Available online at www.sciencedirect.com
ScienceDirect
Journal of Sound and Vibration 301 (2007) 690–700
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Dynamic stability of a thin-walled beam subjected to axial loads and end moments

Ratko Pavlović*, Predrag Kozić, Predrag Rajković, Ivan Pavlović

Mechanical Engineering Faculty, A. Medvedeva 14, 18000 Niš, Serbia

Received 4 May 2005; received in revised form 17 July 2006; accepted 23 October 2006
Available online 22 December 2006

Abstract

The dynamic stability problem of thin-walled beams subjected to combined action of axial loads and end moments is studied. Each force and moment consists of a constant part and a time-dependent stochastic function. Closed form analytical solutions are obtained for simply supported boundary conditions. By using the direct Liapunov method almost sure asymptotic stability and uniform stochastic stability conditions are obtained as the function of stochastic process variance, damping coefficient, and geometric and physical parameters of the beam. The almost sure stability regions for I-cross section and narrow rectangular cross section are shown in the plane of variances of stochastic parts axial force and end moment. Uniform stochastic stability regions are shown in intensity of stochastic loadings and constant parts of axial loads and end moments.

1. Introduction

Many engineering structures like bridges, cranes and aircraft are made up, in one way or another, of a number of thin-walled elements. Such structures find wide application, because of their optimal strength and stiffness. Study of dynamical problems concerning stability and oscillations is very important. These structures often experience a combination of static and dynamic loads, and there are many situations where the dynamic behavior of the structure depends significantly on the static stress field.

The problem of elastic stability of thin-walled beams, cross-sections with two axes of symmetry, subjected to equal and constant end moments, was first solved by Timoshenko [1].

In the case when a thin-walled beam is subjected to an axial load and end moments, Joshi and Suryanarayan [2] obtained solution for coupled flexural–torsional vibration.

Dynamic stability of a simply supported thin, elastic beam subjected to stochastic white-noise excitations is considered by Ariaratnam [3]. By applying Galerkin method, the problem is reduced to consideration of parametric oscillations of the discrete system.

Tylikowski [4], applied direct Liapunov method to uniform stochastic stability analysis of thin-walled double-tee beams loaded by equal end moments.

*Corresponding author. Tel.: +381 18 500 635; fax: +381 18 588 244.
E-mail addresses: ratko@masfak.ni.ac.yu, ratpav@yahoo.com (R. Pavlović).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2006.10.032

Li et al. [5] used an analytical method to perform the flexure–torsion coupled stochastic response analysis of monosymmetric axially loaded Timoshenko thin-walled beam subjected to various kinds of concentrated and distributed stochastic excitations with stationary and ergodic properties.

Almost sure stability of a thin-walled beam subjected to end moments is investigated by Pavlović and Kozić [6]. Stochastic part of the moment is a Gaussian zero mean process or harmonic one. Stability regions as the function of stochastic process variance, damping coefficient, geometric and physical parameters of the beam, are shown.

The intention of the present paper is to investigate almost sure asymptotic stability and uniform stochastic stability of thin-walled beams subjected to time-dependent stochastic axial loads and end moments.

2. Problem formulation

Let us consider the flexural–torsional stability of a homogeneous, isotropic, thin-walled beam with two planes of symmetry. The beam is assumed to be loaded in the plane of greater bending rigidity by two equal couples and axial loads acting at the ends (Fig. 1).

The governing differential equations for the coupled flexural and torsional motion of the beam can be written as [4,7]

$$\rho A \frac{\partial^2 U}{\partial T^2} + \alpha_U \frac{\partial U}{\partial T} + EI_y \frac{\partial^4 U}{\partial Z^4} + \bar{F}(T)\frac{\partial^2 U}{\partial Z^2} + \overline{M}(T)\frac{\partial^2 \theta}{\partial Z^2} = 0, \tag{1}$$

$$\rho I_p \frac{\partial^2 \theta}{\partial T^2} + \alpha_\theta \frac{\partial \theta}{\partial T} - \left(GJ - \overline{F}(T)\frac{I_p}{A}\right)\frac{\partial^2 \theta}{\partial Z^2} + \overline{M}(T)\frac{\partial^2 U}{\partial Z^2} + EI_s \frac{\partial^4 \theta}{\partial Z^4} = 0, \quad Z \in (0, \ell), \tag{2}$$

where U is the flexural displacement in the x-direction, θ is torsional displacement, ρ is mass density, A is area of the cross-section of beam, I_y I_p, I_s are axial, polar and sectorial moment of inertia, J is Saint–Venant torsional constant, E is Young modulus of elasticity, G is shear modulus, α_U, α_θ are viscous damping coefficients, T is time and Z is axial coordinate.

Using the following transformations

$$U = u\sqrt{\frac{I_p}{A}}, \quad Z = z\ell, \quad T = k_t t, \quad \overline{F}(T) = F_{\text{cr}}(F_o + F(t)), \quad \overline{M}(T) = M_{\text{cr}}(M_o + M(t)),$$

$$F_{\text{cr}} = \frac{\pi^2 EI_y}{\ell^2}, \quad M_{\text{cr}} = \frac{\pi}{\ell}\sqrt{EI_y GJ}, \quad k_t^2 = \frac{\rho A \ell^4}{EI_y}, \quad e = \frac{AI_s}{I_y I_p} \tag{3}$$

$$\beta_1 = \frac{1}{2}\alpha_U \frac{\ell^2}{\sqrt{\rho A E I_y}}, \quad \beta_2 = \frac{1}{2}\alpha_\theta \ell^2 \sqrt{\frac{A}{\rho E I_y I_p^2}}, \quad S = \frac{GJA\ell^2}{\pi^2 E I_y I_p},$$

where ℓ is the length of the beam, F_{cr} is Euler critical force, M_{cr} is critical buckling moment for the simply supported narrow rectangular beam, S is slenderness parameter, β_1 and β_2 are reduced viscous damping

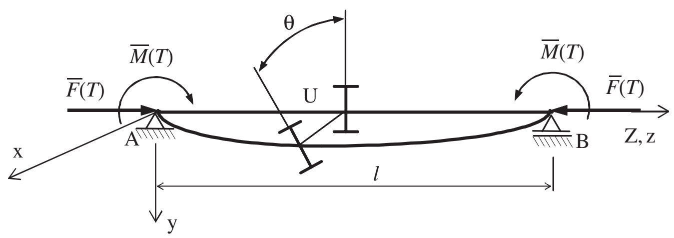

Fig. 1. Geometry of thin-walled I-beam subjected to axial loads and end moments.

coefficients, we get governing equations as

$$\frac{\partial^2 u}{\partial t^2}+2\beta_1\frac{\partial u}{\partial t}+\frac{\partial^4 u}{\partial z^4}+\pi^2(F_o+F(t))\frac{\partial^2 u}{\partial z^2}+\pi^2\sqrt{S}(M_o+M(t))\frac{\partial^2\theta}{\partial z^2}=0, \tag{4}$$

$$\frac{\partial^2\theta}{\partial t^2}+2\beta_2\frac{\partial\theta}{\partial t}-\pi^2(S-F_o-F(t))\frac{\partial^2\theta}{\partial z^2}+\pi^2\sqrt{S}(M_o+M(t))\frac{\partial^2 u}{\partial z^2}+e\frac{\partial^4\theta}{\partial z^4}=0,\quad z\in(0,1). \tag{5}$$

Taking free warping displacement and zero angular displacements into account, boundary conditions for the simply supported beam are

$$u(t,0)=u(t,1)=\frac{\partial^2 u}{\partial z^2}(t,0)=\frac{\partial^2 u}{\partial z^2}(t,1)=0, \tag{6}$$

$$\theta(t,0)=\theta(t,1)=\frac{\partial^2\theta}{\partial z^2}(t,0)=\frac{\partial^2\theta}{\partial z^2}(t,1)=0.$$

The purpose of the present paper is the investigation of almost sure asymptotic and uniform stochastic stability of the thin-walled beam subjected to stochastic time-dependent axial loads and end moments. To estimate perturbated solutions it is necessary to introduce a measure of distance $\|\cdot\|$ of solutions of Eqs. (4) and (5) with nontrivial initial conditions and the trivial one. The equilibrium state of Eqs. (4) and (5) is said to be almost sure stochastically stable, [8] if

$$P\left\{\lim_{t\to\infty}\|\mathbf{w}(.,t)\|=0\right\}=1, \tag{7}$$

where $\mathbf{w}=\mathrm{col}(u,\theta)$ matrix column.

In the case when the loadings are broad-band Gaussian processes which can be treated as white-noises, we investigate the uniform stochastic stability of the trivial solution, i.e. we formulate conditions implying the following logical sentence:

$$\mathop{\wedge}_{\varepsilon>0}\mathop{\wedge}_{\delta>0}\mathop{\vee}_{t>0}||\mathbf{w}(.,t)||r\Rightarrow P\left\{\sup_{t>0}||\mathbf{w}(.,t)||>\delta\right\}<\varepsilon. \tag{8}$$

3. Stability analyses

With the purpose of applying the Liapunov method, we can construct the functional by means of the Parks–Pritchard's method [9]. Thus, let us write Eqs. (4) and (5) in the formal form $\mathbf{Lw}=0$, and introduce the linear operator N which is a formal derivative of the operator $\mathbf{L}$ with respect to $\partial/\partial t$.

Integrating the scalar product of the vectors $\mathbf{LwNw}$ on rectangular $C=\Omega\times\Delta=[z\colon 0\leqslant z\leqslant 1]\times[\tau\colon 0\leqslant\tau\leqslant t]$ with respect to Eqs. (4) and (5), it is clear

$$\begin{aligned}2\int_0^t\int_0^1\Bigg\{&\left[\frac{\partial^2 u}{\partial t^2}+2\beta_1\frac{\partial u}{\partial t}+\frac{\partial^4 u}{\partial z^4}+\pi^2(F_o+F(t))\frac{\partial^2 u}{\partial z^2}+\pi^2\sqrt{S}(M_o+M(t))\frac{\partial^2\theta}{\partial z^2}\right]\left(\frac{\partial u}{\partial t}+\beta_1 u\right)\\&+\left[\frac{\partial^2\theta}{\partial t^2}+2\beta_2\frac{\partial\theta}{\partial t}-\pi^2(S-F_o-F(t))\frac{\partial^2\theta}{\partial z^2}+\pi^2\sqrt{S}(M_o+M(t))\frac{\partial^2 u}{\partial z^2}+e\frac{\partial^4\theta}{\partial z^4}\right]\left(\frac{\partial\theta}{\partial t}+\beta_2\theta\right)\Bigg\}\mathrm{d}z\,\mathrm{d}\tau=0.\end{aligned} \tag{9}$$

After applying partial integration to Eq. (9), the sum of two integrals may be obtained as

$$\mathbf{V}|_0^t-\int_0^t\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t}=0, \tag{10}$$

where

$$\begin{aligned}\mathbf{V}=\int_0^1\Bigg\{&\left(\frac{\partial u}{\partial t}+\beta_1 u\right)^2+\beta_1^2u^2+\left(\frac{\partial^2 u}{\partial z^2}\right)^2-\pi^2F_o\left(\frac{\partial u}{\partial z}\right)^2-2\pi^2\sqrt{S}M_o\frac{\partial u}{\partial z}\frac{\partial\theta}{\partial z}\\&+\left(\frac{\partial\theta}{\partial t}+\beta_2\theta\right)^2+\beta_2^2\theta^2-\pi^2(F_o-S)\left(\frac{\partial\theta}{\partial z}\right)^2+e\left(\frac{\partial^2\theta}{\partial z^2}\right)^2\Bigg\}\mathrm{d}z\end{aligned} \tag{11}$$

and

$$\frac{d\mathbf{V}}{dt} = -\int_0^1 \left[2\beta_1 \left(\frac{\partial u}{\partial t}\right)^2 + 2\beta_1 \left(\frac{\partial^2 u}{\partial z^2}\right)^2 - 2\beta_1 \pi^2 F_o \left(\frac{\partial u}{\partial z}\right)^2 - 2(\beta_1+\beta_2)\pi^2 \sqrt{S} M_o \frac{\partial u}{\partial z}\frac{\partial \theta}{\partial z} \right.$$
$$+ 2\pi^2 F(t) \frac{\partial^2 u}{\partial z^2}\left(\frac{\partial u}{\partial t} + \beta_1 u\right) + 2\pi^2 \sqrt{S} M(t) \frac{\partial^2 \theta}{\partial z^2}\left(\frac{\partial u}{\partial t} + \beta_1 u\right)$$
$$+ 2\beta_2 \left(\frac{\partial \theta}{\partial t}\right)^2 - 2\beta_2 \pi^2 (F_o - S)\left(\frac{\partial \theta}{\partial z}\right)^2 + 2\pi^2 F(t)\frac{\partial^2 \theta}{\partial z^2}\left(\frac{\partial \theta}{\partial t} + \beta_2 \theta\right)$$
$$\left. + 2\pi^2 \sqrt{S} M(t) \frac{\partial^2 u}{\partial z^2}\left(\frac{\partial \theta}{\partial t} + \beta_2 \theta\right) + 2\beta_2 e \left(\frac{\partial^2 \theta}{\partial z^2}\right)^2 \right] dz. \tag{12}$$

Functional $\mathbf{V}$ will be a Liapunov functional if it is a positive definite. As functions u and θ satisfy boundary and continuity conditions (6), by using well-known Steklov's inequalities:

$$\int_0^1 \left(\frac{\partial^2 u}{\partial z^2}\right)^2 dz \geqslant \pi^2 \int_0^1 \left(\frac{\partial u}{\partial z}\right)^2 dz, \quad \int_0^1 \left(\frac{\partial^2 \theta}{\partial z^2}\right)^2 dz \geqslant \pi^2 \int_0^1 \left(\frac{\partial \theta}{\partial z}\right)^2 dz \tag{13}$$

and omitting dynamical terms, we can write

$$\mathbf{V} \geqslant \pi^2 \int_0^1 \left[(1-F_o)\left(\frac{\partial u}{\partial z}\right)^2 - 2\sqrt{S} M_o \frac{\partial u}{\partial z}\frac{\partial \theta}{\partial z} + (S+e-F_o)\left(\frac{\partial \theta}{\partial z}\right)^2 \right] dz. \tag{14}$$

The positive definite condition reduces to relations

$$F_o < \min(1, S+e), \tag{15}$$

$$\frac{S M_o^2}{S+e-F_o} + F_o < 1 \tag{16}$$

which represent static stability conditions for thin-walled beam subjected to constant axial forces F_o and end moments M_o. By putting $e = 0$ in Eq. (16) we get Joshi–Suryanarayan's [2] relation.

Now, as the measure of distance between the perturbated solutions and the trivial ones can be chosen square root of the functional $\|\cdot\| = \mathbf{V}^{1/2}$.

If a thin-walled beam is subjected only to constant loadings, $F(t) = M(t) = 0$, similarly, by using inequalities (13), we can estimate the time-derivative of the Liapunov functional:

$$\frac{dV}{dt} \leqslant -\pi^2 \int_0^1 \left[2\beta_1 (1-F_o)\left(\frac{\partial u}{\partial z}\right)^2 - 2(\beta_1+\beta_2)\sqrt{S} M_o \frac{\partial u}{\partial z}\frac{\partial \theta}{\partial z} + 2\beta_2 (S+e-F_o)\left(\frac{\partial u}{\partial z}\right)^2 \right] dz \tag{17}$$

and it is negative semi-definite if

$$\frac{\beta_1 \beta_2}{(\beta_1 + \beta_2/2)^2} \geqslant \frac{S M_o^2}{(1-F_o)(S+e-F_o)}. \tag{18}$$

If the relation (18) is fulfilled, equilibrium state is stable with respect to measure $\|\cdot\| = \mathbf{V}^{1/2}$. Having in the mind that geometric mean is not greater than arithmetic one, dynamic stability condition (18) is more restrictive than (16). Conditions (16) and (18) are identical when damping coefficients β_1 and β_2 are equal.

For simplicity, further we will take $\beta_1 = \beta_2 = \beta$.

4. Almost-sure stability

Relation (12) can be written in the form

$$\frac{d\mathbf{V}}{dt} = -2\beta \mathbf{V} + 2\mathbf{U}, \tag{19}$$

where **U** is an auxiliary functional defined as

$$\mathbf{U} = \int_0^1 \left[\left(2\beta^2 u - \pi^2 F(t) \frac{\partial^2 u}{\partial z^2} - \pi^2 \sqrt{S} M(t) \frac{\partial^2 \theta}{\partial z^2} \right) \left(\frac{\partial u}{\partial t} + \beta u \right) + \left(2\beta^2 \theta - \pi^2 F(t) \frac{\partial^2 \theta}{\partial z^2} - \pi^2 \sqrt{S} M(t) \frac{\partial^2 u}{\partial z^2} \right) \left(\frac{\partial \theta}{\partial t} + \beta \theta \right) \right] \mathrm{d}z. \tag{20}$$

Now we attempt to construct a bound

$$\mathbf{U} \leqslant \lambda \mathbf{V}, \tag{21}$$

where λ is the unknown function.

Proceeding similarly as Kozin [8], we have to solve an additional variational problem:

$$\delta(\mathbf{U} - \lambda \mathbf{V}) = 0. \tag{22}$$

By using the associated Euler's equations we obtain

$$\begin{aligned}
&2\beta^2 \left(\frac{\partial u}{\partial t} + 2\beta u \right) - \pi^2 F(t) \left(\frac{\partial^3 u}{\partial t \partial z^2} + 2\beta \frac{\partial^2 u}{\partial z^2} \right) - \pi^2 \sqrt{S} M(t) \left(\frac{\partial^3 \theta}{\partial t \partial z^2} + 2\beta \frac{\partial^2 \theta}{\partial z^2} \right) \\
&\quad - 2\lambda \left[\beta \left(\frac{\partial u}{\partial t} + 2\beta u \right) + \frac{\partial^4 u}{\partial z^4} + \pi^2 F_o \frac{\partial^2 u}{\partial z^2} + \pi^2 \sqrt{S} M_o \frac{\partial^2 \theta}{\partial z^2} \right] = 0, \\
&2\beta^2 \left(\frac{\partial \theta}{\partial t} + 2\beta \theta \right) - \pi^2 F(t) \left(\frac{\partial^3 \theta}{\partial t \partial z^2} + 2\beta \frac{\partial^2 \theta}{\partial z^2} \right) - \pi^2 \sqrt{S} M(t) \left(\frac{\partial^3 u}{\partial t \partial z^2} + 2\beta \frac{\partial^2 u}{\partial z^2} \right) \\
&\quad - 2\lambda \left[\beta \left(\frac{\partial \theta}{\partial t} + 2\beta \theta \right) + e \frac{\partial^4 \theta}{\partial z^4} + \pi^2 \sqrt{S} M_o \frac{\partial^2 u}{\partial z^2} + \pi^2 (F_o - S) \frac{\partial^2 \theta}{\partial z^2} \right] = 0, \\
&2\beta^2 u - \pi^2 F(t) \frac{\partial^2 u}{\partial z^2} - \pi^2 \sqrt{S} M(t) \frac{\partial^2 \theta}{\partial z^2} - 2\lambda \left(\frac{\partial u}{\partial t} + \beta u \right) = 0, \\
&2\beta^2 \theta - \pi^2 F(t) \frac{\partial^2 \theta}{\partial z^2} - \pi^2 \sqrt{S} M(t) \frac{\partial^2 u}{\partial z^2} - 2\lambda \left(\frac{\partial \theta}{\partial t} + \beta \theta \right) = 0.
\end{aligned} \tag{23}$$

After simplifying, we get two equations

$$\begin{aligned}
&2\beta^2 f_1 - \pi^2 F(t) \frac{\partial^2 f_1}{\partial z^2} - \pi^2 \sqrt{S} M(t) \frac{\partial^2 f_2}{\partial z^2} - 4\lambda^2 h_1 = 0, \\
&2\beta^2 f_2 - \pi^2 F(t) \frac{\partial^2 f_2}{\partial z^2} - \pi^2 \sqrt{S} M(t) \frac{\partial^2 f_1}{\partial z^2} - 4\lambda^2 h_2 = 0,
\end{aligned} \tag{24}$$

where

$$\begin{aligned}
f_1 &= 2\beta^2 u - \pi^2 F(t) \frac{\partial^2 u}{\partial z^2} - \pi^2 \sqrt{S} M(t) \frac{\partial^2 \theta}{\partial z^2}, \\
f_2 &= 2\beta^2 \theta - \pi^2 F(t) \frac{\partial^2 \theta}{\partial z^2} - \pi^2 \sqrt{S} M(t) \frac{\partial^2 u}{\partial z^2},
\end{aligned} \tag{25}$$

$$\begin{aligned}
h_1 &= \beta^2 u + \frac{\partial^4 u}{\partial z^4} + \pi^2 F_o \frac{\partial^2 u}{\partial z^2} + \pi^2 \sqrt{S} M_o \frac{\partial^2 \theta}{\partial z^2}, \\
h_2 &= \beta^2 \theta + \pi^2 \sqrt{S} M_o \frac{\partial^2 u}{\partial z^2} + \pi^2 (F_o - S) \frac{\partial^2 \theta}{\partial z^2} + e \frac{\partial^4 \theta}{\partial z^4}.
\end{aligned}$$

According to the boundary condition (6), we may write the solution in the form

$$u(z,t) = \sum_{m=1}^{\infty} U_m T_m(t) \sin \alpha_m z,$$
$$\theta(z,t) = \sum_{m=1}^{\infty} \Theta_m T_m(t) \sin \alpha_m z, \tag{26}$$

where $\alpha_m = m\pi$, and from Eq. (24) we obtain algebraic equation of the fourth order:

$$16 A_m \lambda_m^4 - 4 B_m \lambda_m^2 + C_m = 0, \tag{27}$$

where

$$A_m = \left(\beta^2 + \alpha_m^4 - \pi^2 F_o \alpha_m^2\right)\left[\beta^2 - \pi^2 (F_o - S)\alpha_m^2 + e\alpha_m^4\right] - \pi^4 S M_o^2 \alpha_m^4,$$
$$B_m = \left[\left(2\beta^2 + \pi^2 F_{(t)} \alpha_m^2\right)^2 + \pi^4 S M_{(t)}^2 \alpha_m^4\right]\left[2\beta^2 + \alpha_m^4 + \pi^2 (S - 2F_o)\alpha_m^2 + e\alpha_m^4\right]$$
$$+ 4\pi^4 S M_o \left(2\beta^2 + \pi^2 F_{(t)} \alpha_m^2\right) M_{(t)} \alpha_m^4,$$
$$C_m = \left[\left(2\beta^2 + \pi^2 F_{(t)} \alpha_m^2\right)^2 - \pi^4 S M_{(t)}^2 \alpha_m^4\right]^2. \tag{28}$$

By substituting inequality (21) into relation (19) and solving the obtained differential inequality, when the processes $F(t)$ and $M(t)$ are ergodic and stationary, we conclude that the trivial solutions of Eqs. (4) and (5) are almost sure asymptotically stable, if

$$\mathbf{E}\left\{\max_m \lambda_m(t)\right\} \leqslant \beta, \tag{29}$$

where **E** denotes the operator of the mathematical expectation.

5. Uniform stochastic stability

If the forces and moments, $F(t)$ and $M(t)$, acting on the beam are broad-band stochastic processes, which can be modified as "white noise":

$$F(t) = \sigma_1 \dot{\xi}_1(t), \quad M(t) = \sigma_2 \dot{\xi}_2(t), \tag{30}$$

where $\xi_1(t)$ and $\xi_2(t)$ are the standard Wiener processes, and σ_1, σ_2 intensities of stochastic loads, the dynamic Eqs. (4) and (5) can be written in Ito differential form:

$$\mathrm{d}u = v\,\mathrm{d}t,$$
$$\mathrm{d}v = -\left(2\beta_1 v + \frac{\partial^4 u}{\partial z^4} + \pi^2 F_o \frac{\partial^2 u}{\partial z^2} + \pi^2 \sqrt{S} M_o \frac{\partial^2 \theta}{\partial z^2}\right)\mathrm{d}t - \pi^2 \sigma_1 \frac{\partial^2 u}{\partial z^2}\,\mathrm{d}\xi_1(t) - \pi^2 \sqrt{S}\sigma_2 \frac{\partial^2 \theta}{\partial z^2}\,\mathrm{d}\xi_2(t), \tag{31}$$

$$\mathrm{d}\theta = \psi\,\mathrm{d}t,$$
$$\mathrm{d}\psi = -\left[2\beta_2 \psi + e\frac{\partial^4 \theta}{\partial z^4} + \pi^2 (F_o - S)\frac{\partial^2 \theta}{\partial z^2} + \pi^2 \sqrt{S} M_o \frac{\partial^2 u}{\partial z^2}\right]\mathrm{d}t - \pi^2 \sigma_1 \frac{\partial^2 \theta}{\partial z^2}\,\mathrm{d}\xi_1(t) - \pi^2 \sqrt{S}\sigma_2 \frac{\partial^2 u}{\partial z^2}\,\mathrm{d}\xi_2(t),$$

For stability analysis we take functional (11) and applying Ito formula we calculate the differential of **V**:

$$\mathrm{d}V = \int_0^1 \left[-2\beta_1 v^2 - 2\beta_1 u \frac{\partial^4 u}{\partial z^4} - 2\beta_1 \pi^2 F_o u \frac{\partial^2 u}{\partial z^2} - 2\beta_1 \pi^2 \sqrt{S} M_o u \frac{\partial^2 \theta}{\partial z^2}\right.$$
$$- 2\beta_2 \psi^2 - 2\beta_2 \pi^2 (F_o - S)\theta \frac{\partial^2 \theta}{\partial z^2} - 2\beta_2 \pi^2 \sqrt{S} M_o \theta \frac{\partial^2 u}{\partial z^2} - 2\beta_2 e\theta \frac{\partial^4 \theta}{\partial z^4}$$

$$
\begin{aligned}
&+\pi^4\left(\sigma_1^2+S\sigma_2^2\right)\left(\frac{\partial^2 u}{\partial z^2}\right)^2+\pi^4\left(\sigma_1^2+S\sigma_2^2\right)\left(\frac{\partial^2 \theta}{\partial z^2}\right)^2\Bigg]\mathrm{d}z\,\mathrm{d}t\\
&-2\pi^2\int_0^1\left\{\left[(v+\beta_1 u)\frac{\partial^2 u}{\partial z^2}+(\psi+\beta_2\theta)\frac{\partial^2\theta}{\partial z^2}\right]\sigma_1\,\mathrm{d}\xi_1(t)\right.\\
&\left.+\left[(v+\beta_1 u)\frac{\partial^2 \theta}{\partial z^2}+(\psi+\beta_2\theta)\frac{\partial^2 u}{\partial z^2}\right]\sqrt{S}\sigma_2\,\mathrm{d}\xi_2(t)\right\}\mathrm{d}z.
\end{aligned}\tag{32}
$$

On integrating with respect to t from $t=s$ to $t=\tau_\delta(t)$, where $\tau_\delta(t)=\min(t,td)$, $\tau_\delta=\inf[t:\|\cdot\|>\delta>0]$, by averaging and taking into consideration that $\mathbf{E}\{\xi_1(t)\}=\mathbf{E}\{\xi_2(t)\}=0$, it follows that

$$
\begin{aligned}
\mathbf{E}\{V(\tau_\delta(t))\}=\mathbf{V}(s)-\mathbf{E}\Bigg\{&\int_s^{\tau_\delta(t)}\int_0^1\int_0^1\Bigg[2\beta_1 v^2+2\beta_1 u\frac{\partial^4 u}{\partial z^4}+2\beta_1\pi^2 F_o u\frac{\partial^2 u}{\partial z^2}+2\beta_1\pi^2\sqrt{S}M_o u\frac{\partial^2\theta}{\partial z^2}\\
&+2\beta_2\psi^2+2\beta_2\pi^2(F_o-S)\theta\frac{\partial^2\theta}{\partial z^2}+2\beta_2\pi^2\sqrt{S}M_o\theta\frac{\partial^2 u}{\partial z^2}+2\beta_2 e\theta\frac{\partial^4\theta}{\partial z^4}\\
&-\pi^4\left(\sigma_1^2+S\sigma_2^2\right)\left(\frac{\partial^2 u}{\partial z^2}\right)^2-\pi^4\left(\sigma_1^2+S\sigma_2^2\right)\left(\frac{\partial^2 \theta}{\partial z^2}\right)^2\Bigg]\mathrm{d}z\,\mathrm{d}t\Bigg\}.
\end{aligned}\tag{33}
$$

On supposing $\mathbf{E}\{\mathbf{V}(\tau_\delta(t))\}\leqslant V(s)$, it follows that functional $\mathbf{V}$ is supermartingale. By calculating the probability of the event that $\sup_{t\geqslant s}\|\mathbf{w}\|>\delta$ and using Chebyshev inequality we obtain

$$
P\left\{\sup_{t\geqslant s}\|\mathbf{w}\|>\delta\right\}\leqslant\frac{\mathbf{E}\{\mathbf{V}(\tau_\delta(t))\}}{\delta^2}\leqslant\frac{\mathbf{V}(s)}{\delta^2}.\tag{34}
$$

When $s\to 0$ we get the desired definition of uniform stochastic stability.

The Liapunov functional is supermartingale when second term in Eq. (33) is nonnegative. Using relations (13), and omitting positive terms $2\beta_1 v^2$ and $2\beta_2\psi^2$, it follows:

$$
\begin{aligned}
\pi^2\int_0^1\Bigg\{&\left[2\beta_1(1-F_o)-\pi^4(\sigma_1^2+S\sigma_2^2)\right]\left(\frac{\partial u}{\partial z}\right)^2-2(\beta_1+\beta_2)\sqrt{S}M_o\frac{\partial u}{\partial z}\frac{\partial\theta}{\partial z}\\
&+\left[2\beta_2(e+S-F_o)-\pi^4(\sigma_1^2+S\sigma_2^2)\right]\left(\frac{\partial\theta}{\partial z}\right)^2\Bigg\}\mathrm{d}z\leqslant 0.
\end{aligned}\tag{35}
$$

If the auxiliary conditions given by inequities

$$
\begin{aligned}
&2\beta_1 F_o+\pi^4(\sigma_1^2+S\sigma_2^2)<2\beta_1,\\
&2\beta_2 F_o+\pi^4(\sigma_1^2+S\sigma_2^2)<2\beta_2(e+S),
\end{aligned}\tag{36}
$$

are satisfied, then the condition that functional $\mathbf{V}$ is supermartingale has the form

$$
\begin{aligned}
&\pi^8(\sigma_1^2+S\sigma_2^2)^2-2\pi^4(\sigma_1^2+S\sigma_2^2)\left[\beta_1(1-F_o)+\beta_2(e+S-F_o)\right]\\
&+4\beta_1\beta_2(1-F_o)(e+S-F_o)-(\beta_1+\beta_2)^2 SM_o^2\geqslant 0.
\end{aligned}\tag{37}
$$

According to the previous relation we can find regions of uniform stochastic stability as a function of reduced system parameters and loading parameters.

6. Numerical results and discussion

Inequalities (16), (31) and (37) give the possibility to obtain stability regions for thin-walled beams. Numerical results are calculated for steel beams with elastic constant ratio $E/G\approx 2.59$ and two types of cross sections: I-section and narrow rectangular cross section.

With respect to standard I-section we can approximativelly take $h/b\approx 2$, $b/\delta_1\approx 11$, $\delta/\delta_1\approx 1.5$, where h is the depth, b is width, δ is thickness of the flanges and δ_1 is thickness of the rib of I-section. These ratios give us

$S \approx 0.01928(\ell/h)^2$ and $e \approx 1.276$. For the narrow rectangular cross section, according to assumption $\delta/h < 0.1$, for thin-walled cross sections $S \approx 1.88(\ell/h)^2$ and $e = 0$, which are arrived at using the approximation $1 + (\delta/h)^2 \approx 1$.

According to relation (16), static stability regions for both cross sections as function of geometrical ratio ℓ/h can be considered. Euler–Bernoulli theory of beam-column is valid for $\ell/h \geqslant 10$, then is slenderness parameter $S > 1$, and critical axial force is $F_o = 1$ If $\ell/h < 10$ then we must take into account transverse shear and rotary inertia of cross section.

The almost sure stability regions as functions of loading variances, damping coefficient, ratio of beam length to depth of cross section, constant components of loading are calculated numerically by using Schwarz inequality. This calculation is performed for the first mode ($m = 1$).

Almost sure stability regions for I-section are given in Figs. 2a–d, and for narrow rectangular cross section in Figs. 3a–d. From Figs. 2a and 3a it is evident that when the damping coefficient increases, the stability regions become larger so that the curves are on almost equal distance when the damping coefficient difference is equal. When geometrical ratio ℓ/h increases stability regions are smaller (Figs. 2b and 3b), and its influence on end moment variance, in case of narrow rectangular cross section, is more dominant.

Stability regions are larger when constant component of axial loading changes from pressure ($F_o = 0.5$) to tension ($F_o = -0.5$) (Figs. 2c and 3c). The stability region decrease is more expressed when the compressive force increases comparing to its increase when the tensile force grows. If constant component of the end moment grows, stability regions are smaller (Figs. 2d and 3d), and that is marked in the case of the narrow rectangular cross section. It is evident that end moment variances are about ten times higher for I-section than for narrow rectangular section, when axial force vary only a little.

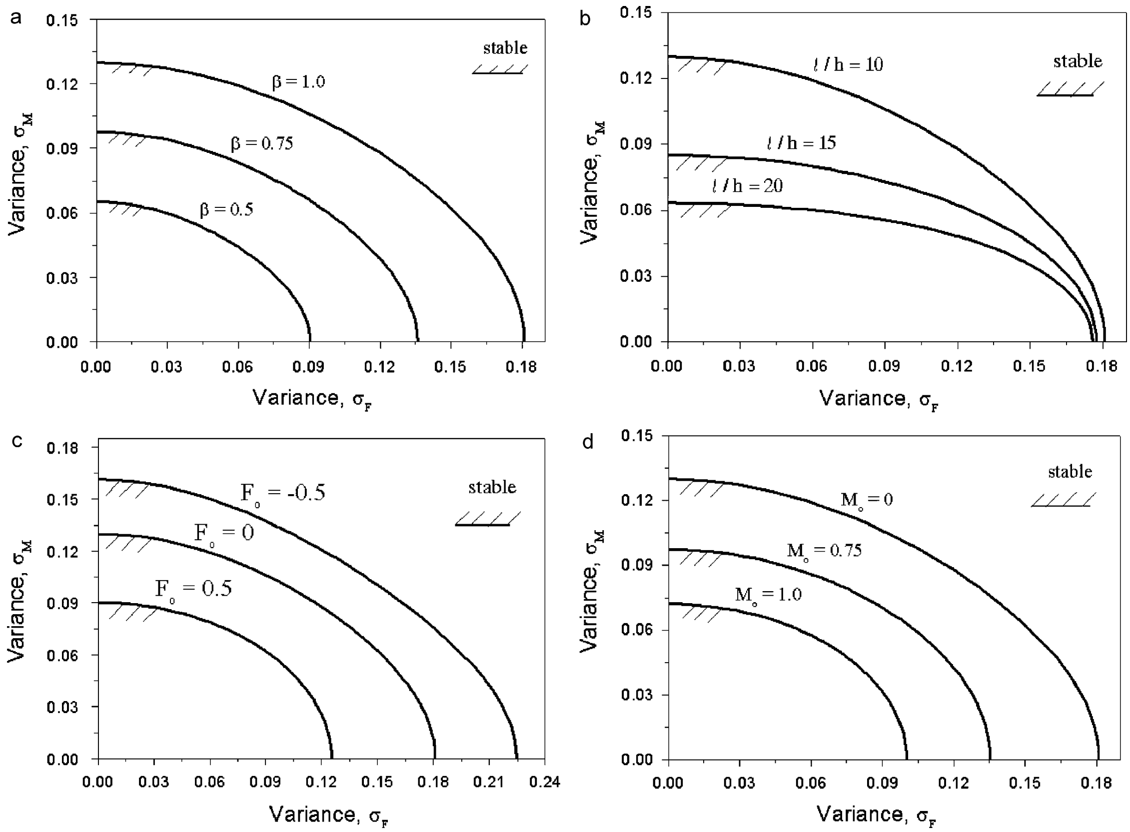

Fig. 2. Almost sure stability regions for I-section as a function of: (a) damping coefficient; (b) ℓ/h ratio; (c) deterministic component of axial force; and (d) deterministic component of end moment.

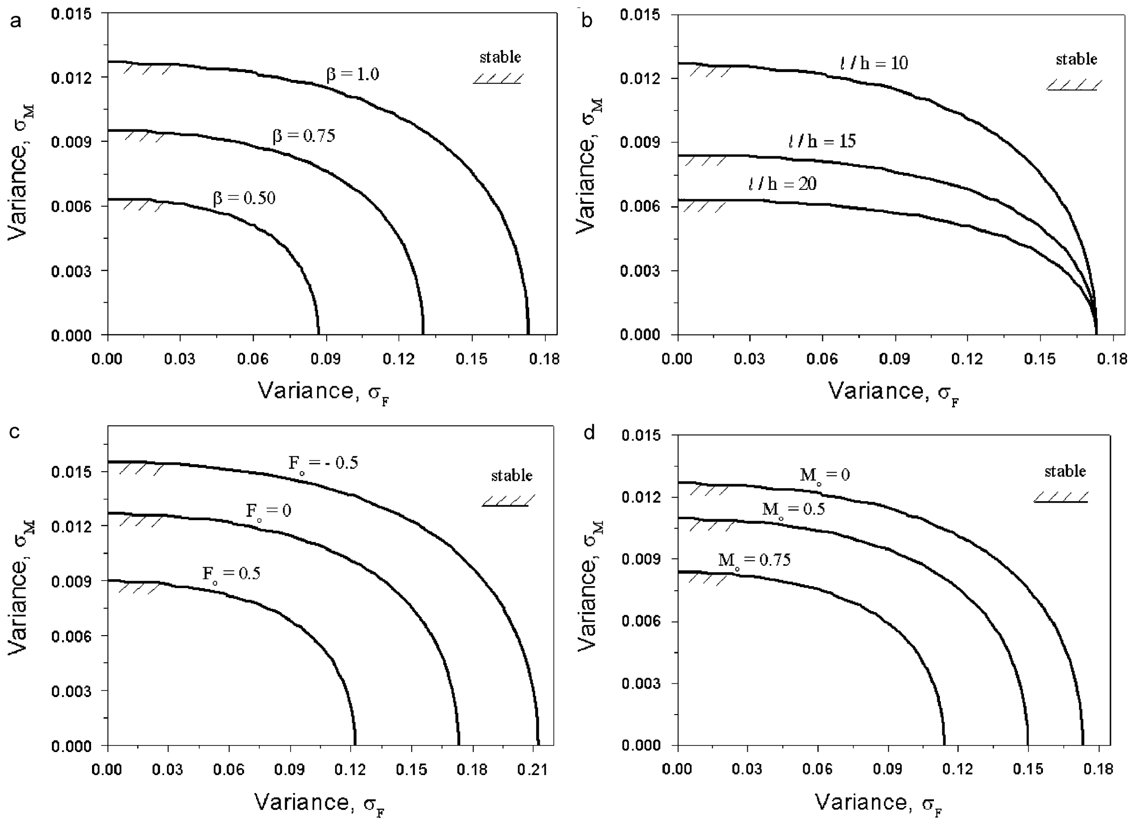

Fig. 3. Almost sure stability regions for narrow rectangular cross section as a function of: (a) damping coefficient; (b) ℓ/h ratio; (c) deterministic component of axial force; and (d) deterministic component of end moment.

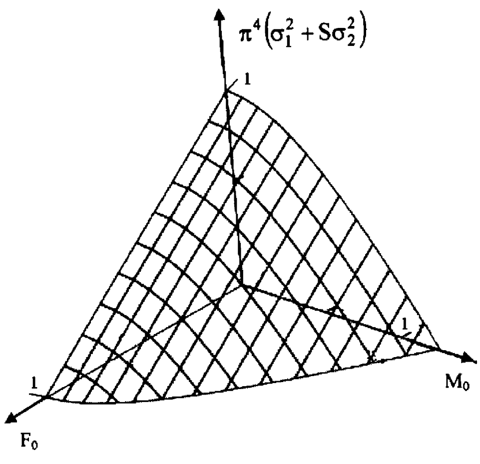

Fig. 4. Uniform stability regions for I-section ($\beta = 0.5$, $\ell/h = 10$).

In Figs. 4 and 5, uniform stability regions are given in the space of constant components of axial force (F_o), end moment (M_o) and grouped intensities of stochastic load $\pi^4(\sigma_1^2 + S\sigma_2^2)$. Horizontal planes ($F_o$, M_o) are given by relation (16). As in the previous case, it is taken that $\beta_1 = \beta_2 = \beta$. Uniform stability regions are given for I-section ($\ell/h = 10$, $\beta = 0.5$) and for the narrow rectangular cross section ($\ell/h = 20$, $\beta = 0.5$). We can conclude that stability regions are significantly enlarged when damping coefficient increases, and stability surfaces are lower when geometric ratio ℓ/h decreases.

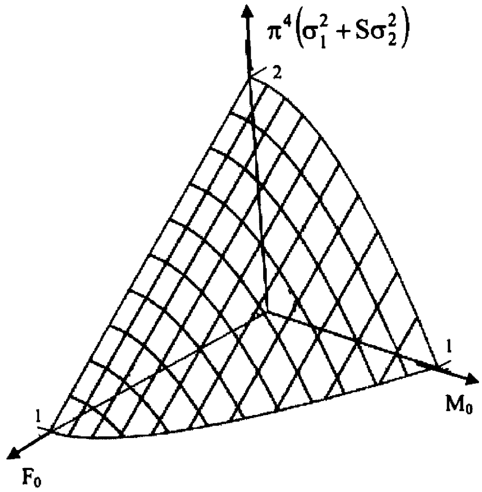

Fig. 5. Uniform stability regions for narrow rectangular cross section ($\beta = 1.0$, $\ell/h = 20$).

7. Conclusions

The paper presents a solution for dynamic stability problem of thin-walled beams subjected to time-dependent stochastic axial loads and end moments. When the stochastic processes are nonwhite almost sure asymptotic stability is investigated, if they are Gaussian white noises, uniform stochastic stability is studied.

Numerical calculations are performed for I-section and narrow rectangular cross section. Beams of narrow rectangular section can be considered as the special case of I-beams with zero warping rigidity EI_s, and then is $e = 0$. On the other hand, very thin-walled I-beams have comparatively low torsional rigidities GJ, and then is slenderness S very small. Both parameters, e and S, depend of beam cross-section geometry, and for known beam material slenderness S is function of the geometric ratio ℓ/h.

The almost sure stability regions as functions of loading variances and uniform stability regions as intensity of stochastic loading are given. The end moment variances are about ten times higher for I-section than for narrow rectangular section while axial force variances differ a little.

The uniform stability regions are significantly enlarged when the damping coefficient increases, and the stability surfaces are lower when geometric ratio ℓ/h decreases.

When parameter e is comparable with S ($10<\ell/h< 20$), the shape of the cross-section noticeably influences the stability regions. When $\ell/h>20$, this influence rapidly decreases.

Generally speaking, the narrow rectangular cross section has smaller stability regions than the I-section and greater sensitivity to end moment variance.

Acknowledgements

This work was supported by Ministry of Science and Environmental protection of the Republic of Serbia, through the Project no. 144023.

References

[1] S. Timoshenko, *Theory of Elastic Stability*, McGraw-Hill, New York, 1936.

[2] A. Joshi, S. Suryanarayan, A unified solution for various boundary condition for the coupled flexural–torsional instability of closed thin-walled beam-columns, *International Journal of Solids and Structures* 20 (2) (1984) 167–178.

[3] S.T. Ariaratnam, Stability of Mechanical Systems under Stochastic Parametric Excitation, Lecture Notes in Mathematics, Vol. 294, Springer, Berlin, Heidelberg, New York, 1972, pp. 291–302.

[4] A. Tylikowski, Dynamiczna stateczność plaskiej postaci zginania dwuteowego preta cienkošciennego, *Mechanika, Tom 2, Zeszyl* 4 (1983) 5–19.
[5] J. Li, H. Hua, R. Shen, X. Jin, Stochastic vibration of axially loaded monosymmetric Timoshenko thin-walled beam, *Journal of Sound and Vibration* 274 (2004) 915–938.
[6] R. Pavlović, P. Kozić, Almost sure stability of the thin-walled beam subjected to end moments, *Theoretical and Applied Mechanics* 30 (3) (2003) 193–207.
[7] W.F. Chen, T. Atsuta, *Theory of Beam-Columns, Vol. 2: Space Behavior and Design*, McGraw-Hill, New York, 1976.
[8] F. Kozin, Stability of the Linear Stochastic Systems, Lecture Notes in Mathematics, Vol. 294, Springer, Berlin, Heidelberg, New York, 1972, pp. 186–229.
[9] P.C. Parks, A.J. Pritchard, On the construction and use of Liapunov functionals*, Proceedings of the 4th IFAC Congress*, Technical Session 20, Warszawa, 1969, pp. 59–73.