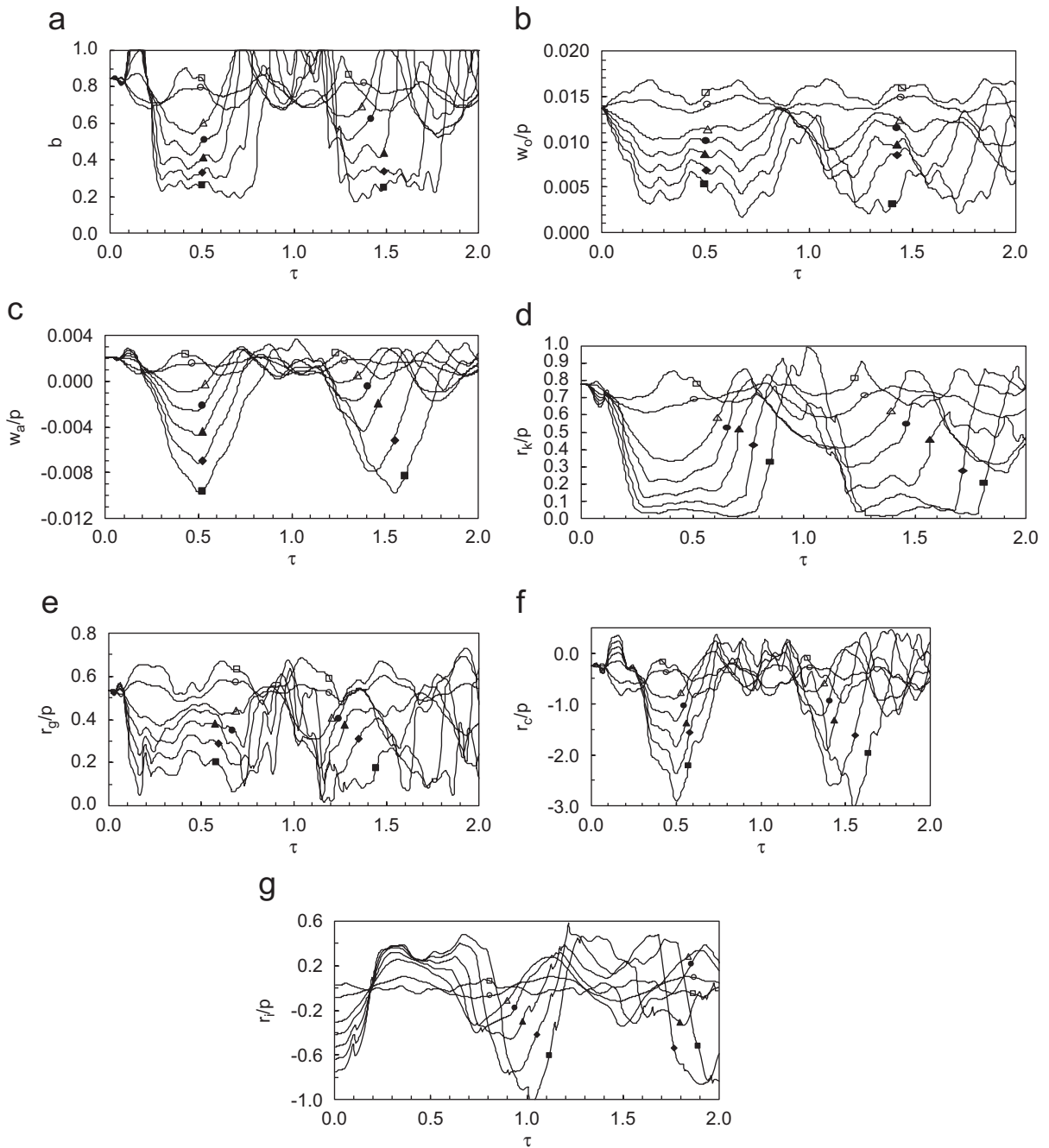


Fig. 5. Time variation of (a) the contact radius  $b$ , (b) the displacement of the center of the plate  $w_o$ , (c) the displacement of the edge of the plate  $w_a$ , (d) the foundation reaction  $r_k$ , (e) the foundation reaction  $r_g$ , (f) the foundation reaction  $r_c$ , (g) the inertia force  $r_i$  for  $g = 2, k = 5$  and  $q/p = 0.1$  and for  $\beta = 0.5, \blacksquare; 0.6, \blacklozenge; 0.7, \blacktriangle; 0.8, \bullet; 0.9, \triangle; 1.1, \circ; 1.2, \square$ .

Figs. 4 and 5 include time variations of the contact radius  $b(\tau)$ , the displacement of the center  $w_o(\tau)$  and the edge  $w_a(\tau)$  of the plate, the sum of the foundation reactions  $r_k(\tau)$  and  $r_g(\tau)$  that are distributed on the contact region, the edge reaction  $r_c(\tau)$  and the sum of the inertia force  $r_i(\tau)$  distributed on the plate. In Fig. 5, these variations are presented for  $g = 10$  and  $k = 50$  by assuming that  $q/p = 0.1$ . For these foundation stiffnesses, the first two non-dimensional periods of the plate are 1.277 and 0.458 for  $r = 0.849$ , i.e., the initial value of the

contact radius. Similarly, these two periods can be detected in Fig. 5, albeit not as easy as in Fig. 4, due to the rapid change of the contact radius during the oscillations.

#### 4. Conclusion

The paper presents analysis of the lift-off problem of a circular elastic plate subjected to a concentrated axial force and rotationally symmetric distributed loading on a two-parameter foundation. Special attention is paid to the non-dimensionalization of the formulation as well as on the boundary conditions of the plate and the foundation. In order to cover a large spectrum of values of the parameters, numerical results are presented by introducing non-dimensional parameters. Although the displacements of the plate and the foundation are assumed to be small, the governing equation of the problem is nonlinear, as the foundation cannot support tension and the plate lifts off the foundation. Solution of the problem is accomplished by applying Galerkin's method and the numerical results are presented comparatively for various values of the parameters of the problem. In static and dynamic cases, the numerical solution is assessed by checking the vertical force equilibrium. From the numerical analysis presented, the following conclusion can be drawn:

- a. In a two-dimensional foundation model, the separation point is to be determined by requiring the continuity of the displacement and its slope. Due to the shortcoming of the model, the foundation pressure displays a discontinuity along the contact circle contrary to the intuitive approach. When the complete contact is established, an edge reaction develops as a result of discontinuity of the slope of the displacement function. In the present formulation, the edge reaction is included into the governing equation of the problem; it is not treated as a boundary condition.
- b. Inclusion of the tensionless character of the foundation softens the static and dynamic behavior of the system due to the decrease in the support flexibility. The approximate period of the oscillations increases, when partial contact develops and the time variations of the parameters of the problem become more complex.
- c. The uplift of the plate is influenced mainly by the fundamental mode and the higher modes have lesser effect on the behavior.

#### References

- [1] A.D. Kerr, Elastic and viscoelastic foundation models, *Journal of Applied Mechanics* 31 (1964) 491–498.
- [2] A.D. Kerr, On the derivations of well-posed boundary value problems in structural mechanics, *International Journal of Solids and Structures* 12 (1) (1976) 1–11.
- [3] Y. Weisman, On foundations that react in compression only, *Journal of Applied Mechanics* 37 (1970) 1019–1030.
- [4] P. Villaggio, A free boundary value problem in plate theory, *Journal of Applied Mechanics* 50 (1983) 297–302.
- [5] J.P. Dempsey, L.M. Keer, N.B. Patel, M.L. Glasser, Contact between plates and unilateral supports, *Journal of Applied Mechanics* 51 (1984) 324–328.
- [6] Z. Celep, D. Turhan, R.Z. Al-Zaid, Circular elastic plates on elastic unilateral edge supports, *Journal of Applied Mechanics* 55 (3) (1988) 624–628.
- [7] Z. Celep, Circular plate on tensionless Winkler foundation, *Journal of Engineering Mechanics* 114 (10) (1988) 1723–1739.
- [8] Z. Celep, D. Turhan, R.Z. Al-Zaid, Contact between a circular plate and a tensionless edge support, *International Journal of Mechanical Sciences* 30 (10) (1988) 733–741.
- [9] T. Hong, J.G. Teng, Y.F. Luo, Axisymmetric shells and plates on tensionless elastic foundations, *International Journal of Solids and Structures* 36 (1999) 5277–5300.
- [10] Z. Celep, Harmonic and seismic responses of a plate-column system on a tensionless Winkler foundation, *Journal of Sound and Vibration* 155 (1) (1992) 47–53.
- [11] K. Güler, Z. Celep, Static and dynamic responses of a circular plate on a tensionless elastic foundation, *Journal of Sound and Vibration* 183 (2) (1995) 185–195.
- [12] Z. Celep, K. Güler, Static and dynamic responses of a rigid circular plate on a tensionless Winkler foundation, *Journal of Sound and Vibration* 276 (1–2) (2004) 449–458.
- [13] Z. Celep, M. Gençoğlu, Forced vibrations of rigid circular plate on a tensionless Winkler edge support, *Journal of Sound and Vibration* 263 (2003) 945–953.

- [14] C.V.G. Vallabhan, W.T. Straughan, Y.C. Das, A refined model for analysis of plates on elastic foundations, *Journal of Engineering Mechanics ASCE* 117 (1991) 2830–2844.
- [15] C.V.G. Vallabhan, A.T. Daloğlu, Consistent FEM-Vlasov model for plates on layered soil, *Journal of Structural Engineering ASCE* 127 (1999) 108–113.
- [16] Y. Ayvaz, A.T. Daloğlu, A. Doğangün, Application of a modified Vlasov model to earthquake analysis of plate on elastic foundations, *Journal of Sound and Vibration* 212 (3) (1998) 499–509.
- [17] A.T. Daloğlu, A. Doğangün, Y. Ayvaz, Dynamic analysis of foundation plates using a consistent Vlasov model, *Journal of Sound and Vibration* 224 (5) (1999) 941–951.
- [18] M. Çelik, A. Saygun, A method for the analysis on plates on a two-parameter foundation, *International Journal of Solids and Structures* 36 (1999) 2891–2916.
- [19] A. Saygun, M. Çelik, Analysis of circular plates on two-parameter elastic foundation, *Structural Engineering and Mechanics* 15 (2) (2003) 249–267.
- [20] R. Buczkowski, W. Torbacki, Finite element modeling of thick plates on two-parameter elastic foundation, *International Journal for Numerical and Analytical Method in Geomechanics* 25 (2001) 1409–1427.
- [21] A.D. Kerr, D.W. Coffin, Beams on a two-dimensional Pasternak base subjected to loads that cause lift-off, *International Journal of Solids and Structures* 28 (4) (1991) 413–422.
- [22] K. Güler, Circular elastic plate resting on tensionless Pasternak foundation, *Journal of Engineering Mechanics ASCE* 130 (10) (2004) 1251–1254.