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Journal of Sound and Vibration 301 (2007) 544–559
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Asymptotic formulas for the acoustic radiation impedance of an elastically supported annular plate

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Received 24 March 2004; received in revised form 11 October 2006; accepted 16 October 2006
Available online 12 December 2006

Abstract

This paper deals with the normalized acoustic radiation impedance of an elastically supported annular plate in axisymmetric vibrations. Cauchy's theorem about residues and the stationary phase method have been used to approximate the corresponding integrals. As a result, elementary asymptotic formulas valid for axisymmetric boundary configurations of clamped, guided, simply supported and free annular plates as well as for all the intermediate boundary configurations have been obtained.

1. Introduction

Thin flat annular plate idealization is often used for analysis of numerous real-life vibrating systems such as setting centrifuges, pipe-reducing elements in petroleum industry, transport, electro-acoustic devices, sound probes, microphones, etc. In most cases, the plates are excited mechanically and acoustically and become sound sources. The acoustic radiation impedance is an important acoustic measure to describe such sources. It is difficult or impossible to find exact analytical solutions for this quantity. Therefore, approximate solutions have been presented only in a few cases [1–8].

The main aim of this study is to present high-frequency asymptotic formulas of radiation impedance of an elastically supported annular plate valid for all axisymmetric boundary conditions in full recognition of real-life conditions. Annular plates are seldom clamped, guided, simply supported or free. Often, the plate edge satisfies some intermediate boundary conditions.

2. Governing equations

A flat thin elastically supported annular plate is embedded into a flat rigid infinite baffle (cf., Fig. 1). The plate vibrations as well as the radiated acoustic waves are time harmonic and axisymmetric. Low fluid loading has been assumed. Material damping has been assumed to be small enough to be neglected. The mode shape

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2006.10.031

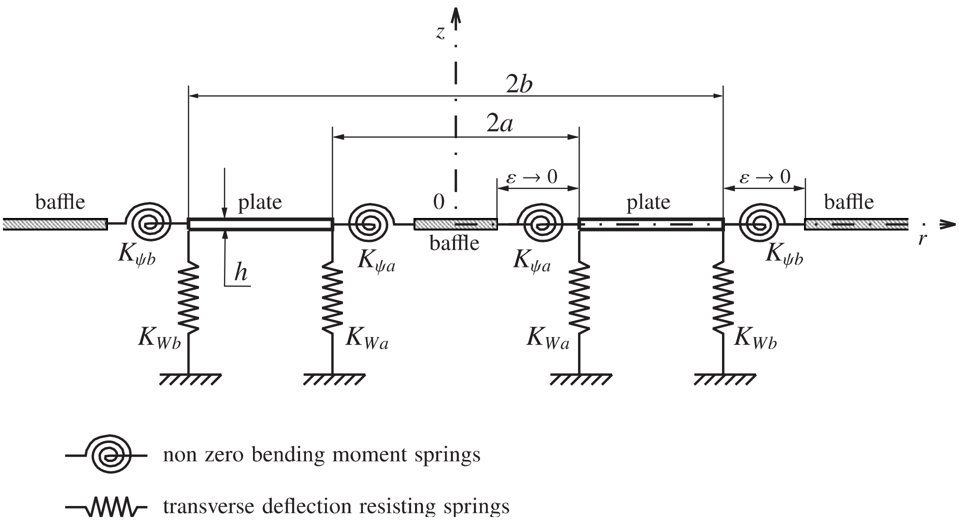

Fig. 1. Acoustic system containing an elastically supported annular plate embedded into a flat infinite baffle.

for the axisymmetric problem is [9,10]

$$W_n(r) = A_n\,[J_0(k_n r) + B_n I_0(k_n r) - C_n Y_0(k_n r) - D_n K_0(k_n r)], \tag{1}$$

where $r \in [a,b]$ is the radial variable, a,b are the internal and external radii, $J_n(\cdot)$, $I_n(\cdot)$, $Y_n(\cdot)$, $K_n(\cdot)$ are the nth order Bessel, modified Bessel, Neumann, McDonald functions, $k_n^4 = \omega_n^2 \varrho h/D_E$, ω_n is the eigenfrequency, $D_E = Eh^3/12\,(1-\nu^2)$ is the bending stiffness, ϱ, h, E, ν are the plate density, thickness, Young modulus and Poisson ratio, respectively, and n is the number of nodal circles while the number of nodal diameters is assumed to be zero. The axisymmetric boundary conditions are [10]:

$$K_{W\mu} W_n(\mu) = D_E \frac{\mathrm{d}}{\mathrm{d}r} \nabla_r^2 W_n(r)\Big|_{r=\mu}, \tag{2a}$$

$$K_{\psi\mu} \frac{\mathrm{d}}{\mathrm{d}r} W_n(r)\Big|_{r=\mu} = -D_E \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\nu}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right) W_n(r)\Big|_{r=\mu}, \tag{2b}$$

where $\nabla_r^2 = \mathrm{d}^2/\mathrm{d}r^2 + (1/r)\,\mathrm{d}/\mathrm{d}r$, $\mu \in \{a,b\}$, and K_W, K_ψ are boundary stiffness values associated with deflections and rotations of the plate edges, respectively. It is useful to define the following vector: $\mathbf{K} = (K_1, K_2, K_3, K_4)$ where $K_1 = K_{Wa} a^3/D_E$, $K_2 = K_{Wb} b^3/D_E$, $K_3 = K_{\psi a} a/D_E$, $K_4 = K_{\psi b} b/D_E$. Eq. (1) is a solution to the homogeneous equation of motion $(k_n^{-4}\nabla_r^4 - 1)\, W_n(r) = 0$ where $\nabla_r^4 = \nabla_r^2 \nabla_r^2$. Inserting Eq. (1) into Eqs. (2) gives four algebraic equations. Three of them are linearly independent. Solving them produces three constants:

$$C_n = N^{(1)}/D^{(1)} = N^{(2)}/D^{(2)},$$

$$\begin{aligned} B_n &= s\lambda_n Q_b\{(q_b p_b - 1)[N(s\lambda_n) - C_n R(s\lambda_n)] + 2q_b K_0(s\lambda_n) G_0(s\lambda_n) + 2p_b K_1(s\lambda_n) G_1(s\lambda_n)\} \\ &= \lambda_n Q_a\{(q_a p_a - 1)\,[N(\lambda_n) - C_n R(\lambda_n)] + 2q_a K_0(\lambda_n) G_0(\lambda_n) + 2p_a K_1(\lambda_n) G_1(\lambda_n)\}, \end{aligned}$$

$$\begin{aligned} D_n &= s\lambda_n Q_b\{(q_b p_b - 1)[S(s\lambda_n) - C_n T(s\lambda_n)] + 2q_b I_0(s\lambda_n) G_0(s\lambda_n) - 2p_b I_1(s\lambda_n) G_1(s\lambda_n)\} \\ &= \lambda_n Q_a\{(q_a p_a - 1)\,[S(\lambda_n) - C_n T(\lambda_n)] + 2q_a I_0(\lambda_n) G_0(\lambda_n) - 2p_a I_1(\lambda_n) G_1(\lambda_n)\}, \end{aligned}$$

where

$$N^{(1)} = sQ_b[(1-q_bp_b)\,S(s\lambda_n) - 2q_bJ_0(s\lambda_n)I_0(s\lambda_n) + 2p_bJ_1(s\lambda_n)I_1(s\lambda_n)] - Q_a[(1-q_ap_a)\,S(\lambda_n) - 2q_aJ_0(\lambda_n)I_0(\lambda_n) + 2p_aJ_1(\lambda_n)I_1(\lambda_n)],$$

$$D^{(1)} = sQ_b\,[(1-q_bp_b)\,T(s\lambda_n) - 2q_bN_0(s\lambda_n)I_0(s\lambda_n) + 2p_bN_1(s\lambda_n)I_1(s\lambda_n)] - Q_a\,[(1-q_ap_a)\,T(\lambda_n) - 2q_aN_0(\lambda_n)I_0(\lambda_n) + 2p_aN_1(\lambda_n)I_1(\lambda_n)],$$

$$N^{(2)} = sQ_b\,[(1-q_bp_b)\,N(s\lambda_n) - 2q_bJ_0(s\lambda_n)K_0(s\lambda_n) - 2p_bJ_1(s\lambda_n)K_1(s\lambda_n)] - Q_a\,[(1-q_ap_a)\,N(\lambda_n) - 2q_aJ_0(\lambda_n)K_0(\lambda_n) - 2p_aJ_1(\lambda_n)K_1(\lambda_n)],$$

$$D^{(2)} = sQ_b\,[(1-q_bp_b)\,R(s\lambda_n) - 2q_bN_0(s\lambda_n)K_0(s\lambda_n) - 2p_bN_1(s\lambda_n)K_1(s\lambda_n)] - Q_a\,[(1-q_ap_a)\,R(\lambda_n) - 2q_aN_0(\lambda_n)K_0(\lambda_n) - 2p_aN_1(\lambda_n)K_1(\lambda_n)],$$

$s = b/a$, $\lambda_n = k_na$ is the eigenvalue, $Q_a = 1/(1+q_ap_a)$, $Q_b = 1/(1+q_bp_b)$, $p_a = (K_3 - 1 + v)/\lambda_n$, $p_b = (K_4 - 1 + v)/s\lambda_n$, $q_a = K_1/\lambda_n^3$, $q_b = K_2/(s\lambda_n)^3$, $S(s\lambda) = J_1(s\lambda)I_0(s\lambda) + J_0(s\lambda)I_1(s\lambda)$, $T(s\lambda) = Y_1(s\lambda)I_0(s\lambda) + Y_0(s\lambda)I_1(s\lambda)$, $N(s\lambda) = J_1(s\lambda)K_0(s\lambda) - J_0(s\lambda)K_1(s\lambda)$, $R(s\lambda) = Y_1(s\lambda)K_0(s\lambda) - Y_0(s\lambda)K_1(s\lambda)$, and $G_0(s\lambda) = J_0(s\lambda) - C_nY_0(s\lambda)$, $G_1(s\lambda) = J_1(s\lambda) - C_nY_1(s\lambda)$.

The fourth constant A_n has been determined using the orthogonality condition $\int_a^b W_n^2(r)r\,\mathrm{d}r = a^2(s^2-1)/2$ [11]:

$$\begin{aligned} A_n^{-2} &= \frac{1}{a^2(s^2-1)}\left\{2\int_a^b [J_0(k_nr) + B_nI_0(k_nr) - C_nY_0(k_nr) - D_nK_0(k_nr)]^2 r\,\mathrm{d}r\right\} \\ &= \frac{1}{s^2-1}\,(s^2\{G_1^2(s\lambda_n) + G_0^2(s\lambda_n) + [2G(s\lambda) - G_0(s\lambda_n)]^2 - [2H(s\lambda_m) + G_1(s\lambda_n)]^2 \\ &\quad + (4/s\lambda_n)\,[G_1(s\lambda_n)G(s\lambda_n) + G_0(s\lambda)H(s\lambda_n)]\} \\ &\quad - \{G_1^2(\lambda_n) + G_0^2(\lambda_n) + [2G(\lambda) - G_0(\lambda_n)]^2 - [2H(\lambda_m) + G_1(\lambda_n)]^2 \\ &\quad + (4/\lambda_n)[G_1(\lambda_n)G(\lambda_n) + G_0(\lambda)H(\lambda_n)]\}), \end{aligned}$$

where $G(\lambda_n) = Q_a[p_aG_1(\lambda_n) + G_0(\lambda_n)]$, $G(s\lambda_n) = Q_b[p_bG_1(s\lambda_n) + G_0(s\lambda_n)]$, and $H(\lambda_n) = Q_a[q_aG_0(\lambda_n) - G_1(\lambda_n)]$, $H(s\lambda_n) = Q_b[q_bG_0(s\lambda_n) - G_1(s\lambda_n)]$.

The frequency equation takes the form $N^{(1)}/D^{(1)} = N^{(2)}/D^{(2)}$.

The normalized radiation impedance related to axisymmetric mode $(0, n)$ has been formulated in its Hankel representation [3,12]

$$\zeta_n = \theta_n - \mathrm{i}\chi_n = 4\delta_n^4q_n^2\int_0^\infty \psi_n^2(x)\,\frac{x\,\mathrm{d}x}{\gamma}, \tag{3}$$

where θ_n, χ_n are the normalized acoustic radiation resistance and reactance, respectively, $\mathrm{i} = \sqrt{-1}$, $\gamma = \sqrt{1-x^2}$ for $x \leqslant 1$ and $\gamma = \mathrm{i}\sqrt{x^2-1}$ for $x > 1$, $x \in \mathbb{R}$, $\delta_n = k_n/k$, k is the acoustic wavenumber, $q_n^2 = 2s^2G_0^2(s\lambda_n)A_n^2/(s^2-1)$, and

$$\psi_n(x) = \frac{2\beta}{a^2}\,\frac{1}{sG_0(s\lambda_n)}\int_a^b \frac{W_n(r)}{A_n}J_0(krx)r\,\mathrm{d}r = \frac{\psi_{1,n}(x)}{\delta_n^4 - x^4} + \frac{\psi_{2,n}(x)}{\delta_n^2(\delta_n^2 + x^2)}, \tag{4}$$

$$\psi_{1,n}(x) = \delta_na_{s,n}\,J_0(s\beta x) - xJ_1(s\beta x) - d_n\,[\delta_na_{1,n}\,J_0(\beta x) - xJ_1(\beta x)],$$

$$\psi_{2,n}(x) = Q_b\,[\delta_n(q_b - a_{s,n})\,J_0(s\beta x) + x(1 + p_ba_{s,n})\,J_1(s\beta x)] - d_n\,\{Q_a[\delta_n(q_a - a_{1,n})J_0(\beta x) + x(1 + p_aa_{1,n})\,J_1(\beta x)]\},$$

$a_{s,n} = G_1(s\lambda_n)/G_0(s\lambda_n)$, $d_n = G_0(\lambda_n)/sG_0(s\lambda_n)$, $\beta = ka$.

Integrating in Eq. (3) has been performed along the real axis. Cauchy's principal value computed within the limits of $(0, 1)$ represents radiation resistance, whereas Cauchy's principal value computed within the limits of $(1, \infty)$ represents radiation reactance.

3. Asymptotic formulas

It is possible to perform numerical computations using Eq. (3). However, this study focuses on its analytical calculation to obtain an elementary asymptotic formula. The closed contour integral technique and the stationary phase method have been used for this purpose [6,7]. First, Eq. (3) has been expressed as

$$\zeta_n = 4\delta_n^4 q_n^2 [\zeta_n^{(1)} + \zeta_n^{(2)} + \zeta_n^{(3)}], \tag{5}$$

where

$$\zeta_n^{(1)} = \int_0^\infty \frac{\psi_{1,n}^2(x)}{(x^4 - \delta_n^4)^2} \frac{x\,\mathrm{d}x}{\gamma}, \tag{6a}$$

$$\zeta_n^{(2)} = -\frac{2}{\delta_n^2} \int_0^\infty \frac{\psi_{1,n}(x)\psi_{2,n}(x)}{(x^4 - \delta_n^4)(x^2 + \delta_n^2)} \frac{x\,\mathrm{d}x}{\gamma}, \tag{6b}$$

$$\zeta_n^{(3)} = \frac{1}{\delta_n^4} \int_0^\infty \frac{\psi_{2,n}^2(x)}{(x^2 + \delta_n^2)^2} \frac{x\,\mathrm{d}x}{\gamma}. \tag{6c}$$

Further, Cauchy's principal values representing radiation resistance computed within definite limits in Eq. (6) have been substituted with residues and the corresponding infinite limits integrals (with no poles, cf., Fig. 2). While integrating (6a) within the definite limits, the following function has been used:

$$\begin{aligned} F_1(z) = {} & \delta_n^2 a_n^2(s\lambda_n) J_0(s\beta z) H_0^{(1)}(s\beta z) + z^2 J_1(s\beta z) H_1^{(1)}(s\beta z) \\ & + \delta_n^2 d_n^2 a_n^2(\lambda_n) J_0(\beta z) H_0^{(1)}(\beta z) + d_n^2 z^2 J_1(\beta z) H_1^{(1)}(\beta z) \\ & + \delta_n a_n^2(s\lambda_n) z [J_0(s\beta z) H_1^{(1)}(s\beta z) + H_0^{(1)}(s\beta z) J_1(s\beta z)] \\ & - \delta_n d_n^2 a_n^2(\lambda_n) z [J_0(\beta z) H_1^{(1)}(\beta z) + H_0^{(1)}(\beta z) J_1(\beta z)] \\ & - 2 d_n [\delta_n^2 a_n(s\lambda_n) a_n(\lambda_n) H_0^{(1)}(s\beta z) J_0(\beta z) + z^2 H_1^{(1)}(s\beta z) J_1(\beta z)] \\ & + 2\delta_n d_n z [a_n(s\lambda_n) H_0^{(1)}(s\beta z) J_1(\beta z) + a_n(\lambda_n) H_1^{(1)}(s\beta z) J_0(\beta z)], \end{aligned} \tag{7}$$

where $z = x + \mathrm{i}y \in \mathbb{C}$, $x, y \in \mathbb{R}$, such that Re $F_1(x) = \psi_{1,n}^2(x)$ where $H_n^{(1)}$ is the nth order Hankel function of the first kind. Cauchy's theorem provides

$$\oint_C \frac{zF_1(z)\,\mathrm{d}z}{\sqrt{1 - z^2}(z^4 - \delta_n^4)^2} = 0, \tag{8}$$

where the integrand is homogeneous and regular along and inside contour C. There are two branching points: $z = 1$ for term $\sqrt{1 - z^2}$ and $z = 0$ for the Hankel function. Furthermore, integrals computed along small arcs vanish as their radii approach zero whereas the integral computed along the big arc also vanishes as its radius grows infinitely. Contributions at the small arcs about the pole singularities have been expressed using functions:

$$\mathscr{F}_1^{(1)}(z) = \frac{zF_1(z)}{\sqrt{1 - z^2}(z + \delta_n)^2(z^2 + \delta_n^2)^2} \quad \text{at } z = \delta_n, \tag{9a}$$

$$\mathscr{F}_1^{(2)}(z) = \frac{zF_1(z)}{\sqrt{1 - z^2}(z + \mathrm{i}\delta_n)^2(z^2 - \delta_n^2)^2} \quad \text{at } z = \mathrm{i}\delta_n \tag{9b}$$

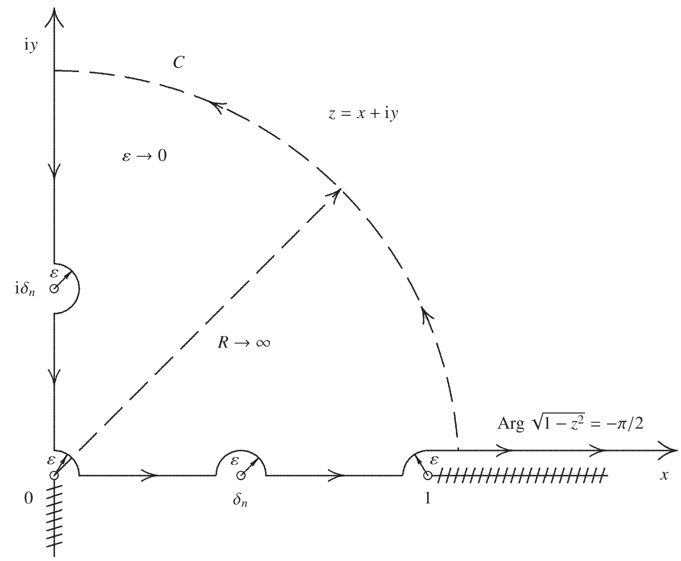

Fig. 2. Integration contour C.

and Eq. (8) has been rewritten as [7]

$$\begin{aligned}\mathrm{Re}⨍_0^1 \frac{xF_1(x)\,\mathrm{d}x}{\sqrt{1-x^2}(x^4-\delta_n^4)^2} &= \int_0^1 \frac{\psi_{1,n}^2(x)}{(x^4-\delta_n^4)^2}\frac{x\,\mathrm{d}x}{\sqrt{1-x^2}} \\ &= \mathrm{Re}\left\{\pi\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}z}\mathscr{F}_1^{(1)}(z)\bigg|_{z=\delta_n} + \pi\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}z}\mathscr{F}_1^{(2)}(z)\bigg|_{z=\mathrm{i}\delta_n}\right\} \\ &\quad + \int_1^\infty \frac{x\ \mathrm{Im}\ F_1(x)\,\mathrm{d}x}{\sqrt{x^2-1}(x^4-\delta_n^4)^2} + ⨍_\infty^0 \frac{y\ \mathrm{Re}\ F_1(\mathrm{i}y)\,\mathrm{d}y}{\sqrt{1+y^2}\,(y^4-\delta_n^4)^2},\end{aligned} \tag{10}$$

where $⨍$ denotes Cauchy's principal value. The value of Im $F_1(x)$ has been obtained from Eq. (7) by taking Neumann function as the imaginary part of Hankel function. For purely imaginary argument iy in Eq. (7), considering that $J_0(\mathrm{i}y) = I_0(y)$, $J_1(\mathrm{i}y) = \mathrm{i}I_1(y)$, $H_0^{(1)}(\mathrm{i}y) = -(2\mathrm{i}/\pi)K_0(y)$, $H_1^{(1)}(\mathrm{i}y) = -(2/\pi)K_1(y)$, it has been obtained that

$$\begin{aligned}F_1(\mathrm{i}y) = 0 - \mathrm{i}\frac{2}{\pi}\,\{&\delta_n^2 a_n^2(s\lambda_n)I_0(s\beta y)K_0(s\beta y) - y^2 I_1(s\beta y)K_1(s\beta y) \\ &+ \delta_n^2 d_n^2 a_n^2(\lambda_n)I_0(\beta y)K_0(\beta y) - d_n^2 y^2 I_1(\beta y)K_1(\beta y) \\ &+ \delta_n a_n^2(s\lambda_n)y\,[I_0(s\beta y)K_1(s\beta y) - K_0(s\beta y)I_1(s\beta y)] \\ &- \delta_n d_n^2 a_n^2(\lambda_n)y\,[I_0(\beta y)K_1(\beta y) - K_0(\beta y)I_1(\beta y)] \\ &- 2d_n\,[\delta_n^2 a_n(s\lambda_n)a_n(\lambda_n)K_0(s\beta y)I_0(\beta y) - y^2K_1(s\beta y)I_1(\beta y)] \\ &- 2\delta_n d_n y\,[a_n(s\lambda_n)K_0(s\beta y)I_1(\beta y) - a_n(\lambda_n)K_1(s\beta y)I_0(\beta y)]\}.\end{aligned} \tag{11}$$

Thus, it has been deduced that Re $F_1(\mathrm{i}y) = 0$ and that the last integral in Eq. (10) computed along the imaginary axis is also equal to zero. So, the first integral in Eq. (10) has been expressed as the sum of residues at $z = \delta_n, z = \mathrm{i}\delta_n$ and the integral computed along the real axis within the limits $(1, \infty)$. Further, the following values have been computed:

$$F_1(\delta_n) = 0, \tag{12a}$$

$$\mathrm{Im}\ F_1'(\delta_n) = -\frac{2}{\pi}\delta_n\{1 + a_n^2(s\lambda_n) - d_n^2[1 + a_n^2(\lambda_n)]\}, \tag{12b}$$

$$\mathrm{Im}\ F_1(\mathrm{i}\delta_n) = -\frac{2}{\pi}\,\delta_n^2(AB - 2Ab + ab), \tag{12c}$$

$$F_1'(\mathrm{i}\delta_n) = \frac{2}{\pi}\,\beta\delta_n^2[s(BG - AH) + bg - ah - 2sGb + 2Ah], \tag{12d}$$

where $A = a_n(s\lambda_n)K_0(s\lambda_n) - K_1(s\lambda_n)$, $a = d_n\,[a_n(\lambda_n)K_0(\lambda_n) - K_1(\lambda_n)]$, $B = a_n(s\lambda_n)I_0(s\lambda_n) + I_1(s\lambda_n)$, $b = d_n\,[a_n(\lambda_n)\ I_0(\lambda_n) + I_1(\lambda_n)]$, $G = a_n(s\lambda_n)K_1(s\lambda_n) - K_0(s\lambda_n)$, $g = d_n\,[a_n(\lambda_n)K_1(\lambda_n) - K_0(\lambda_n)]$, $H = a_n(s\lambda_n)I_1(s\lambda_n) + I_0(s\lambda_n)$, $h = d_n\,[a_n(\lambda_n)I_1(\lambda_n) + I_0(\lambda_n)]$.

Using Eqs. (9) and (12a) gives

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathscr{F}_1^{(1)}(z)\bigg|_{z=\delta_n} = \frac{1}{16\delta_n^5\sqrt{1-\delta_n^2}}\frac{\mathrm{d}}{\mathrm{d}z}F_1(z)\bigg|_{z=\delta_n}, \tag{13a}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathscr{F}_1^{(2)}(z)\bigg|_{z=\mathrm{i}\delta_n} = \frac{1}{16\,\delta_n^6\sqrt{1+\delta_n^2}}\left[-\mathrm{i}\delta_n\frac{\mathrm{d}}{\mathrm{d}z}F_1(z)\bigg|_{z=\mathrm{i}\delta_n} + \left(2 + \frac{\delta_n^2}{1+\delta_n^2}\right)F_1(\mathrm{i}\delta_n)\right]. \tag{13b}$$

The sum of residues at the poles from Eq. (10) has been denoted as

$$\begin{aligned}\bar{\theta}_n^{(1)} &= \mathrm{Re}\{\pi\mathrm{i}[\mathscr{F}_1^{(1)'}(\delta_n) + \mathscr{F}_1^{(2)'}(\mathrm{i}\delta_n)]\} = \frac{1 + a_n^2(s\lambda_n) - d_n^2[1 + a_n^2(\lambda_n)]}{8\delta_n^4\sqrt{1-\delta_n^2}}\\ &\quad + \frac{1}{8\delta_n^4\sqrt{1+\delta_n^2}}\left\{\lambda_n[s(BG - AH - 2Gb) + bg - ah + 2Ah]\right.\\ &\quad \left. + \left(2 + \frac{\delta_n^2}{1+\delta_n^2}\right)(AB - 2Ab + ab)\right\}.\end{aligned} \tag{14}$$

While integrating (6b) within the limits (0, 1) the following function has been used:

$$\begin{aligned}F_2(z) = Q_b\{&\tfrac{1}{2}\delta_n[a_n(s\lambda_n)F_n - G_n]z[J_1(s\beta z)H_0^{(1)}(s\beta z) + J_0(s\beta z)H_1^{(1)}(s\beta z)]\\ &+ \delta_n^2 a_n(s\lambda_n)G_nJ_0(s\beta z)H_0^{(1)}(s\beta z) - F_nz^2J_1(s\beta z)H_1^{(1)}(s\beta z)\\ &+ d_n[\delta_nG_nH_0^{(1)}(s\beta z) + F_nzH_1^{(1)}(s\beta z)][zJ_1(\beta z) - \delta_na_n(\lambda_n)J_0(\beta z)]\}\\ &+ Q_ad_n\,\{\tfrac{1}{2}\delta_nd_n[a_n(\lambda_n)f_n - g_n]z[J_1(\beta z)H_0^{(1)}(\beta z) + J_0(\beta z)H_1^{(1)}(\beta z)]\\ &+ d_n[\delta_n^2a_n(\lambda_n)g_nJ_0(\beta z)H_0^{(1)}(\beta z) - f_nz^2J_1(\beta z)H_1^{(1)}(\beta z)]\\ &+ [\delta_ng_nJ_0(\beta z) + f_nzJ_1(\beta z)]\,[zH_1^{(1)}(s\beta z) - \delta_na_n(s\lambda_n)H_0^{(1)}(s\beta z)]\},\end{aligned} \tag{15}$$

where $F_n = 1 + p_ba_n(s\lambda_n)$, $f_n = 1 + p_aa_n(\lambda_n)$, $G_n = q_b - a_n(s\lambda_n)$, $g_n = q_a - a_n(\lambda_n)$, so that $\mathrm{Re}\ F_2(x) = \psi_{1,n}(x)\,\psi_{2,n}(x)$. The following contour integral has been formulated:

$$\oint_C \frac{zF_2(z)\,\mathrm{d}z}{\sqrt{1-z^2}\,(z^4 - \delta_n^4)(z^2 + \delta_n^2)} = 0. \tag{16}$$

The integration contour is identical to the former one (cf., Fig. 2). The integrand has a first-order pole at $z = \delta_n$ and a second-order pole at $z = \mathrm{i}\delta_n$. The corresponding residues have been computed using the following functions:

$$\mathscr{F}_2^{(1)}(z) = \frac{zF_2(z)}{\sqrt{1-z^2}\,(z+\delta_n)(z^2+\delta_n^2)^2} \quad \text{at } z = \delta_n, \tag{17a}$$

$$\mathscr{F}_2^{(2)}(z) = \frac{zF_2(z)}{\sqrt{1-z^2}\,(z^2-\delta_n^2)(z+\mathrm{i}\delta_n)^2} \quad \text{at } z=\mathrm{i}\delta_n. \tag{17b}$$

Eq. (16) has been rewritten as

$$\begin{aligned}
\mathrm{Re}\,⨍_0^1 \frac{xF_2(x)\,\mathrm{d}x}{\sqrt{1-x^2}\,(x^4-\delta_n^4)\,(x^2+\delta_n^2)} &= \int_0^1 \frac{\psi_{1,n}(x)\psi_{2,n}(x)}{(x^4-\delta_n^4)(x^2+\delta_n^2)}\frac{x\,\mathrm{d}x}{\sqrt{1-x^2}} \\
&= \mathrm{Re}\left\{\pi\mathrm{i}\mathscr{F}_2^{(1)}(\delta_n) + \pi\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}z}\mathscr{F}_2^{(2)}(z)\bigg|_{z=\mathrm{i}\delta_n}\right\} \\
&\quad + \int_1^\infty \frac{x\;\mathrm{Im}\;F_2(x)\,\mathrm{d}x}{\sqrt{x^2-1}(x^4-\delta_n^4)\,(x^2+\delta_n^2)} + ⨍_\infty^0 \frac{y\;\mathrm{Re}\;F_2(\mathrm{i}y)\,\mathrm{d}y}{\sqrt{1+y^2}\,(y^4-\delta_n^4)^2}
\end{aligned} \tag{18}$$

and the last integral is equal to zero as integrated along imaginary axis since

$$\begin{aligned}
F_2(\mathrm{i}y) = 0 + \mathrm{i}\frac{2}{\pi}\big(&Q_b\{\tfrac{1}{2}\delta_n y[a_n(s\lambda_n)F_n - G_n]y[I_1(s\beta y)K_0(s\beta y) - I_0(s\beta y)K_1(s\beta y)] \\
&- \delta_n^2 a_n(s\lambda_n)G_n I_0(s\beta y)K_0(s\beta y) - F_n y^2 I_1(s\beta y)K_1(s\beta y) \\
&+ d_n[\delta_n G_n K_0(s\beta y) + F_n y K_1(s\beta y)]\,[yI_1(\beta y) + \delta_n a_n(\lambda_n)I_0(\beta z)]\} \\
&+ Q_a d_n\{\tfrac{1}{2}\delta_n d_n y\,[a_n(\lambda_n)f_n - g_n]y[I_1(\beta y)K_0(\beta y) - I_0(\beta y)K_1(\beta y)] \\
&- d_n[\delta_n^2 a_n(\lambda_n)g_n I_0(\beta y)K_0(\beta y) + f_n y^2 I_1(\beta y)K_1(\beta y)] \\
&-[\delta_n g_n I_0(\beta y) - f_n y I_1(\beta y)][yK_1(s\beta y) - \delta_n a_n(s\lambda_n)K_0(s\beta y)]\}\big),
\end{aligned} \tag{19}$$

which implies that $\mathrm{Re}\;F_2(\mathrm{i}y) = 0$. The following values have been obtained:

$$\mathrm{Im}\;F_2(\delta_n) = \frac{1}{\pi}\frac{\delta_n}{s\beta}\,\{Q_b[p_b a_n^2(s\lambda_n) + q_b] - sQ_a d_n^2\,[p_a a_n^2(\lambda_n) + q_a]\}, \tag{20a}$$

$$\begin{aligned}
\mathrm{Im}\;F_2(\mathrm{i}\delta_n) = \frac{2}{\pi}\,\delta_n^2\{&Q_b\,[A\,(B-b) + bu_b + \tfrac{1}{2}(v_b A - u_b B)] \\
&+ Q_a[b(a-A) - v_a A + \tfrac{1}{2}(v_a a - u_a b)]\},
\end{aligned} \tag{20b}$$

where $v_b = p_b a_n(s\lambda_n)I_1(s\lambda_n) - q_b I_0(s\lambda_n)$, $v_a = d_n\,[p_a a_n(\lambda_n)I_1(\lambda_n) - q_a I_0(\lambda_n)]$, $u_b = p_b a_n(s\lambda_n)K_1(s\lambda_n) + q_b K_0(s\lambda_n)$, $u_a = d_n[p_a a_n(\lambda_n)K_1(\lambda_n) + q_a K_0(\lambda_n)]$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z}F_2(z)\bigg|_{z=\mathrm{i}\delta_n} &= sQ_b[A(H+y_b) - G(B+v_b) + b(G-x_b) + h(-A+u_b) \\
&\quad + Q_a\,[-A(h+y_a) - sG(b+v_a) - b(g-x_a) + h\,(a-u_a)]
\end{aligned} \tag{21}$$

and $x_a = d_n[p_a a_n(\lambda_n)K_0(\lambda_n) + q_a K_1(\lambda_n)]$, $x_b = p_b a_n(s\lambda_n)K_0(s\lambda_n) + q_b K_1(s\lambda_n)$, $y_a = d_n\,[a_n(\lambda_n)I_0(\lambda_n) - q_a I_1(\lambda_n)]$, $y_b = p_b a_n(s\lambda_n)I_0(s\lambda_n) - q_b I_1(s\lambda_n)$.

The values of

$$\mathscr{F}_2^{(1)}(\delta_n) = \frac{F_2(\delta_n)}{8\delta_n^4\sqrt{1-\delta_n^2}}, \tag{22a}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathscr{F}_2^{(2)}(z)\bigg|_{z=\mathrm{i}\delta_n} = \frac{1}{8\delta_n^4\sqrt{1+\delta_n^2}}\left[\mathrm{i}\delta_n\frac{\mathrm{d}}{\mathrm{d}z}\,F_2(z)\bigg|_{z=\mathrm{i}\delta_n} - \left(1 + \frac{\delta_n^2}{1+\delta_n^2}\right)F_2(\mathrm{i}\delta_n)\right] \tag{22b}$$

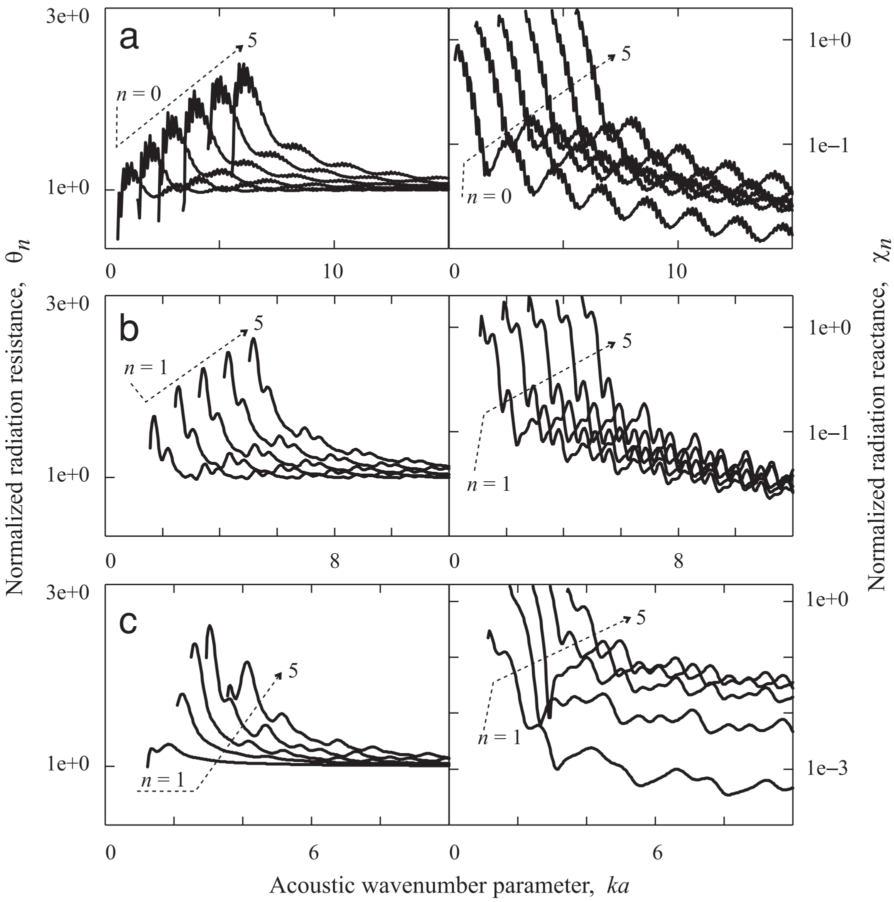

Fig. 3. Normalized acoustic radiation impedance $\zeta_n = \theta_n - \mathrm{i}\chi_n$ for $\mathbf{K} = (20, 30, 50, 40)$, and: (a) $s = 1.2$, (b) $s = 2.0$, (c) $s = 5.0$.

have been obtained from Eqs. (17). The contribution of residues at the poles to integral (18) is

$$\bar{\theta}_n^{(2)} = \mathrm{Re}\left\{\pi\mathrm{i}\mathscr{F}_2^{(1)}(\delta_n) + \pi\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z}\,\mathscr{F}_n^{(2)}(z)\bigg|_{z=\mathrm{i}\delta_n}\right\}$$

$$= -\frac{\pi}{8\delta_n^4}\left\{\frac{\mathrm{Im}\;F_2(\delta_n)}{\sqrt{1-\delta_n^2}} + \frac{1}{\sqrt{1+\delta_n^2}}\left[\delta_n\frac{\mathrm{d}}{\mathrm{d}z}\,F_2(z)\bigg|_{z=\mathrm{i}\delta_n} - \left(1+\frac{\delta_n^2}{1+\delta_n^2}\right)\mathrm{Im}\;F_2(\mathrm{i}\delta_n)\right]\right\}. \tag{23}$$

While computing integral (6c) within the limits $(0, 1)$ the following function has been used:

$$\begin{aligned} F_3(z) = {} & Q_b^2\,\{\delta_n^2 G_n^2 J_0(s\beta z)H_0^{(1)}(s\beta z) + F_n^2 z^2 J_1(s\beta z)H_1^{(1)}(s\beta z) \\ & + \delta_n G_n F_n z\,[J_0(s\beta z)H_1^{(1)}(s\beta z) + J_1(s\beta z)H_0^{(1)}(s\beta z)]\} \\ & + Q_a^2 d_n^2\,\{\delta_n^2 g_n^2 J_0(\beta z)H_0^{(1)}(\beta z) + f_n^2 z^2 J_1(\beta z)H_1^{(1)}(\beta z) \\ & + \delta_n g_n f_n z\,[J_0(\beta z)H_1^{(1)}(\beta z) + J_1(\beta z)H_0^{(1)}(\beta z)]\} \\ & + 2Q_a Q_b d_n\,\{\delta_n^2 g_n G_n H_0^{(1)}(s\beta z)J_0(\beta z) + d_n G_n f_n z H_0^{(1)}(s\beta z)J_1(\beta z) \\ & + \delta_n g_n F_n z H_1^{(1)}(s\beta z)J_0(\beta z) + F_n f_n z^2 H_1^{(1)}(s\beta z)J_1(\beta z)\} \end{aligned} \tag{24}$$

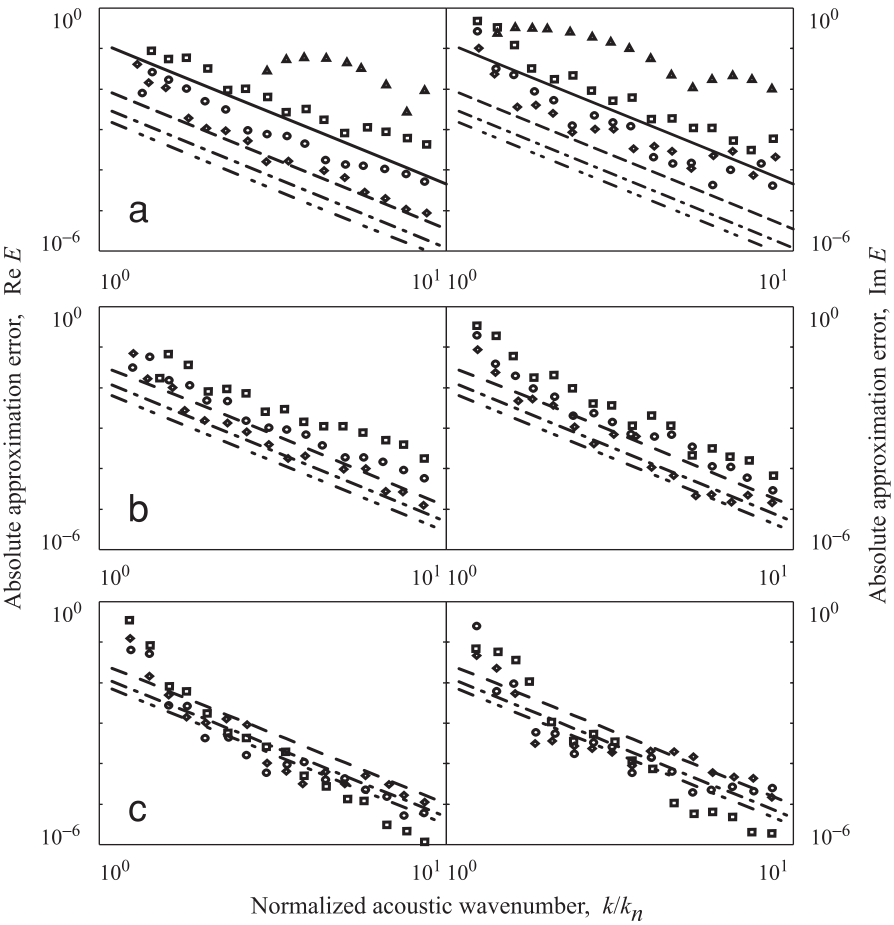

Fig. 4. Absolute approximation error for the radiation impedance $E = \text{Re } E - \text{i Im } E$ for $\mathbf{K} = (20, 30, 50, 40)$ and: (a) $s = 1.2$, (b) $s = 2.0$, (c) $s = 5.0$. Theoretical value has been plotted with lines and the estimated one with empty symbols. Key: ______ and △△△△, (0,0) mode; – – – and □□□□, (0,1) mode; ----- and ○○○○, (0,2) mode; ------ and ◇◇◇◇, (0,3) mode.

such that $\text{Re } F_3(x) = \psi^2_{2,n}(x)$. Integrating in

$$\oint_{C'} \frac{zF_3(z)\,\mathrm{d}z}{\sqrt{1-z^2}\,(z^2+\delta_n^2)^2} = 0 \tag{25}$$

has been performed along a closed contour C', similar to C (cf., Fig. 2). The integrand has a second-order pole at $z = \mathrm{i}\delta_n$ and no singularity at $z = \delta_n$. The residue has been computed using

$$\mathscr{F}_3(z) = \frac{zF_3(z)}{\sqrt{1-z^2}\,(z+\mathrm{i}\delta_n)^2} \quad \text{at } z = \mathrm{i}\delta_n. \tag{26}$$

Eq. (25) assumes the form of

$$\begin{aligned} \text{Re}\,⨍_0^1 \frac{xF_3(x)\,\mathrm{d}x}{\sqrt{1-x^2}\,(x^2+\delta_n^2)^2} &= \int_0^1 \frac{\psi^2_{2,n}(x)}{(x^2+\delta_n^2)^2}\,\frac{x\,\mathrm{d}x}{\sqrt{1-x^2}} \\ &= \text{Re}\left\{\pi\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z}\,\mathscr{F}_3(z)\bigg|_{z=\mathrm{i}\delta_n}\right\} \\ &\quad + \int_1^\infty \frac{x \text{ Im } F_3(x)\,\mathrm{d}x}{\sqrt{x^2-1}(x^2+\delta_n^2)^2} + ⨍_\infty^0 \frac{y \text{ Re } F_3(\mathrm{i}y)\,\mathrm{d}y}{\sqrt{1+y^2}\,(y^4-\delta_n^4)^2} \end{aligned} \tag{27}$$

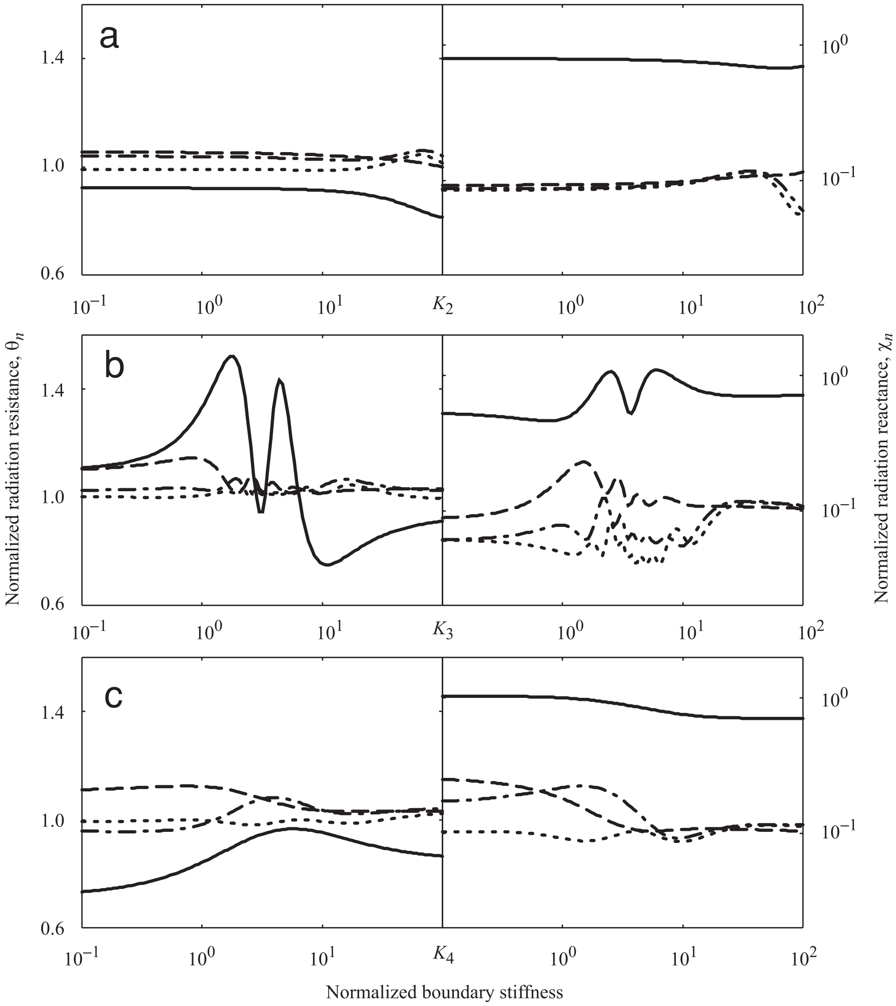

Fig. 5. Normalized acoustic radiation impedance $\zeta_n = \theta_n - i\chi_n$ for $s = 2.0$ and: (a) (0, 0) mode, $\mathbf{K} = (20, -, 50, 40)$; (b) (0, 2) mode, $\mathbf{K} = (20, 30, -, 40)$; (c) (0, 1) mode, $\mathbf{K} = (20, 30, 50, -)$. Key: ______, $k/k_n = 1.5$; – – –, $k/k_n = 3.0$; –·–·–, $k/k_n = 4.5$; . . . . , $k/k_n = 6.0$.

and the last integral is equal to zero as integrated along imaginary axis since

$$
\begin{aligned}
F_3(\mathrm{i}y) = 0 - \mathrm{i}\,\frac{2}{\pi}\,(Q_b^2\{\delta_n^2 G_n^2 I_0(s\beta y)K_0(s\beta y) - F_n^2 y^2 I_1(s\beta y)K_1(s\beta y) \\
+ \delta_n G_n F_n y\,[I_0(s\beta y)K_1(s\beta y) - I_1(s\beta y)K_0(s\beta y)]\} \\
+ Q_a^2 d_n^2\,\{\delta_n^2 g_n^2 I_0(\beta y)K_0(\beta y) - f_n^2 y^2 I_1(\beta y)K_1(\beta y) \\
+ \delta_n g_n f_n y\,[I_0(\beta y)K_1(\beta y) - I_1(\beta y)K_0(\beta y)]\} \\
+ 2Q_a Q_b d_n\,\{\delta_n^2 g_n G_n K_0(s\beta y)I_0(\beta y) - d_n G_n f_n y K_0(s\beta y)I_1(\beta y) \\
+ \delta_n g_n F_n y K_1(s\beta y)I_0(\beta y) - F_n f_n y^2 K_1(s\beta y)I_1(\beta y)\}),
\end{aligned} \tag{28}
$$

which implies that Re $F_3(\mathrm{i}y) = 0$. The following value has been obtained from Eq. (26)

$$
\mathrm{Im}\,\mathscr{F}_3(\mathrm{i}\delta_n) = \frac{\delta_n^2}{\pi}\,\{\tfrac{1}{2}(a - A)(B - b) + Q_a[(v_a + b)(A - a) + (u_a - a)(B - b)]\}, \tag{29}
$$

where $Q_b(u_b - A) = Q_a(u_a - a) + \frac{1}{2}(a - A)$ and $Q_b(v_b - A) = Q_a(v_a + b) + \frac{1}{2}(B - b)$.

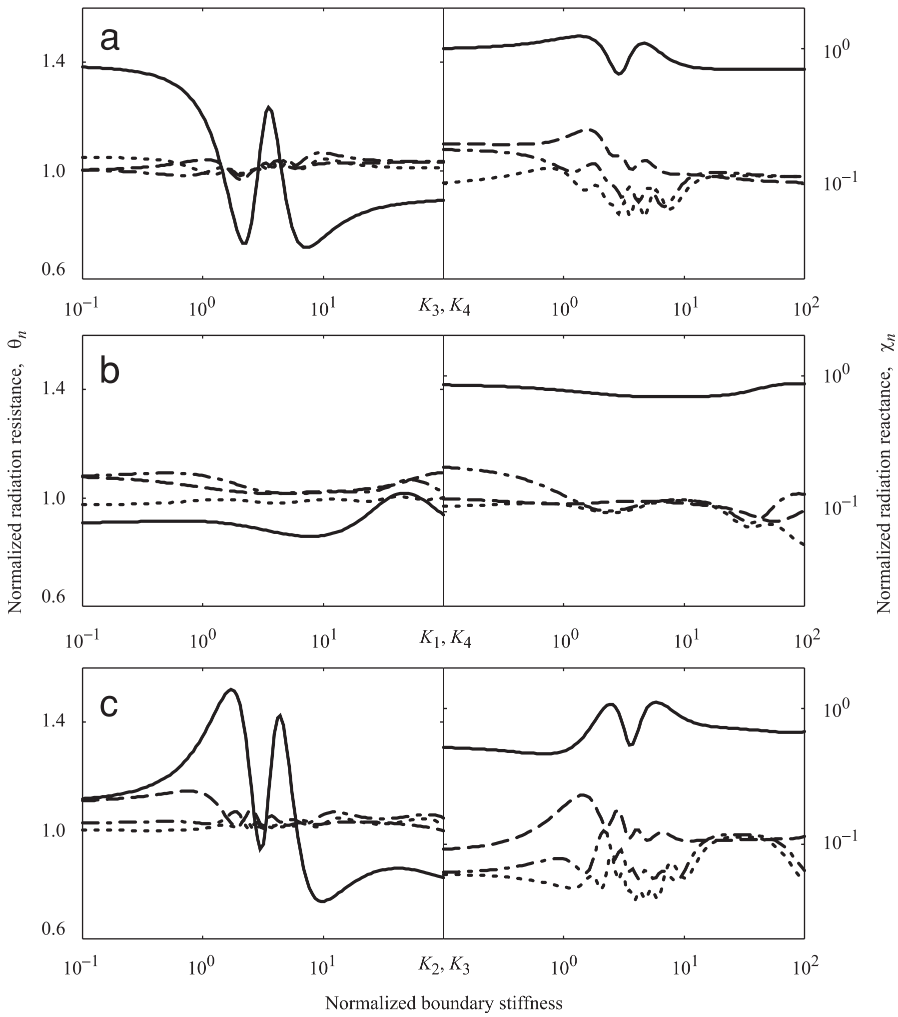

Fig. 6. Normalized acoustic radiation impedance $\zeta_n = \theta_n - \mathrm{i}\chi_n$ for $s = 2.0$ and: (a) (0, 3) mode, $\mathbf{K} = (20, 30, -, -)$; (b) (0, 2) mode, $\mathbf{K} = (-, 30, 50, -)$; (c) (0, 2) mode, $\mathbf{K} = (20, -, -, 40)$. Key: ______, $k/k_n = 1.5$; – – –, $k/k_n = 3.0$; –·–·–, $k/k_n = 4.5$; . . . . , $k/k_n = 6.0$.

The following values have been used

$$\frac{\pi}{2\beta\delta_n^2}\frac{\mathrm{d}}{\mathrm{d}z}\mathscr{F}_3(z)\bigg|_{z=\mathrm{i}\delta_n}$$
$$= sQ_b\{Q_b[-(BG+AH)+2(Bx_b-Ay_b)+v_bx_b+u_by_b]+(b-B)(x_b-G)\}$$
$$+Q_a\{Q_a[bg+ah+2(gv_a-hu_a)-(v_ax_a+u_ay_a)]+(A-a)(y_a+h)\}, \tag{30}$$

$$\mathrm{Im}\,[\tfrac{1}{4}F_1(\mathrm{i}\delta_n)+F_2(\mathrm{i}\delta_n)+F_3(\mathrm{i}\delta_n)]=0. \tag{31}$$

Summing up all the residues in Eqs. (10), (18), (27) provides an asymptotic formula for the normalized non-oscillating acoustic radiation resistance of an elastically supported annular plate

$$\bar{\theta}_n \simeq \frac{1}{\sqrt{1+\delta_n^2}}+\frac{q_n^2u_n}{2}\left(\frac{1}{\sqrt{1-\delta_n^2}}-\frac{1}{\sqrt{1+\delta_n^2}}\right) \tag{32}$$

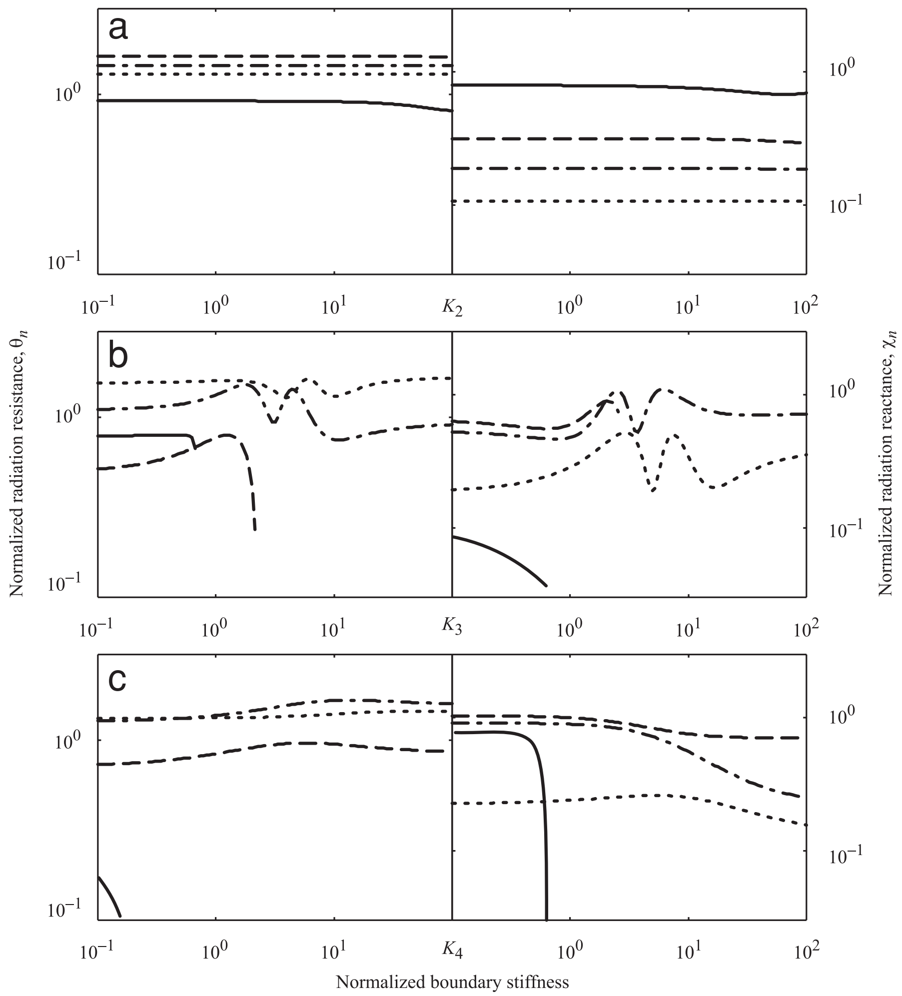

Fig. 7. Normalized acoustic radiation impedance $\zeta_n = \theta_n - i\chi_n$ for $\mathbf{K} = (20, 30, 50, 40)$, $s = 2.0$, $k/k_n = 1.5$. Key: ______, (0,0) mode; – – –, (0,1) mode; –·–·–, (0,2) mode; . . . . , (0,3) mode.

valid for axisymmetric acoustic waves, free of any oscillations, where $u_n = (1/2)[1 + a_n^2(s\lambda_n)] + (Q_b/s\lambda_n)[q_b + p_b a_n^2(s\lambda_n)] - d_n^2\{(1/2)[1 + a_n^2(\lambda_n)] + (Q_a/\lambda_n)[q_a + p_a a_n^2(\lambda_n)]\}$.

Integrating all the imaginary terms in Eqs. (10), (18), (27) has made it necessary to use some asymptotic expansion series in the same way as Levine and Leppington presented for a clamped circular plate in Ref. [7]. It is worth noticing that functions $F_m(z)$ in Eqs. (7), (19) and (24) have been chosen in such a way that the zero expansion term for radiation resistance is equal to zero and the residues from the poles at δ_n and $i\delta_n$ are the only contribution to $\bar{\theta}_n$. This expansion series has been improved by computing the oscillating contribution to the radiation resistance from Eqs. (10), (18) and (27) integrated within the limits $(1, \infty)$. The asymptotic stationary phase method has been used giving

$$\tilde{\theta}_n \simeq \frac{2q_n^2}{\beta\sqrt{\pi\beta}} \frac{\delta_n^4}{(1+\delta_n^2)^2} \left\{ (b_0^2 - b_1^2)\cos w_1 + 2b_0 b_1 \sin w_1 \right.$$
$$\left. + \frac{1}{s\sqrt{s}} [(h_0^2 - h_1^2)\cos w_2 + 2h_0 h_1 \sin w_2] \right.$$

$$+\frac{2\sqrt{2}}{\sqrt{s}\sqrt{s-1}}[(h_1b_0-h_0b_1)\cos w_3-(h_1b_1+h_0b_0)\sin w_3]$$
$$\left.+\frac{2\sqrt{2}}{\sqrt{s}\sqrt{s+1}}[(h_1b_1-h_0b_0)\cos w_4-(h_1b_0+h_0b_1)\sin w_4]\right\},\tag{33}$$

where $w_1=2\beta+\pi/4$, $w_2=2s\beta+\pi/4$, $w_3=(s-1)\beta+\pi/4$, $w_4=(s+1)\beta+\pi/4$,

$$h_0=\frac{1}{\delta_n}\left\{\frac{\delta_n}{1-\delta_n^2}+\frac{Q_b}{\delta_n}[1+p_ba_n(s\lambda_n)]\right\},\quad h_1=\frac{\delta_na_n(s\lambda_n)}{1-\delta_n^2}-\frac{Q_b}{\delta_n}[q_b-a_n\ (s\lambda_n)],$$
$$b_0=\frac{d_n}{\delta_n}\left\{\frac{\delta_n}{1-\delta_n^2}+\frac{Q_a}{\delta_n}[1+p_aa_n(\lambda_n)]\right\},\quad b_1=d_n\left\{\frac{\delta_na_n(\lambda_n)}{1-\delta_n^2}-\frac{Q_a}{\delta_n}[q_a-a_n(\lambda_n)]\right\}$$

because there were no poles within the integration limits [6,7]. Summing up Eqs. (32) and (33) gives an elementary asymptotic formula for the normalized radiation resistance

$$\theta_n=\bar{\theta}_n+\tilde{\theta}_n+\mathcal{O}(\delta_n^4\beta^{-3/2}),\tag{34}$$

where $\mathcal{O}(\cdot)$ is the approximation error. Using some corresponding asymptotic formulas, integrating within the limits $(1,\infty)$ in Eqs. 6, and summing up give a rough asymptotic formula for the normalized radiation reactance which does not contain any oscillations

$$\bar{\chi}_n=\frac{q_n^2}{\pi s\beta}\left[\frac{\alpha_{1n}}{1+\delta_n^2}+\frac{\alpha_{2n}\arcsin\delta_n}{2\delta_n(1-\delta_n^2)^{3/2}}+\frac{\alpha_{3n}\operatorname{arsh}\delta_n}{2\delta_n(1+\delta_n^2)^{3/2}}\right],\tag{35}$$

with the same denotations as given after Eq. (33) and

$$\begin{aligned}\alpha_{1n}=&(1-\delta_n^2)^{-1}\{1+\delta_n^2a_n^2(s\lambda_n)+sd_n^2[1+\delta_n^2a_n^2(\lambda_n)]\}\\&+2Q_b\{[1+p_ba_n(s\lambda_n)](Q_b[1+p_ba_n(s\lambda_n)]-1)\\&-[q_b-a_n(s\lambda_n)](Q_b[q_b-a_n(s\lambda_n)]+a_n(s\lambda_n))\}\\&+2sd_n^2Q_a\{[1+p_aa_n(\lambda_n)](Q_a[1+p_aa_n(\lambda_n)]-1)\\&-[q_a-a_n(\lambda_n)](Q_a[q_a-a_n(\lambda_n)]+a_n(\lambda_n))\},\end{aligned}\tag{36a}$$

$$\begin{aligned}\alpha_{2n}=&-(3-4\delta_n^2)[a_n^2(s\lambda_n)+sd_n^2a_n^2(\lambda_n)]-(1-2\delta_n^2)(1+sd_n^2)\\&+4(1-\delta_n^2)\{Q_b[1+a_n^2(s\lambda_n)+(p_b-q_b)a_n(s\lambda_n)]\\&+sd_n^2Q_a[1+a_n^2(\lambda_n)+(p_a-q_a)a_n(\lambda_n)]\},\end{aligned}\tag{36b}$$

$$\begin{aligned}\alpha_{3n}=&(3+4\delta_n^2)[a_n^2(s\lambda_n)+sd_n^2a_n^2(\lambda_n)]-(1+2\delta_n^2)(1+sd_n^2)\\&+4(2+3\delta_n^2)\{Q_ba_n(s\lambda_n)[q_b-a_n(s\lambda_n)]+sd_n^2Q_aa_n(\lambda_n)[q_a-a_n(\lambda_n)]\}\\&+4\delta_n^2\{Q_b[1+p_ba_n(s\lambda_n)]+sd_n^2Q_a[1+p_aa_n(\lambda_n)]\}\\&+4(1+2\delta_n^2)\{Q_b^2[q_b-a_n(s\lambda_n)]^2+sd_n^2Q_a^2[q_a-a_n(\lambda_n)]^2\}\\&+4\{Q_b^2[1+p_ba_n(s\lambda_n)]^2+sd_n^2Q_a^2[1+p_aa_n(\lambda_n)]^2\}.\end{aligned}\tag{36c}$$

Integrating some additional oscillating terms using the stationary phase method provides

$$\tilde{\chi}_n=\frac{2q_n^2}{\beta\sqrt{\pi\beta}}\frac{\delta_n^4}{(1+\delta_n^2)^2}\left\{(b_1^2-b_0^2)\sin w_1+2b_1b_0\cos w_1\right.$$
$$+\frac{1}{s\sqrt{s}}[(h_1^2-h_0^2)\sin w_2+2h_1h_0\cos w_2]$$

$$
\left. \begin{aligned}
& -\frac{2\sqrt{2}}{\sqrt{s}\sqrt{s-1}}\,[(h_1b_0-h_0b_1)\sin w_3+(h_1b_1+h_0b_0)\cos w_3] \\
& -\frac{2\sqrt{2}}{\sqrt{s}\sqrt{s+1}}\,[(h_1b_1-h_0b_0)\sin w_4+(h_1b_0+h_0b_1)\cos w_4]
\end{aligned} \right\}. \tag{37}
$$

Also, it considerably improves the computation accuracy giving an asymptotic formula for the normalized acoustic axisymmetric radiation reactance [6,7]

$$\chi_n = \bar{\chi}_n + \tilde{\chi}_n + \mathcal{O}(\delta_n^4\beta^{-3/2}), \tag{38}$$

with the same order of approximation error as in Eq. (34).

4. Numerical analysis

A number of curves have been plotted for the normalized acoustic radiation impedance of an elastically supported annular plate with fixed coordinates of vector **K** using Eqs. (34) and (38) and shown in Fig. 3. The lower values of the plate geometric parameter s imply a greater number of oscillations per ka unit (cf., Fig. 3(a)). The values of the normalized radiation resistance and reactance tend to unity and zero, respectively, for $k/k_n \geqslant 10$ and successive axisymmetric mode numbers n. However, they assume values much different from unity and zero for $k/k_n \in (1 \ldots 10)$ where they must be computed numerically using, e.g. the asymptotic formulas presented herein. It can be noted that the radiation impedance tends to its limiting values much faster for the zero axisymmetric mode than for the higher ones for $k/k_n > 2$.

The absolute approximation error in Eqs. (34) and (38) has been presented in Fig. 4. The numerical value of the absolute error has been estimated as

$$E = |\theta_I - \theta_A| - \mathrm{i}|\chi_I - \chi_A|, \tag{39}$$

where θ_I, χ_I have been computed using Eq. (3), and θ_A, χ_A have been computed using Eqs. (34) and (38). Fig. 4 shows that the estimated error value does not considerably exceed its theoretical measure $\delta_n^4\beta^{-3/2}$ within a wide range of wave parameter ka. This means that the approximation error can be well estimated by its theoretical value within the mentioned ka range.

The normalized radiation impedance has also been presented as a function of the normalized boundary stiffness in Figs. 5–7 using Eqs. (34) and (38). In Fig. 5, the boundary stiffness values **K** are fixed except for one of them varying within the range of $10^{-10} \ldots 10^{2}$. The biggest influence of the boundary stiffness values on the radiation impedance can be noticed for $K_3 \in (10^0 \ldots 10^1)$ (cf., Fig. 5(b)). A small influence can also be noticed for $K_2 > 20$ and for $K_4 < 20$.

In Fig. 6, two values of the boundary stiffness vary between 10^{-1} and 10^2 while the remaining two are fixed. A big influence of the change in both values on the radiation impedance appears within their range of $10^0 \ldots 10^1$ since one of them is K_3 (cf., Figs. 6(a) and (c)). The influence of a simultaneous change in the two boundary stiffness values K_2 and K_3 has been shown in Fig. 6(c). The curves in Figs. 5(b) and 6(c) are nearly identical with the exception of the range of $K_2, K_3 \geqslant 50$ where K_2 considerably influences the radiation impedance. The influence of K_1 and K_4 on the radiation impedance (Fig. 6(b)) is not much different than that of K_4 shown in Fig. 5(c).

The curves presented in Figs. 5 and 6 have been prepared for a fixed axisymmetric vibration mode $(0, n) \equiv n$ and $s = 2.0$, whereas the ones presented in Fig. 7 have been prepared for a fixed value of $k/k_n = 1.5$ and for some lower modenumbers $n = 0, 1, 2, 3$. It can be noticed that the values assumed by the normalized radiation resistance and reactance tend to zero for $n = 0$ and $K_4 > 0.5$ as well as for $n = 0, 1$, $K_3 > 5$, $k/k_n = 1.5$ and $s = 2.0$. A big influence of the boundary stiffness value K_3 on the radiation impedance occurs within the range of $K_3 \in (10^0 \ldots 10^1)$.

Fig. 8 presents the radiation resistance and reactance plotted as functions of s for a fixed modenumber $n = 1$ and for some sample values of k/k_n (cf., Fig. 8(a)) as well as for a fixed value of $k/k_n = 1.5$ parameter and for some sample values of modenumber n (cf., Fig. 8(b)). It is worth noticing that the radiation resistance and reactance values oscillate with a change in parameter s and that the oscillation amplitude decreases for higher

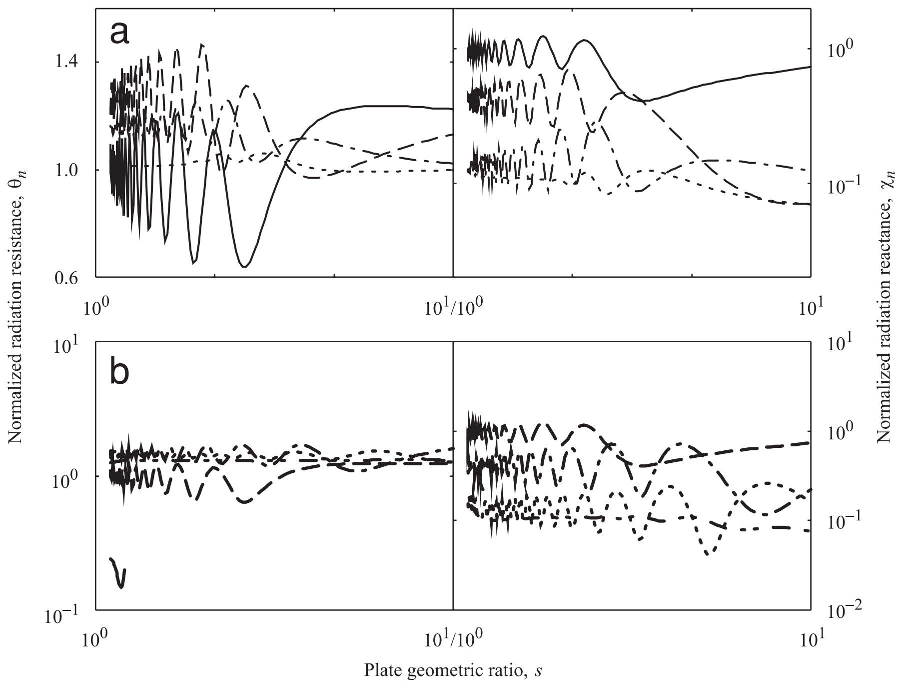

Fig. 8. Normalized acoustic radiation impedance $\zeta_n = \theta_n - i\chi_n$ for $\mathbf{K} = (20, 30, 50, 40)$. Key: ______ (a) (0, 1) mode, $k/k_1 = 1.5$, (b) (0, 0) mode, $k/k_1 = 1.5$, – – – (a) (0, 1) mode, $k/k_1 = 2.0$, (b) (0, 1) mode, $k/k_1 = 1.5$, –·–·– (a) (0, 1) mode, $k/k_1 = 2.5$, (b) (0, 2) mode, $k/k_1 = 1.5$, ... (a) (0, 1) mode, $k/k_1 = 3.0$, (b) (0, 3) mode, $k/k_1 = 1.5$.

modenumbers. Moreover, it can be noted that the radiation impedance values tend to zero for the fixed value of $k/k_n = 1.5$ and for $s > 1.5$ for $n = 0$ (cf., solid lines in Fig. 8(b)).

5. Concluding remarks

The integral formulas for the normalized acoustic radiation impedance of an axisymmetric mode of an annular plate in its Hankel form have been analyzed theoretically. The residues at poles have been computed and the corresponding integrals within infinite limits have been computed analytically using the stationary phase method. As a result, asymptotic formulas have been obtained, valid for $k/k_n > 1$. The presented formulas have been expressed in their elementary form. The estimated approximation error value does not considerably exceed its theoretical measure for high frequencies. The asymptotic axisymmetric formulas presented herein are valid for any homogeneous axisymmetric boundary configurations of a thin annular plate.

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