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The minimum support stiffness required to raise the fundamental natural frequency of plate structures

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Abstract

The minimum stiffness and optimum position of a flexible point support is calculated that raises the fundamental natural frequency of plate structures. For a single support the maximum fundamental natural frequency of the supported structure is equal to the second natural frequency of the unsupported structure. The structure is modelled using finite element analysis allowing a wide range of applications and boundary conditions. If a support is positioned within a finite element then the shape functions are used to calculate the contribution of the support to the system stiffness and also the slope of the mode at the support. Efficient methods are used to calculate the minimum support stiffness required. Numerical results demonstrate that a system with flexible supports may be designed with the same fundamental natural frequency as that of a system with rigid supports. Examples of plate structures with one or two point supports are analysed and show that the method is efficient in these cases.

1. Introduction

The free vibration of thin plate structures with point supports has attracted significant attention from researchers since the addition of point supports has numerous potential industrial applications. Thus far, a vast literature exists for the analysis of the natural frequencies and mode shapes. Because the exact solution of the transverse vibration is not readily available for a rectangular plate with point supports, a variety of numerical approaches have been adopted to obtain the natural frequencies and mode shapes, for example by using the Rayleigh–Ritz method [1–6], the superposition method [7,8], the finite strip method [9,10], or by finite element analysis [11,12]. Narita [1] demonstrated that the position of a point constraint has a considerable impact on the dynamic performance of a cantilever plate. It was observed that both the natural frequencies and the mode shapes of the plate can be significantly affected by the position of a rigid simple support, modelled as a point support with infinite stiffness. However, no support is absolutely rigid, and the zero deflection of the supported point cannot be achieved physically.

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A survey of the early literature reveals that the effect of the support stiffness on the flexural vibration of a rectangular plate has received little attention, although the natural frequencies of plates on elastic supports have been studied [2,10]. The lowest natural frequency of a rectangular plate may be increased to between the first and the second natural frequencies of the original system with a simple support [13]. Won and Park [11] showed that there exists a certain minimum support stiffness that increases the first natural frequency of the supported structure to the second natural frequency of the unsupported structure, provided that it is positioned appropriately. Increasing the support stiffness above this minimum value does not raise the lowest natural frequency any further. Thus an elastic support with a finite stiffness can achieve similar results to a rigid support when it is located optimally, and this has great potential for practical designs. More recently, Wang et al. [14] computed the minimum stiffness and the corresponding optimal position of an intermediate support for a uniform beam.

A rectangular plate with one boundary edge conventionally supported (either clamped or simply supported) and others free has been an important analytical model for structural problems. When designing a support, such as a column of a slab in civil engineering or for a circuit board in electrical engineering, to increase the natural frequencies of a structure, the minimum stiffness and the optimal position of the support are of most interest. In this paper, the minimum stiffness and the optimal position are calculated numerically for one or two elastic supports lying along the free edge opposite to the restrained boundary edge of the plate. For the rectangular plates with simply supported or clamped edges, a single support along the axis of symmetry is also designed. Finite element analysis is used to model the plate and the minimum stiffness for a fixed location is obtained as a solution to an eigenvalue problem. The optimal support position may be obtained where the gradient of the fundamental mode shape at the support location is zero. The use of the finite element analysis approach allows the design of a range of supports for general plate structures, and this is demonstrated on a cantilever plate with a slot. Often a support will be located within an element, and in this case the stiffness matrix of the support is obtained using the shape functions. This approach has been used previously for model updating [15] and support optimization [12]. Damping is neglected in this study; for lightly damped structures damping has a negligible effect on the resonance frequencies.

2. Modelling the plate and supports

Assuming that a thin plate lying in the x–y plane undergoes free vibration, there are a number of approaches to estimate the natural frequencies and mode shapes of the structure. In the Rayleigh–Ritz approach the transverse deflection, $w(x, y)$, may be approximated by a general series solution of the form

$$w(x, y) = \sum_{n=1}^{N} q_n \phi_n(x, y), \tag{1}$$

where the admissible functions $\phi_n(x, y)$ are usually chosen to satisfy the geometric boundary conditions. The equations of motion of the structure are obtained in terms of the unknown generalized coordinates and the natural frequencies and mode shapes are calculated from the associated eigenvalue problem. Bhat [16], Mundkur and Bhat [17] suggested this form of displacement model for the whole plate, although for complex geometries suitable admissible functions are difficult to determine. Furthermore, the accuracy and convergence of the solutions depend critically on the choice of admissible functions. The finite element method [18] may also be written in this form where each admissible function is only defined over a single element. This approach is considered in more detail in Section 5. However, the use of admissible functions is convenient in this paper since the position and slopes at the support positions may be expressed easily.

The support is modelled as a single translational spring, neglecting inertia and damping effects. Since the in-plane stiffness of the plate is relatively large, including in-plane springs in the support model will have little effect on the lower modes of the structure. The spring model of the support assumes that the rotational stiffness is small, and the connection to the plate is essentially pinned. Often this is a good approximation, although the addition of rotational springs is considered later. Suppose a pinned support of translational stiffness k_s is located at (x_s, y_s). The term *simple* is used here to refer to a point support that prevents lateral displacement but offers no resistance to the rotation of the plate about any axis. Then the potential energy of the support spring is given by

$$U = \frac{1}{2}k_s w(x_s, y_s)^2 = \frac{1}{2}k_s\left(\sum_{n=1}^{N} q_n\phi_n(x_s, y_s)\right)^2 = \frac{1}{2}k_s\mathbf{q}^{\mathrm{T}}\mathbf{K}_s\mathbf{q}, \tag{2}$$

where $\{\mathbf{q}\}_n = q_n$. The (m,n)th element of the stiffness matrix corresponding to the support, $\mathbf{K}_s$, is

$$[\mathbf{K}_s]_{mn} = \phi_m(x_s, y_s)\phi_n(x_s, y_s), \tag{3}$$

and this stiffness matrix may be written as

$$\mathbf{K}_s = \mathbf{v}\mathbf{v}^{\mathrm{T}}, \quad \text{where } \{\mathbf{v}\}_n = \phi_n(x_s, y_s). \tag{4}$$

Note that $\mathbf{K}_s = \mathbf{K}_s(x_s, y_s)$, that is the equivalent stiffness matrix for the support is a function of the support location. Furthermore, the stiffness matrix $\mathbf{K}_s$ is a dyadic (from Eq. (4)) and hence has rank one.

The eigenvalue problem for the plate supported by a single spring is given by

$$\left[\mathbf{K}_p - \omega^2\mathbf{M}_p + k_s\mathbf{K}_s(x_s, y_s)\right]\mathbf{q} = 0, \tag{5}$$

where $\mathbf{q}$ now represents the mode shape and ω is the natural frequency. For multiple supports the stiffness matrices for each support are simply added together. If the support stiffness is assumed to be identical for all supports then the eigenvalue problem is of the form given in Eq. (5) except that the rank of $\mathbf{K}_s$ is now equal to the number of supports.

The stiffness matrix of a rotational spring has a similar form. For example, the stiffness matrix of a rotational spring about the y-axis is given by

$$\mathbf{K}_s = \mathbf{v}\mathbf{v}^{\mathrm{T}}, \quad \text{where } \{\mathbf{v}\}_n = \frac{\partial\phi_n}{\partial x}(x_s, y_s). \tag{6}$$

Only the supports modelled using translational springs will be considered further in this paper. However the extension to rotational springs is straight-forward based on the definition of the stiffness matrix given in Eq. (6).

3. Requirements for an optimal support position

When a support is optimally located, it is well known that a natural frequency can be raised to a certain value with a minimum stiffness [11,12]. To determine the optimal position of a flexible support, the sensitivity of the natural frequency to the position of the support must be estimated. This sensitivity information allows both the search direction and the optimal position of the support to be determined. The derivative of a natural frequency of the plate with respect to a support position has been developed by the discrete method [12] and the ith natural frequency of the plate is given by

$$\left.\frac{\partial\omega_i^2}{\partial x}\right|_{\substack{x=x_s\\y=y_s}} = -2k_s w_i(x_s, y_s)\theta_{yi}(x_s, y_s), \tag{7a}$$

$$\left.\frac{\partial\omega_i^2}{\partial y}\right|_{\substack{x=x_s\\y=y_s}} = 2k_s w_i(x_s, y_s)\theta_{xi}(x_s, y_s), \tag{7b}$$

where $w_i(x_s, y_s)$ is the transverse deflection of the associated *mass normalized* vibration mode, and $\theta_{xi}(x_s, y_s)$ and $\theta_{yi}(x_s, y_s)$ are the rotations (or slopes) about the x and y axes respectively, at the support position. From Eq. (7), to increase the first natural frequency ($i = 1$), the optimal position of an additional support must be located where the first mode shape of the supported structure either has a zero displacement or has zero slopes at the support. Note that the zero displacement criterion is of little use in the current application since this requires a rigid support or the placement of the support at a node of the mode. An alternative interpretation is that if the structure with an elastic support has a first mode with zero slopes (or gradient) at the supported point, then the support stiffness is the minimum required to raise the fundamental natural frequency of the plate to the value attained [14].

The conditions for the minimum support stiffness may be applied to the eigenvalue solutions obtained from Eq. (5) as

$$\theta_{yi}(x_s, y_s) = \left.\frac{\partial w}{\partial x}\right|_{\substack{x=x_s\\y=y_s}} = \sum_{n=1}^{N} q_n \left.\frac{\partial \phi_n}{\partial x}\right|_{\substack{x=x_s\\y=y_s}} = 0, \tag{8a}$$

$$\theta_{xi}(x_s, y_s) = \left.\frac{\partial w}{\partial y}\right|_{\substack{x=x_s\\y=y_s}} = \sum_{n=1}^{N} q_n \left.\frac{\partial \phi_n}{\partial y}\right|_{\substack{x=x_s\\y=y_s}} = 0. \tag{8b}$$

4. Approaches to support optimization

The equations to estimate the natural frequencies and mode shapes for the supported plate have been developed in Eq. (5), and conditions for the optimum support location established in Eq. (8). For each support the quantities to be determined are the support location and stiffness. If the support location were known then the support stiffness may be obtained using Eq. (5). The standard approach is to set up an optimization problem to calculate the stiffness required to raise the fundamental natural frequency to the required value. Won and Park [11] set $\omega = \omega_i$ in Eq. (5) and found the support stiffness using the determinants as a solution of

$$\det\left[\mathbf{K}_p - \omega_i^2 \mathbf{M}_p + k_s \mathbf{K}_s(x_s, y_s)\right] = 0. \tag{9}$$

A small or zero determinant is a very poor test for singularity since there is little correlation between the magnitude of the determinant and the condition of the corresponding set of equations [19]. The optimum support stiffness calculated using the zero determinant as an objective function will often be inaccurate. Here we propose to solve an eigenvalue problem directly. Suppose we want to raise the first natural frequency of the supported plate to the second natural frequency of the unsupported plate. Then Eq. (5) becomes

$$\left[\mathbf{K}_p - \omega_2^2 \mathbf{M}_p + k_s \mathbf{K}_s(x_s, y_s)\right]\mathbf{q} = 0. \tag{10}$$

We can regard Eq. (10) as an eigenvalue problem for k_s, since the dynamic stiffness matrix of the plate, $\mathbf{D}_p = \mathbf{K}_p - \omega_2^2 \mathbf{M}_p$, and the support stiffness matrix, $\mathbf{K}_s$, are both known. The lowest non-zero eigenvalue gives the minimum stiffness and the corresponding eigenvector, $\mathbf{q}$, gives the fundamental mode shape that may be used to compute the slope at the supports. Note that the rank of the support stiffness matrix, $\mathbf{K}_s$, is equal to the number of supports and hence there will be a significant number of infinite eigenvalues. Also the rank of the dynamic stiffness matrix of the plate, $\mathbf{D}_p$, will be one less than the number of degrees of freedom (dof) and hence there will usually be one zero eigenvalue. Sometimes there is no value of support stiffness that will raise the fundamental natural frequency to ω_2 and in this case there will be no eigenvalues that are positive, non-zero and finite.

Won and Park [11] used matrix identities to transform the eigenvalue problem of Eq. (9) to an equivalent problem of much smaller dimension. Suppose that the support stiffness matrix has rank M. Then $\mathbf{K}_s = \mathbf{P}\mathbf{P}^{\mathrm{T}}$ where $\mathbf{P}$ has dimensions (N, M). Suppose that $\mathbf{D}_p = \mathbf{K}_p - \omega_d^2 \mathbf{M}_p$ is non-singular, where ω_d is the desired fundamental natural frequency. Then Eq. (9) becomes [20]

$$\det\left[\mathbf{I}_M + k_s \mathbf{P}^{\mathrm{T}} \mathbf{D}_p^{-1} \mathbf{P}\right] = 0, \tag{11}$$

where $\mathbf{I}_M$ is the $M \times M$ identity matrix. Notice that the determinant of a much smaller matrix is required. In the case of a single support, so that $\mathbf{P} = \mathbf{v}$, where $\mathbf{v}$ is the vector defined in Eq. (4), then the support stiffness required is immediately obtained as

$$k_s = -\frac{1}{\mathbf{v}^{\mathrm{T}} \mathbf{D}_p^{-1} \mathbf{v}}. \tag{12}$$

The major difficulty with Eq. (11) is the requirement that $\mathbf{D}_p$ is non-singular. If the objective is to raise the fundamental natural frequency of the supported structure to the second natural frequency of the unsupported structure, then $\mathbf{D}_p$ will be singular. Won and Park [11] solved this problem using a transformation based on the modes of the unsupported structure, where the mode corresponding to the desired fundamental natural frequency is not used. However this only works if the supports are located on the nodal line (line of zero deflection) of the neglected mode. Often this is a good solution, although for general plate structures calculating this nodal line is difficult. An alternative is to apply a shift, k_{s0}, to the support stiffness, so that Eq. (11) becomes

$$\det\left[\mathbf{I}_M + (k_s - k_{s0})\mathbf{P}^{\mathrm{T}}\left[\mathbf{D}_p + k_{s0}\mathbf{K}_s\right]^{-1}\mathbf{P}\right] = 0. \tag{13}$$

k_{s0} should be chosen so that the inversion of $\mathbf{D}_p + k_{s0}\mathbf{K}_s$ is well conditioned and $k_{s0} < k_s$.

To optimize the support location a numerical procedure may be used based on the location alone. Since the minimum support stiffness at a given location may be obtained using the above procedure, the optimum location is calculated by minimising support stiffness directly. Alternatively, the support location that gives zero slopes at the supports of the fundamental mode shape may be estimated using Eq. (8). These slopes may be combined to find the zeros of the single objective function

$$J(x_s, y_s) = \left(\sum_{n=1}^{N} q_n \left.\frac{\partial \phi_n}{\partial x}\right|_{\substack{x=x_s \\ y=y_s}}\right)^2 + \left(\sum_{n=1}^{N} q_n \left.\frac{\partial \phi_n}{\partial y}\right|_{\substack{x=x_s \\ y=y_s}}\right)^2, \tag{14}$$

where $\mathbf{q}$ is the normalized mode shape and is a function of the support location. This mode shape is the eigenvector corresponding to the optimum support stiffness from Eq. (10). Note that the mode shape slopes are proportional to the natural frequency gradients through Eq. (7) and give the direction to the optimum location.

5. Finite element models

The finite element method may also be described using admissible functions, and this section highlights some of the implementation details required. In the finite element method the displacement within each element is approximated using shape functions. This will be demonstrated using rectangular plate elements, but other formulations could be used. The plate is modelled using rectangular finite elements, with dimensions (a, b) and four nodes, shown in Fig. 1. The ith node has three dof, namely out of plane displacement, w_i, and two rotations about the x and y axes, θ_{xi} and θ_{yi}. The elements and shape functions used here are given by Dawe [18] in Section 11.3. If the coordinates of the centre of the element are (x_e, y_e), then the transverse displacement at location (x, y) within the element, given by $w(x, y)$, is approximated using the shape functions $N_j(\xi, \eta)$ as

$$w(x,y) = [N_1(\xi,\eta) \quad N_2(\xi,\eta)\ldots N_{12}(\xi,\eta)]\begin{Bmatrix} w_1 \\ \theta_{x1} \\ \theta_{y1} \\ w_2 \\ \vdots \\ \theta_{y4} \end{Bmatrix}, \tag{15}$$

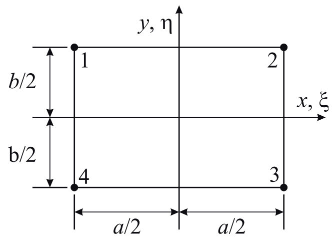

Fig. 1. The rectangular plate element. The numbers correspond to nodes.

where

$$\xi = \frac{x - x_e}{a}, \quad \eta = \frac{y - y_e}{b}.$$

Dawe [18] gives the shape functions explicitly.

The mass and stiffness matrices for the plate, $\mathbf{M}_p$ and $\mathbf{K}_p$, are computed using the standard expressions for the kinetic and potential energy [18]. The kinetic and potential energy for each element is estimated, based on the approximation given in Eq. (15). This gives the element mass and stiffness matrices that are then assembled into the global matrices.

Consider now the addition of a support within an element. Once the element containing the support has been identified the displacement of the support is approximated as $w(x_s, y_s)$ using Eq. (15) with,

$$\xi = \frac{x_s - x_e}{a}, \quad \eta = \frac{y_s - y_e}{b}.$$

The elements of the vector $\mathbf{v}$ in Eq. (4) then correspond the shape function terms in Eq. (15) placed at the global dof corresponding to the element dof. Note that only the stiffness sub-matrix corresponding to the dof for the nodes defining the finite element where the support is located are non-zero. The slopes defined in Eq. (8) are calculated using $\partial \mathbf{v}/\partial x$ and $\partial \mathbf{v}/\partial y$, whose elements correspond to the derivatives of the shape functions, for example $(\partial N_j/\partial x) = (1/a)(\partial N_j/\partial \xi)$.

6. Numerical examples

The examples will consider the rectangular plate, shown schematically for a single support in Fig. 2, with length L and width W. The final example considers a slotted plate. The boundary at $x = 0$ is either clamped or simply supported and all other edges are free. Most of the results will be given in terms of the non-dimensional natural frequency $\lambda = \omega L^2 \sqrt{\rho h/D}$ and the non-dimensional support stiffness $\gamma = k_s L^2/D$, where ρ is the mass density and h is the uniform thickness of the plate. $D = Eh^3/12(1 - v^2)$ is the constant flexural rigidity of the plate, where E is the Young's modulus and v is the Poisson's ratio of the material. The finite element method is used to model the plates based on the element formulation given in the previous section. The computation was performed using specially written code developed in MATLAB.

A square plate and a rectangular plate with an aspect ratio (defined as L/W) of 1.5 are used, and clamped and simply supported boundary conditions are modelled. The plate thickness is 3.28 mm, the Young's modulus is 73.1 GN/m^2, the mass density is 2821 kg/m^3 and Poisson's ratio is 0.3 [12]. Table 1 gives the dimensions of the plates, the number of elements for the different case studies and the natural frequencies (both dimensional and non-dimensional) for the unsupported plates.

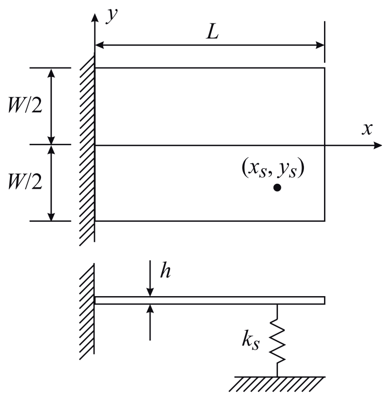

Fig. 2. A uniform rectangular plate with one edge clamped or simply supported with an elastic support.

Table 1
The dimensions and natural frequencies for the unsupported rectangular plates

	Clamped		Simply supported	
Aspect ratio	1.0	1.5	1.0	1.5
Length, L (m)	0.305	0.305	0.305	0.305
Width, W (m)	0.305	0.203	0.305	0.203
Number elements	10×10	15×10	10×10	15×10
Natural frequency (Hz)				
1 (1B)	30.005	29.854	0	0
2 (1T)	73.555	100.772	57.448	85.115
3 (2B)	184.394	185.761	128.987	128.794
Non-dimensional natural frequency				
1 (1B)	3.4710	3.4535	0	0
2 (1T)	8.5088	11.6573	6.6457	9.8461
3 (2B)	21.3307	21.4889	14.9213	14.8989

1B and 2B represent the first and second bending modes, and 1T the first torsional mode.

6.1. One elastic support on the free edge

An elastic point support along the free edge opposite to the clamped or simply supported edge can be used to increase the fundamental frequency of the rectangular plate. Because of the symmetry, the requirement of the zero slope of the fundamental mode shape at the support in the y direction is readily satisfied by locating the support at the midpoint of the free edge. Because the support is located on the nodal line of the second mode, the first natural frequency (corresponding to the first bending mode) can only be increased to the second natural frequency (corresponding to the first torsional mode) of the unsupported structure [13]. Increasing the support stiffness above the minimum value cannot raise the fundamental natural frequency of the supported plate due to switching of the mode shape order [11].

Table 2 lists the minimum non-dimensional stiffness required for a rectangular plate with aspect ratios of 1 (a square plate) and 1.5 and the corresponding non-dimensional natural frequencies. As expected, the minimum support stiffness depends upon the aspect ratio and the constrained edge conditions. Additionally, it is found that the minimum stiffness for a square plate with the simply supported edge is greater than that with the edge clamped. This is because the required change in natural frequency is larger for the simply supported case. Fig. 3 shows the corresponding mode shapes for the simply supported and clamped edges with optimal support stiffness. By using an elastic support, the fundamental natural frequency is increased to its upper limit and becomes equal to the torsional mode giving a repeated natural frequency. Note that the mode shapes corresponding to the repeated natural frequency form a space of dimension 2, and here we have chosen a basis for this space of mode shapes consisting of a bending mode and a torsional mode.

In this case there is no solution for the simply supported case with aspect ratio 1.5. Even a rigid support placed at the centre of the free end of the plate only raises the fundamental natural frequency to 78.3 Hz rather than the 85.1 Hz required. This highlights that the maximum increase in the fundamental natural frequency attainable by adding a support at the free edge is limited. Of course this support location is not optimum for this structure and this is highlighted by the non-zero slope of the mode at the support location, clearly shown in Fig. 3 for the square plate. The optimum support location for this structure is found in Section 6.3.

6.2. Two elastic supports on free edge

This example assumes there are two elastic supports with the same stiffness located symmetrically along the free edge of the plate. The objective is to increase the fundamental natural frequency of the supported plate to the second natural frequency of the unsupported plate. Fig. 4 shows the variation of the minimum support stiffness as a function of the distance of the support from the centre for the square plate, for the clamped and simply supported boundary edges. Clearly the optimum position is not at the centre of the

Table 2
The minimum support stiffnesses and corresponding natural frequencies for a rectangular plate with a support at the center of the free edge

	Clamped		Simply supported
Aspect ratio	1.0	1.5	1.0
Non-dimensional natural frequency			
1 (1B)	8.5088	11.6573	6.6457
2 (1T)	8.5088	11.6573	6.6457
3 (2B)	23.7338	27.6186	16.1827
Non-dimensional minimum stiffness, γ	23.9606	47.8070	35.7646

1B and 2B represent the first and second bending modes, and 1T the first torsional mode.

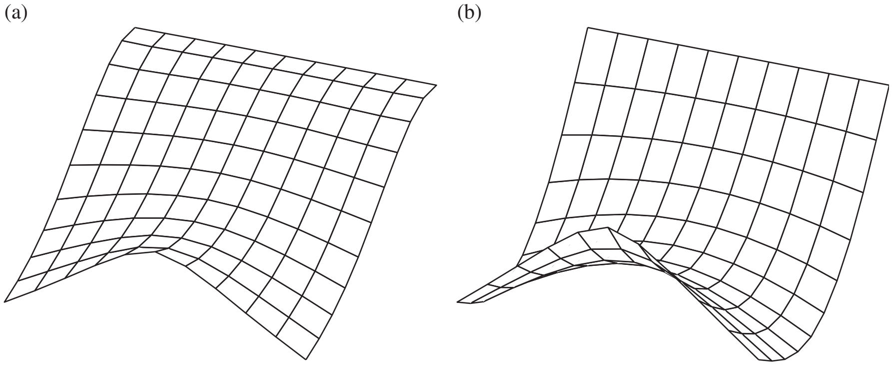

Fig. 3. The first mode shape of the square plate with a single minimum stiffness elastic support: (a) the rear edge clamped; and (b) the rear edge simply supported.

plate and Table 3 gives the optimum support location and the minimum non-dimensional stiffness. Fig. 4 also shows the slope of the fundamental mode shape at the support in the y-direction as the support location is varied and clearly shows that the slope is zero at the optimum location. Note that because the slopes are computed using the finite element shape functions the slopes will be continuous at the nodes, but the curvatures (second derivatives of displacement) will not be continuous. Fig. 5 shows the fundamental mode shapes corresponding to the optimum support location and stiffness and Table 3 gives the natural frequencies. There is no solution for the simply supported case with aspect ratio 1.5, for the reasons discussed in Section 6.1.

6.3. *One elastic support on the line of symmetry*

It has already been highlighted that rigid supports at the free edge of a rectangular plate with aspect ratio 1.5 and a simply supported edge at $x = 0$ are not able to raise the fundamental natural frequency to the second natural frequency of the unsupported plate. Given that the plate is symmetric and that the second mode of the unsupported plate is a torsional mode, the nodal line of this mode is the axis of symmetry, $y = 0$. Suppose that a single flexible support is placed along the axis $y = 0$. Fig. 6 shows the optimum support stiffness as the position of the support changes. It is clear that for locations less than 0.54 or greater than 0.96 even a rigid support is unable to raise the fundamental natural frequency sufficiently. There is also an optimum support position close to 0.79. Fig. 6 also shows the slope of the fundamental mode shape in the x direction at the support and clearly shows that the slope is zero at the optimum location. Fig. 6 also shows the results for the cantilever plate with aspect ratio 1.5 and shows that the minimum stiffness occurs at a support location of approximately 0.90. Table 4 gives the natural frequencies and support location and stiffness for the optimum solutions. For completeness Table 4 also shows the results for a square plate (aspect ratio 1).

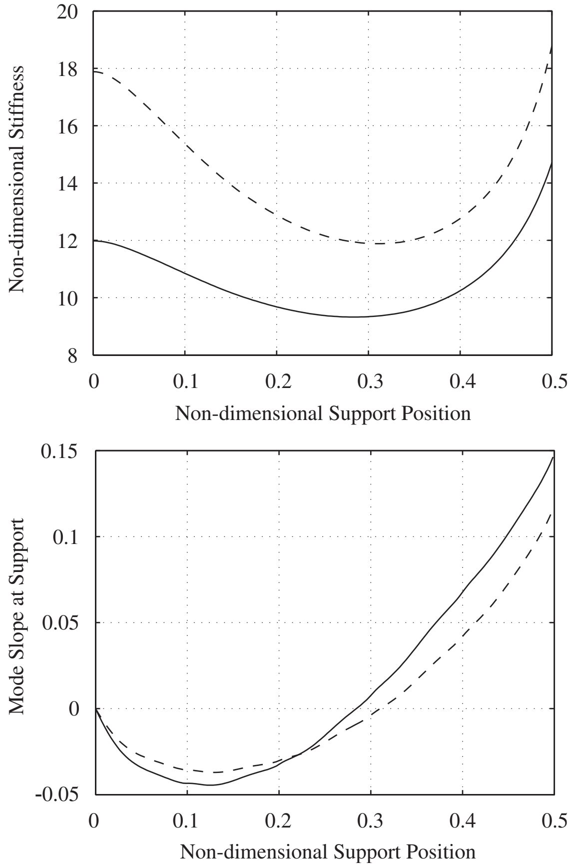

Fig. 4. The minimum support stiffness and mode shape slope in the y direction at different positions on the free edge for the square plate for the clamped (solid) and simply supported (dashed) boundary.

Table 3
The optimum supports and corresponding natural frequencies for a rectangular plate with two symmetric supports on the free edge

	Clamped		Simply supported
Aspect ratio	1.0	1.5	1.0
Non-dimensional Natural frequency			
1 (1B)	8.5088	11.6573	6.6457
2 (1T)	10.9957	16.4054	9.7728
3 (2B)	23.0004	26.9371	18.021
Non-dimensional minimum stiffness, γ ($\times$ 2)	9.3262	18.2840	11.8846
Optimum non-dimensional position (from centre)	$\pm$0.284	$\pm$0.316	$\pm$0.310

1B and 2B represent the first and second bending modes, and 1T the first torsional mode.

6.4. A slotted plate

The final example is a slotted cantilever plate to demonstrate the method on a more general structure without any lines of symmetry. Estimating the nodal lines of the mode shapes is now difficult. The plate is square with the same dimensions as that given in Table 1, and modelled with 100 square finite elements. Three elements are removed to create a slot, as shown in Fig. 7, which gives the first two mode shapes of the structure. The modes of the unsupported plate are no longer purely bending or purely torsional, and the

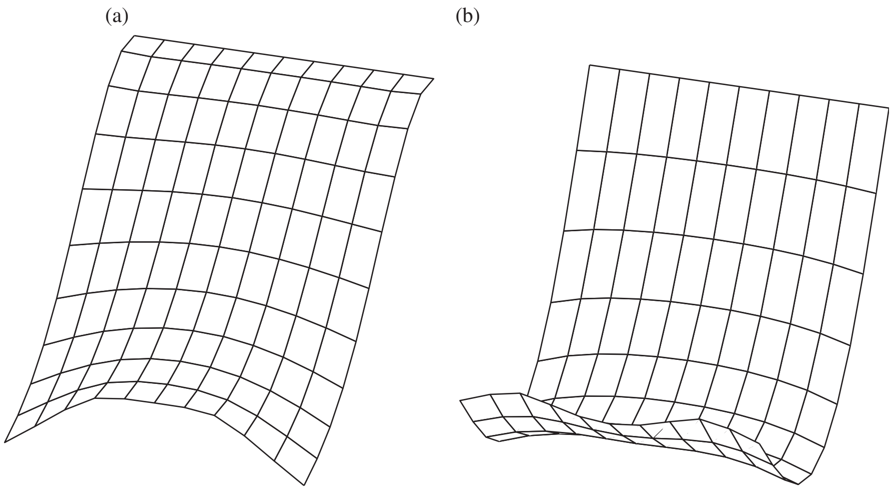

Fig. 5. The first mode shape of the square plate with two minimum stiffness elastic supports: (a) the rear edge clamped; and (b) the rear edge simply supported.

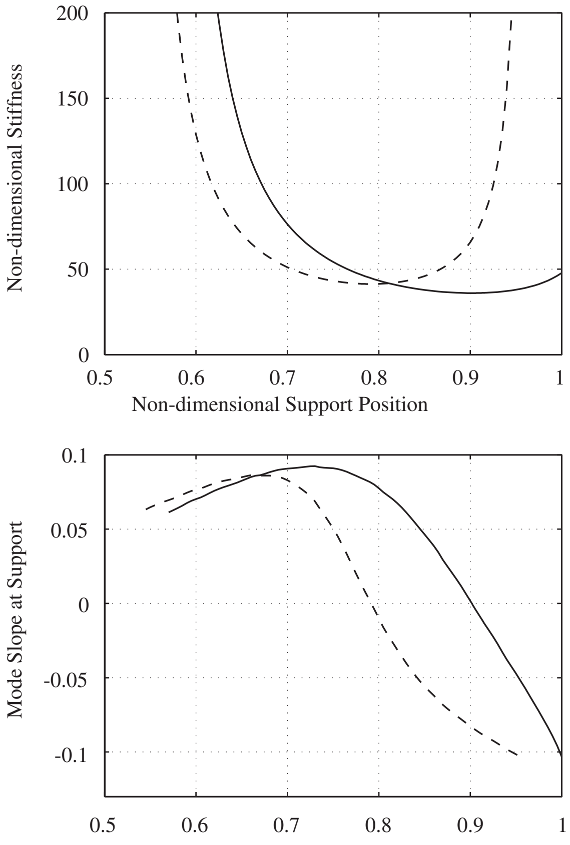

Fig. 6. The minimum support stiffness and mode shape slope in the x-direction at different positions on the line of symmetry for the rectangular plate with aspect ratio 1.5 for the clamped (solid) and simply supported (dashed) boundary.

Table 4
The optimum supports and corresponding natural frequencies for a rectangular plate with a single support on the plate centre line

	Clamped		Simply supported	
Aspect ratio	1.0	1.5	1.0	1.5
Non-dimensional natural frequency				
1 (1B)	8.5088	11.6573	6.6457	9.8461
2 (1T)	8.5088	11.6573	6.5457	9.8461
3 (2B)	23.3674	23.4554	16.1148	15.5690
Non-dimensional minimum stiffness, γ	23.6313	36.0017	26.2139	41.2976
Optimum non-dimensional position (x from supported edge)	0.9734	0.9017	0.8711	0.7917

1B and 2B represent the first and second bending modes, and 1T the first torsional mode.

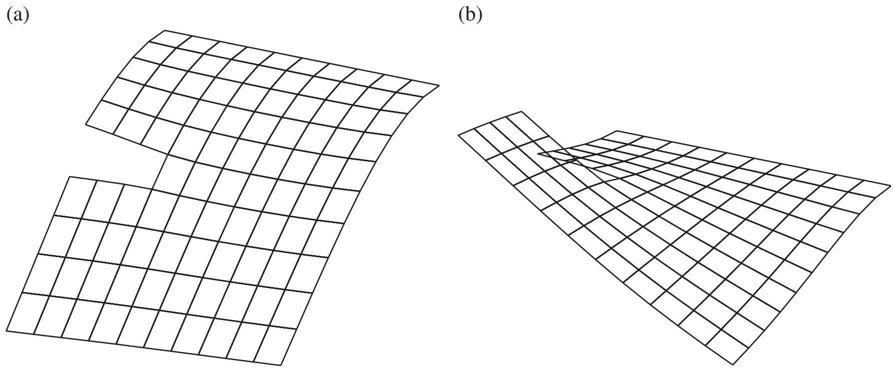

Fig. 7. The first two mode shapes of the square cantilever plate with a slot. The rear edge is clamped. Mode 1 (left) has natural frequency 29.0178 Hz and mode 2 (right) 65.0253 Hz.

natural frequencies have reduced compared to those for the plate without a slot. Fig. 8 shows the support stiffness required to raise the fundamental natural frequency to 50 Hz. For this example a rigid support is unable to raise the fundamental natural frequency of the supported plate to the second natural frequency of the unsupported plate. Fig. 8 shows that the minimum support stiffness required is 15.47 kN/m, when the support is located on the free edge, 12 mm from the midpoint of the edge. This location occurs on the nodal line of the second mode of the unsupported structure and the second natural frequency is unaffected by the introduction of the support.

7. Conclusions

When a natural frequency of a plate structure needs to be increased, an optimally located flexible support can achieve a similar effect to a rigid support. In this study, the minimum stiffnesses of the additional supports are calculated to increase the fundamental natural frequency of the supported plate when one boundary edge is either clamped or simply supported. Numerical examples confirm the existence of the minimum support stiffness and verify that the procedure presented can find this minimum stiffness for different boundary conditions.

The present approach models the structures using the finite element method and may be used in optimization schemes to design practical supports. If the support location is specified a priori, then the minimum stiffness required to attain a given natural frequency is immediately obtained. The alternative is to use optimization methods to minimize the support stiffness and to find the optimum support location. However the proposed approach is more efficient.

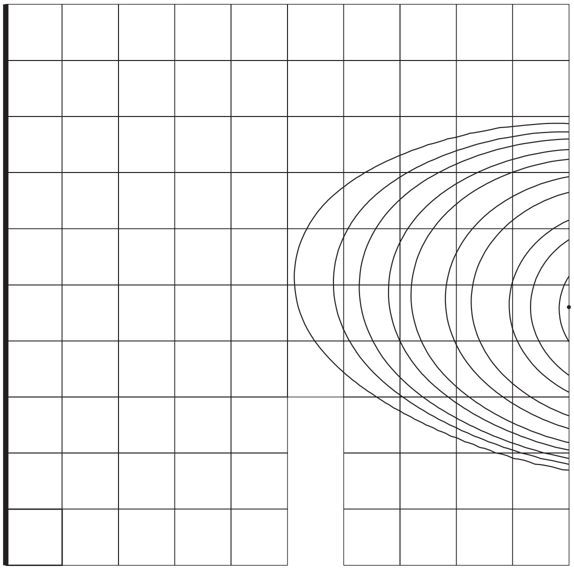

Fig. 8. A contour plot of the support stiffness required to increase the fundamental natural frequency of the slotted cantilever plate to 50 Hz. The thick line represents the clamped edge. The dot represents the optimum position with support stiffness 15.47 kN/m. The contours are at 16, 18, 20, 25, 30, 40, 50, 70, 100, 200 kN/m.

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