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Validation of a modal method by use of an appropriate static potential for a plate coupled to a water-filled cavity

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Abstract

This paper deals with the validation of a modal projection method for the low-frequency vibroacoustic response of an elastic structure coupled to a cavity filled with a dense compressible fluid (water).

When using a modal approach for internal vibroacoustic problems with heavy fluids, it is shown that a formulation with internal pressure is not correct because it does not take into account the zero-frequency static solutions of the fluid.

It is better to use the internal displacement potential involving both the static displacement potential and static pressure terms, although the static displacement potential must be evaluated very carefully. For that purpose, the proposed modal method is validated by comparisons with experiments, a fully analytical solution and a medium-frequency numerical approach, for the case of an elastic rectangular plate coupled to a water-filled parallelepipedic cavity.

The validation of the method is focused on the evaluation of the first resonant frequencies of the coupled system, the vibratory response of the plate and the acoustic pressure inside the cavity for a large frequency band.

1. Introduction

The aim of this paper is to present a validation of a modal approach formulated in potential, by a comparison with experiments and other approaches, for the vibroacoustic response of an elastic homogeneous plate coupled to an acoustic water-filled cavity, within the frequency band [0, 5000 Hz]. The vibroacoustic system tested experimentally is a closed box which contains a parallelepipedic cavity, filled with water. This cavity has five rigid walls and is closed at its sixth face by a steel elastic rectangular clamped plate. Within the analysed band this system has modal behaviour. An analytical solution of the overall coupled system can be constructed.

In the low-frequency (LF) domain, a classical method to solve a vibroacoustic system composed of an elastic structure containing an internal fluid, is to use a modal approach. The method uses a variational formulation of the coupled system, and a Ritz–Galerkin projection of the dynamics on modal bases to reduce the size of the problem. Usually, two separated bases of modes are used: one basis of structural modes in

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vacuo and a second basis of acoustic modes in rigid motionless cavity (rigid-walled cavity). These two bases are extracted separately and both are truncated to a reduced number of modes. The method is powerful because the resolution of the coupled system is only performed on a small number of degrees of freedom, which is the number of retained modes (structural and acoustic) in the modal splitting. Several formulations for the fluid are possible and the choice of a formulation depends on the fluids contained in cavities.

- For a cavity containing a light fluid (i.e. air), the classical modal approach is based on a $(\mathbf{u}, p)$ formulation, $\mathbf{u}$ being the physical displacement vector field of the structure and p the internal pressure field within the fluid. This standard approach is the most commonly used and it is fully summarized in Ref. [1, Chapter 8]. The $(\mathbf{u}, p)$ formulation leads to an unsymmetric matrix reduced system.
- When the cavity contains a dense slightly compressible fluid like water, the classical $(\mathbf{u}, p)$ formulation of Ref. [1] is not adapted since it does not take into account the zero-frequency static solutions of the fluid. Internal coupling with heavy fluids for the LF range was studied in Refs. [2–10] for example, and in Refs. [8,11] for the medium-frequency (MF) range.

In Ref. [6], interesting formulations for $(\mathbf{u}, p)$ and $(\mathbf{u}, \phi)$ (where ϕ is the fluid displacement potential) were introduced. In this paper, a new strategy for solving the unsymmetrical coupled system (coming from a modeling by the finite element method (FEM)) was proposed. But the validation of the formulations is limited to a one-dimensional and to a two-dimensional fluid–structure interaction problems. In Ref. [9] and more recently in Ref. [10], a modified $(\mathbf{u}, p)$ formulation is used and the matrix reduced system is symmetric. The author introduces one of the ‘‘quasi static corrections’’ terms, i.e. the static pressure term to take into account the zero-frequency stiffness effect of the fluid. The static pressure term yields an added stiffness matrix for the reduced stiffness matrix of the structure. The mass of the fluid also adds a mass matrix to the reduced mass matrix of the structure. Furthermore, the structural modal basis introduced in this formulation is not the basis of eigenmodes for the structure in vacuo but the basis of the structure coupled to an incompressible fluid. This basis is determined numerically using a finite element model. These papers are essentially theoretical and no applications are shown for validating these approaches. In Ref. [8, Chapter XIII], the authors introduce explicitly the term modeling the ‘‘added mass effect of the fluid’’ but it is only a theoretical approach and the effect of a heavy fluid is not tested and validated. In Ref. [7], this effect was tested and validated on a partially or completely water-filled vertical cylindrical shell, by using a simplest form of the added mass matrix. In this paper, the overall internal vibroacoustic problem is not treated because the internal fluid is considered to be incompressible and it is acting on the structure as an ‘‘apparent dynamic added mass’’. The dynamic problem is treated by FEM and the added mass matrix introduced by the authors leads to good results for the eigenfrequencies of the cylindrical shell filled with water. In Ref. [11], the author uses a $(\mathbf{u}, \psi)$ formulation (where ψ is the velocity potential) with a reduction basis constructed by narrow MF sub-bands. Only the term of static pressure is considered and the application shown deals with a light fluid. In Refs. [2,3,5] the static pressure term is not taken into account in the modal approach for heavy fluids.

This paper presents a $(\mathbf{u}, \phi)$ formulation, using both the static displacement potential term and the static pressure term. This formulation is useable for any fluid (dense or light). The method relies on the use of the two modal bases: one basis for the structure in vacuo and the other for the internal acoustics of a rigid motionless cavity. The $(\mathbf{u}, \phi)$ formulation leads to a symmetric matrix reduced model, although the method does not converge rapidly when using a classical expression of the static displacement potential. For the particular case of a rectangular plate coupled to a parallelepipedic cavity, the convergence of the method can be accelerated accurately if the exact analytical solution of the static displacement potential is introduced. In that case, the static displacement potential can be split into two terms: one term represents the exact static potential of the incompressible fluid and the second term is a particular solution (accessible for a parallelepipedic cavity) for the ‘‘non iso-volumic modes’’. The presence of this term is necessary to drastically improve the convergence of the modal method. This particular solution allows the method to be validated since it can be compared directly to other approaches.

In Section 2, the system tested experimentally is presented. This system is a rigid cylindrical box containing a parallelepipedic cavity entirely filled with water. The cavity is defined by five rigid walls and it is closed at one end by a clamped elastic homogeneous plate. The vibratory response of the plate and the internal pressure within the cavity were measured.

Section 3 is devoted to the presentation of the different methods (theoretical and numerical) used for the problem of internal vibroacoustic response of an elastic rectangular plate coupled to a parallelepipedic cavity, containing an acoustic fluid.

- The geometry of the general problem of the internal coupling between a structure and an acoustic cavity is introduced, and the different domains which appear in the coupling are defined.
- The modal method using a $(\mathbf{u}, \phi)$ formulation, which takes into account the presence of both the static pressure and static displacement potential terms, is then presented in full detail. The formulation uses a classical expression of the static displacement potential. An exact analytical solution of this potential can be constructed for the particular case, tested experimentally.
- The two other solutions used for validating the modal approach for this particular case are presented rapidly. Among these solutions, a fully analytical solution of the overall vibroacoustic problem can be directly constructed. Its results will constitute the reference solution and it will also allow all the other approaches to be validated.

Section 4 is concerned with the simulations performed by all the methods, presented herein. In this section, the convergence of the modal method is studied for different parameters, and the overall comparison between the experiments and all the approaches used is finally presented. This comparison allows us to validate the $(\mathbf{u}, \phi)$ modal approach proposed in this paper for a dense compressible fluid.

This paper is a continuation of a previous paper published in 2003 [12]. In this previous paper, a validation with experiments of the MF numerical method of computation used herein for comparisons, was performed on the same vibroacoustic system, studied here for validating the modal approach.

2. Description of the experimental case

A steel elastic homogeneous plate which has 170 mm length, 150 mm width and 4 mm thickness is clamped by its four edges to a massive box shaped from a 320 mm steel cylinder (see Fig. 1). One end of the box can be clogged by a 50 mm thick plate in order to obtain a parallelepipedic cavity of 310 mm height. The cavity has five rigid walls up to a frequency of about 5000 Hz, and is filled with either air or water. The elastic plate constitutes the sixth face of the cavity and is located at the top of the box. The size of the box is relatively small and was chosen to minimize the mass of the structure (150 kg) which made the handling of the box easier for the test. Within the frequency band [0, 5000 Hz], the overall system has a LF (modal) dynamic behaviour.

The plate is excited by one mechanical force on a point, as shown in Fig. 2. Five points were chosen for comparisons; two points on the structure:

- *Excitation point* of coordinates (0.051, 0.105, 0.31 m) and point of coordinates (0.1385, 0.0625, 0.31 m) (called *Second structural point*), and three points inside the cavity:
- one point just under the excitation point of coordinates (0.051, 0.105, 0.31 m), one point just under the plate of coordinates (0.02, 0.02, 0.31 m) (called *Point* 20 mm $\times$ 20 mm) and another point at the bottom of the cavity of coordinates (0.02, 0.02, 0.0124 m). This latter point is at the vertical of the previous point.

The measured quantities are the frequency-response-functions (FRF) $|\gamma(\omega)/F|$ for the structural points and $|P(\omega)/F|$ for the fluid points within the frequency band [0, 5000 Hz]. The FRF were measured only when the cavity was filled with water.

For the comparisons, the measured quantities are presented in Section 4 by curves in *full black lines*.

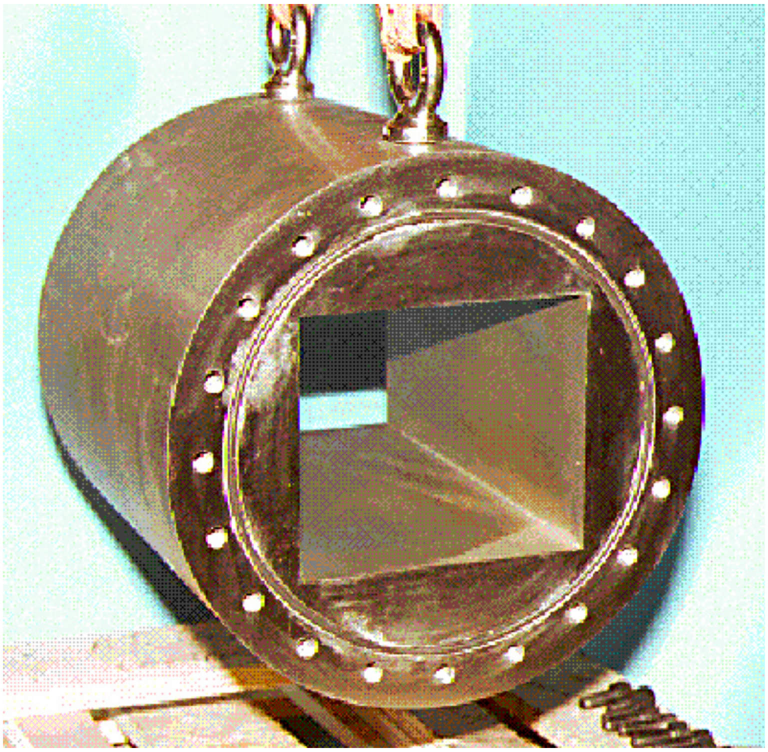

Fig. 1. View of the test structure and cavity.

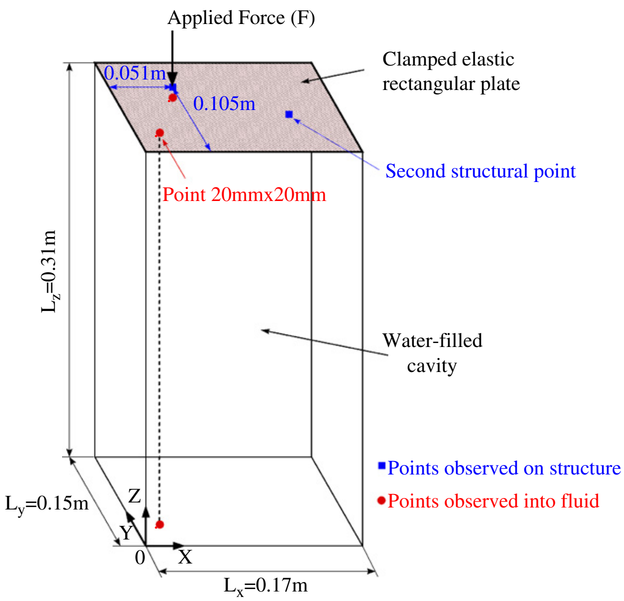

Fig. 2. Description of the analysed vibroacoustic system and selection of measurement points.

3. Theoretical and numerical approaches

3.1. Geometry and notations

Two coupled domains are considered: an elastic structure with volume Ω_s of boundaries Σ_s and Σ_F, and a bounded cavity Ω_F containing an internal fluid. Σ_F is the common interface between the structure and the cavity. The structure is excited by a set of forces F^e over part Σ_e of the boundary Σ_s, as shown in Fig. 3.

When this problem of coupling is composed of a rectangular plate coupled to a parallelepipedic cavity, an analytical solution of the overall vibroacoustic system can be constructed directly. An MF numerical method of computation was also used. These approaches are compared to a modal approach.

3.2. Modal approach using a $(\mathbf{u}, \phi)$ variational formulation with appropriate static behaviour

The $(\mathbf{u}, \phi)$ formulation uses the physical variables: $\mathbf{u} = (u, v, w)$, the displacement of the structure and ϕ, the displacement potential of the fluid within the cavity.

3.2.1. Introduction of static pressure

A static pressure term $\pi(\omega)$, which occurs at $\omega = 0$ when the internal fluid is dense and slightly compressible (like water), was introduced in Ref. [4] in order to take into account the "zero-frequency stiffness effect of the fluid". This term comes from the wall motion effects. It is introduced into the governing equations for structures coupled to internal heavy fluids because internal acoustic cavities are considered with rigid faces in the coupling. The expression of the static pressure with respect to the normal displacement of walls is

$$\pi(\omega = 0) = p_0 = -\frac{\rho_F c_F^2}{V_F} \int_{\Sigma_F} \mathbf{u}.n_F \, \mathrm{d}s. \tag{1}$$

ρ_F, c_F are the fluid mass density and the sound speed in the fluid, V_F being the volume of Ω_F and n_F the outgoing normal from the fluid.

In this way, the relationship between p and ϕ is

$$p = \rho_F \omega^2 \phi + p_0. \tag{2}$$

3.2.2. Equations of motion in $(\mathbf{u}, \phi)$ for the coupled problem

When the pressure p is replaced by a displacement potential ϕ, the governing equations of motion for a structure containing a fluid are:

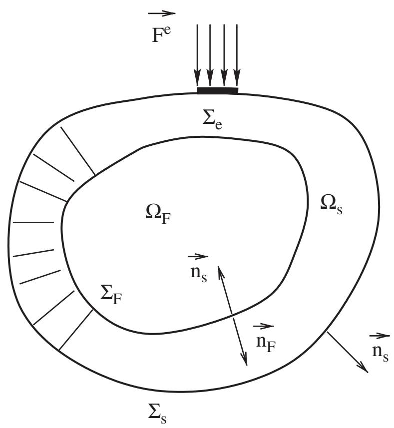

Fig. 3. General case of a structure coupled to an internal cavity.

- for the structure:

$$\begin{cases} \sigma_{ij,j}(\mathbf{u}) + \omega^2 \rho_s u_i = 0 & \text{in } \Omega_s, \quad \text{(a)} \\ \sigma_{ij}(\mathbf{u}) n_{s,j} = F_i^e & \text{on } \Sigma_e, \quad \text{(b)} \\ \sigma_{ij}(\mathbf{u}) n_{s,j} = p n_{F,i} & \text{on } \Sigma_F, \quad \text{(c)} \end{cases} \tag{3}$$

where ρ_s is the structure mass density
- for the fluid:

$$\begin{cases} \Delta\phi + \dfrac{\omega^2}{c_F^2}\phi = \frac{1}{V_F}\int_{\Sigma_F} \mathbf{u}.n_F \, \mathrm{d}s & \text{in } \Omega_F, \quad \text{(a)} \\ \dfrac{\partial\phi}{\partial n_F} = \mathbf{u}.n_F & \text{on } \Sigma_F, \quad \text{(b)} \\ l(\phi) = \int_{\Omega_F} \phi \, \mathrm{d}V = 0 & \text{with } l(1) \neq 0. \quad \text{(c)} \end{cases} \tag{4}$$

3.2.3. Variational formulation

Eliminating p by using Eqs. (2) and (1) for p_0, the $(\mathbf{u}, \phi)$ variational formulation of the coupled problem (3)–(4) can be expressed, for all test regular functions $\partial\mathbf{u}$ on Ω_s and $\partial\phi$ on Ω_F, as follows:

$$\int_{\Omega_s} \sigma_{ij}(\mathbf{u})\varepsilon_{ij}.(\partial\mathbf{u}) \, \mathrm{d}V + \frac{\rho_F c_F^2}{V_F}\left(\int_{\Sigma_F} \mathbf{u}.n_F \, \mathrm{d}s\right)\left(\int_{\Sigma_F} \partial\mathbf{u}.n_F \, \mathrm{d}s\right)$$

$$-\omega^2 \int_{\Omega_s} \rho_s \mathbf{u}.(\partial\mathbf{u}) \, \mathrm{d}V - \omega^2 \int_{\Sigma_F} \rho_F \phi(\partial\mathbf{u}).n_F \, \mathrm{d}s = \int_{\Sigma_e} F^e.(\partial\mathbf{u}) \, \mathrm{d}s, \quad \text{(a)} \tag{5}$$

$$\int_{\Omega_F} \nabla\phi.\nabla(\partial\phi) \, \mathrm{d}V - \frac{\omega^2}{c_F^2}\int_{\Omega_F} \phi.(\partial\phi) \, \mathrm{d}V - \int_{\Sigma_F} \mathbf{u}.n_F(\partial\phi) \, \mathrm{d}s = 0. \quad \text{(b)}$$

The solution of the coupled problem (5) is constructed by projection of the displacement of the structure **u** and the internal potential ϕ on the two modal bases of the structure in vacuo and of the acoustic cavity with rigid walls.

(1) The projection of **u** is

$$\mathbf{u} = \sum_{\beta=1}^{N_s} u_\beta \varphi_\beta, \tag{6}$$

N_s being the number of structural eigenmodes in vacuo retained in the projection, φ_β the structural mode shape and u_β the structural generalized coordinate of mode β.

(2) For the projection of ϕ, a static displacement potential ϕ_0 is introduced, such that:

$$\phi = \phi_0 + \sum_{\beta=1}^{N_F} \phi_\beta \psi_\beta, \tag{7}$$

N_F being the number of acoustic eigenmodes in the rigid motionless cavity retained in the projection, ψ_β the acoustic mode shape, ϕ_β the acoustic generalized coordinate of mode β and ϕ_0 the solution of the acoustic problem (4) for $\omega = 0$.

3.2.4. Classical expression of static displacement potential ϕ_0

One way to introduce the static displacement potential ϕ_0 is to use the expression (7.55), p. 144 of Ref. [1] which is a modal expansion of ϕ_0 on the acoustic eigenmodes ψ_α. In Ref. [1], ϕ_0 has the following expression:

$$\phi_0 = \sum_{\alpha \geqslant 1} \frac{1}{\mu_\alpha^F}\left(\int_{\Sigma_F} \rho_F \psi_\alpha \mathbf{u}.n_F \, \mathrm{d}s\right)\psi_\alpha, \tag{8}$$

where μ_α^F is the generalized mass of acoustic eigenmode ψ_α (μ_α^F will be defined later).

By using projection (6) of the structural displacement $\mathbf{u}$ and a truncature $\alpha = \{1, \ldots, N_F\}$ of the acoustic modes, the following approximation of ϕ_0 can be constructed:

$$\phi_0 \simeq \sum_{\beta=1}^{N_s} \sum_{\alpha=1}^{N_F} \frac{1}{\mu_\alpha^F} \left(\int_{\Sigma_F} \rho_F \psi_\alpha \varphi_\beta . n_F \, \mathrm{d}s \right) \psi_\alpha u_\beta. \quad (9)$$

ϕ_0 is not exact because it is truncated and it depends on the number of retained modes in the projection.

3.2.5. Projection of variational formulation on modal bases

Structural eigenmodes φ_β, for $\beta = \{1, \ldots, N_s\}$ are orthogonal. Acoustic eigenmodes ψ_β, for $\beta = \{1, \ldots, N_F\}$ are also orthogonal.

For the studied test case (a clamped rectangular plate coupled to a parallelepipedic cavity), the eigenfunctions are known analytically. The structural mode shapes are taken from Ref. [13] and the acoustic mode shapes are given in Appendix B.

When using Eq. (9) for ϕ_0, Eq. (7) for ϕ, $\partial \mathbf{u} = \varphi_\gamma$ and $\partial \phi = \psi_\gamma$ as test functions, the variational formulation (5) of coupled problems (3)–(4), projected on structural and acoustic modal bases, becomes:

$$\begin{cases} \lambda_\gamma^s \mu_\gamma^s u_\gamma + \dfrac{\rho_F c_F^2}{V_F} \sum\limits_{\beta=1}^{N_s} \left(\int_{\Sigma_F} \varphi_\gamma \, \mathrm{d}s \right) \left(\int_{\Sigma_F} \varphi_\beta \, \mathrm{d}s \right) u_\beta \\ \quad = \omega^2 \left\{ \mu_\gamma^s u_\gamma + \sum\limits_{\beta=1}^{N_s} \left[\sum\limits_{\alpha=1}^{N_F} \dfrac{C_{\beta\alpha} C_{\gamma\alpha}}{\mu_\alpha^F} \right] u_\beta + \sum\limits_{\beta=1}^{N_F} C_{\gamma\beta} \phi_\beta \right\} + F_\gamma \quad \forall \gamma = \{1, \ldots, N_s\}, \quad \text{(a)} \\ \lambda_\gamma^F \mu_\gamma^F \phi_\gamma = \omega^2 \left\{ \mu_\gamma^F \phi_\gamma + \sum\limits_{\beta=1}^{N_s} C_{\beta\gamma} u_\beta \right\} \quad \forall \gamma = \{1, \ldots, N_F\} \quad \text{(b)} \end{cases} \quad (10)$$

with:

- $\mu_\gamma^s = \int_{\Omega_s} \rho_s \varphi_\gamma . \varphi_\gamma \, \mathrm{d}V$, the generalized mass of structural mode φ_γ;
- $\lambda_\gamma^s = \omega_\gamma^{s2}$, the squared eigenfrequency of structural mode φ_γ;
- $\lambda_\gamma^F = \omega_\gamma^{F2}$, the squared eigenfrequency of acoustic mode ψ_γ;
- $\mu_\gamma^F = \rho_F \int_{\Omega_F} \nabla \psi_\gamma . \nabla \psi_\gamma \, \mathrm{d}V = (\lambda_\gamma^F \rho_F / c_F^2) \int_{\Omega_F} \psi_\gamma . \psi_\gamma \, \mathrm{d}V$;
- $F_\gamma = \int_{\Sigma_e} F^e \varphi_\gamma \, \mathrm{d}s$, the generalized force of structural mode φ_γ;
- $C_{\beta\gamma} = \int_{\Sigma_F} \rho_F \varphi_\beta . n_F \psi_\gamma \, \mathrm{d}s$, the coupling term between structural mode φ_β and acoustic mode ψ_γ.

By introducing the following matrices and vectors:

$\mathbf{M}_s = [M_s]_{\alpha\beta} = \mu_\alpha^s \delta_{\alpha\beta}$, the generalized mass matrix of structure, size $N_s \times N_s$;
$\mathbf{K}_s = [K_s]_{\alpha\beta} = \lambda_\alpha^s \mu_\alpha^s \delta_{\alpha\beta}$, the generalized stiffness matrix of structure, size $N_s \times N_s$;
$\mathbf{M}_F = [M_F]_{\alpha\beta} = \mu_\alpha^F \delta_{\alpha\beta}$, the generalized mass matrix of internal fluid, size $N_F \times N_F$;
$\mathbf{K}_F = [K_F]_{\alpha\beta} = \lambda_\alpha^F \mu_\alpha^F \delta_{\alpha\beta}$, the generalized stiffness matrix of internal fluid, size $N_F \times N_F$;
$\mathbf{M}_{\mathrm{ad}} = [M_{\mathrm{ad}}]_{\alpha\beta} = \sum_{\gamma=1}^{N_F} \dfrac{C_{\alpha\gamma} C_{\beta\gamma}}{\mu_\gamma^F}$, size $N_s \times N_s$;
$\mathbf{K}_{\mathrm{ad}} = [K_{\mathrm{ad}}]_{\alpha\beta} = \dfrac{\rho_F c_F^2}{V_F} \left(\int_{\Sigma_F} \varphi_\alpha \, \mathrm{d}s \right) \left(\int_{\Sigma_F} \varphi_\beta \, \mathrm{d}s \right)$, size $N_s \times N_s$;
$\mathbf{C} = [C]_{\alpha\beta} = C_{\alpha\beta}$, the matrix of coupling, size $N_s \times N_F$;
$\mathbf{F} = \{F_\alpha\}$, the vector of generalized forces (which loads the structure), size N_s;
$\mathbf{U} = \{u_\alpha\}$, the vector of unknown generalized displacement of structure, size N_s;
$\mathbf{\Phi} = \{\phi_\alpha\}$, the vector of unknown generalized internal potential, size N_F;
the above reduced system (10) can be put under the following symmetric matrix form:

$$\begin{bmatrix} -\omega^2 \{\mathbf{M}_s + \mathbf{M}_{\mathrm{ad}}\} + \{\mathbf{K}_s + \mathbf{K}_{\mathrm{ad}}\} & -\omega^2 \mathbf{C} \\ -\omega^2 \mathbf{C}^{\mathrm{T}} & \mathbf{K}_F - \omega^2 \mathbf{M}_F \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{\Phi} \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ 0 \end{bmatrix}. \quad (11)$$

3.2.6. Introduction of damping matrices

If we denote by $\mathbf{C}_s$ the generalized damping matrix of the structure and by $\mathbf{C}_F$ the generalized damping matrix of internal acoustics, these matrices usually are defined by:

$$\mathbf{C}_s = [C_s]_{\alpha\beta} = \eta_s \omega_\alpha^s [M_s]_{\alpha\beta},$$

$$\mathbf{C}_F = [C_F]_{\alpha\beta} = \tau_F [K_F]_{\alpha\beta},$$

where η_s is the mean structural damping loss factor, and coefficient τ_F is due to the viscosity of the internal fluid and is defined in Ref. [8, p. 181], by

$$\tau_F = \frac{4}{3}\frac{\eta_F}{\rho_F c_F^2}, \tag{12}$$

with ρ_F, c_F defined previously and η_F the fluid dynamic viscosity.

By using the damping matrices defined above, the matrix reduced system (11) to be solved finally becomes:

$$\begin{bmatrix} -\omega^2\{\mathbf{M}_s + \mathbf{M}_{\text{ad}}\} + \mathrm{j}\omega\mathbf{C}_s \ldots & -\omega^2\mathbf{C} \\ +\{\mathbf{K}_s + \mathbf{K}_{\text{ad}}\} & \\ -\omega^2\mathbf{C}^{\mathrm{T}} & \mathbf{K}_F - \omega^2\mathbf{M}_F + \mathrm{j}\omega\mathbf{C}_F \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{\Phi} \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ 0 \end{bmatrix}. \tag{13}$$

Matrices $\mathbf{M}_s$, $\mathbf{K}_s$, $\mathbf{C}_s$, $\mathbf{M}_F$, $\mathbf{K}_F$ and $\mathbf{C}_F$ are all diagonal, but matrices $\mathbf{M}_{\text{ad}}$, $\mathbf{K}_{\text{ad}}$ and $\mathbf{C}$ are full. Matrices $\mathbf{M}_{\text{ad}}$ and $\mathbf{K}_{\text{ad}}$ characterize a change in the behaviour of the structure. In this formulation, mass of the structure is modified by an added mass, and its stiffness by an added stiffness which characterize the effect on the structure of the quasi-incompressibility of the fluid. When the fluid is light the fluid mass and stiffness effects are insignificant, but when the fluid is dense like water these effects are not negligible.

3.2.7. Solutions of the coupled problem

The matrix reduced system (13) is the general modal problem to be solved in the LF domain for the coupling between a structure and an internal fluid for any structure and cavity geometry. This modal system is solved for any circular frequency ω by classical matrix inversion (or matrix decomposition) of the first member. The solution is the generalized coordinates $\mathbf{U}$ and $\mathbf{\Phi}$.

The computation of vectors $\mathbf{U}$ and $\mathbf{\Phi}$ leads to the solution for the displacement of the structure $\mathbf{u}$ by using the modal splitting (6) and for the internal pressure p inside the cavity by using Eq. (2), where ϕ and p_0 are expressed as follows:

$$\phi = \sum_{\alpha=1}^{N_F} \left\{ \left(\sum_{\beta=1}^{N_s} \frac{C_{\beta\alpha}}{\mu_\alpha^F} u_\beta \right) + \phi_\alpha \right\} \psi_\alpha, \tag{14}$$

$$p_0 = -\frac{\rho_F c_F^2}{V_F} \sum_{\beta=1}^{N_s} \left(\int_{\Sigma_F} \varphi_\beta \, \mathrm{d}s \right) u_\beta. \tag{15}$$

3.2.8. Particular case

For the studied case (a rectangular plate coupled to a parallelepipedic cavity), an exact analytical solution of ϕ_0 can be constructed. This solution accelerates the convergence of the eigenfrequencies of the coupled system and accurately improves the vibroacoustic response of the system.

The new formulation of the reduced model is fully detailed in Appendix A.

3.3. Other solutions

3.3.1. Analytical solution of the overall system

For the tested system, an analytical solution of the overall vibroacoustic problem can be constructed explicitly. This solution has already been fully presented in Appendix C of a previous paper [12]. The method was adapted to the studied test case from the formulation contained in Ref. [14].

Two simulations were performed: the first simulation with water (see Ref. [12]) and the second simulation with air. The results obtained by this method are presented in Section 4 by curves in *dashed black lines*.

3.3.2. MF numerical computation

A computation using an adapted medium-frequency/finite element method was performed on the system in order to predict the internal noise and vibration levels within the frequency band [100, 5000 Hz].

This method is fully detailed in Refs. [15,16]. It is specific to MF domain and, for the proposed test case, it was validated on measurements in Ref. [12] for a LF dynamic behaviour of the system. In this paper, the method itself is summarized in Appendix B.

A finite element model of the overall structure was developed and two simulations were performed; the first with water and the second with air. The results obtained by this approach are presented in Section 4 by curves in *full dark-grey lines*.

4. Simulations performed

4.1. Plate mechanical parameters introduced in simulations

- For simulations when the cavity contains air, the plate mechanical parameters have the following values: $E = 1.7 \times 10^{11}$ Pa, $\rho_s = 7800\,\mathrm{kg/m^3}$ and $\nu = 0.3$.
- For simulations with water, the parameter values are: $E = 1.62 \times 10^{11}$ Pa, $\rho_s = 7800\,\mathrm{kg/m^3}$ and $\nu = 0.3$.

The gap of 18% with respect to the design value of 2.1×10^{11} Pa of steel Young's modulus is due to the imperfect clamping of the plate. The change of value for E when the cavity contains water instead of air is due to the modification of the plate dynamics after having taken on and taken down the box.

4.2. Calibration of the analytical and numerical models

The plate mechanical parameters used above allow us to calibrate the analytical and numerical models on measurements, by comparing the first six eigenfrequencies of the plate which are given in Table 1.

As one can see in Table 1, six structural modes of the plate in vacuo are contained in the frequency band [0, 5000 Hz].

4.3. Parameters introduced for acoustic cavity

The values of ρ_F, c_F and η_F introduced in the simulations are the following:

- $\rho_F = 1000\,\mathrm{kg/m^3}$, $c_F = 1500\,\mathrm{m/s}$ and $\eta_F = 0.001\,\mathrm{kg/(m \times s)}$ when internal fluid is water.
- $\rho_F = 1.3\,\mathrm{kg/m^3}$, $c_F = 340\,\mathrm{m/s}$ and $\eta_F = 1.7 \times 10^{-3}\,\mathrm{kg/(m \times s)}$ when internal fluid is air.

4.4. Damping values introduced in simulations

The mean structural damping factor η_s introduced for the structure was taken at a constant value of 0.8% over the whole frequency band [0, 5000 Hz]. This value represents the averaged value of modal dampings measured for each mode during a LF identification of the first modes of the plate fixed to the box.

The acoustic coefficient τ_F is calculated by Eq. (12) from the value of the internal fluid dynamic viscosity η_F.

4.5. Structural modal basis introduced

Simulations with the modal approach were performed using the first 22 structural modes of the plate in vacuo. Their mode shapes and eigenfrequencies are analytical and are given by Warburton in Ref. [13].

Table 1
First eigenfrequencies of the plate

	Plate coupled to air (or in vacuo)		Plate coupled to water	
	Measured frequency (Hz)	Analytical frequency (Hz)	Measured frequency (Hz)	Analytical frequency (Hz)
Mode 1	1293.	1286.	1129.	1154.
Mode 2	2424.	2426.	1654.	1641.
Mode 3	2804.	2825.	1952.	1952.
Mode 4	3866.	3884.	2687.	2686.
Mode 5	4224.	4259.	2908.	2877.
Mode 6	NI	5184.	3386.	3422.

4.6. Results when using the classical $(\mathbf{u}, p)$ formulation

4.6.1. Acoustic modal basis introduced for air

For the modal simulations, 196 analytical acoustic mode shapes and associated frequencies were used, and they are given in Appendix B. The fundamental acoustic frequency is 548.4 Hz and the frequency band [0, 5000 Hz] contains about 150 acoustic modes.

4.6.2. Acoustic modal basis introduced for water

For the modal simulations, 49 analytical acoustic modes and associated frequencies were used. The fundamental acoustic frequency is 2419.4. Hz and the frequency band [0, 5000 Hz] contains only four acoustic modes.

4.6.3. Comparison of $(\mathbf{u}, p)$ results with other approaches

We give herein the comparisons of the acoustic results for an air and a water-filled cavity, obtained when using the classical $(\mathbf{u}, p)$ formulation for the modal approach. This formulation is the standard formulation commonly used for light fluids and coming from Ref. [1, Chapter 8].

In Fig. 4 for an air-filled cavity, we have added (for comparison) the results coming from the modal simulation using the $(\mathbf{u}, \phi)$ formulation we have presented. One can see that all the methods agree with each other and there is no noticeable difference between the $(\mathbf{u}, p)$ and the $(\mathbf{u}, \phi)$ formulations for the internal pressure.

One can also see that the acoustic response within the cavity is controlled by about 150 internal modes contained in the analysed frequency band. This acoustic behaviour is perfectly well-restored by all the methods. Finally, these results show a weak coupling between the plate and the cavity.

In Fig. 5 for a water-filled cavity, one can see that the acoustic response within the cavity is controlled by modified structural modes; this characterizes a strong coupling between the plate and the cavity.

The analytical and MF numerical methods agree well with experiments, although the internal acoustic behaviour is not well simulated by the modal approach when using the classical $(\mathbf{u}, p)$ formulation.

From this we can deduce that the modal method using a $(\mathbf{u}, p)$ formulation is not the correct approach for dense slightly compressible fluids like water. A more appropriate formulation is a $(\mathbf{u}, \phi)$ formulation which is also relevant for light fluids as one can see on the results of Fig. 4, for the case of coupling with air.

4.7. Results for water when using $(\mathbf{u}, \phi)$ formulation

4.7.1. Convergence of eigenfrequencies of the coupled system

4.7.1.1. Influence of expression used for static potential. Table 2 gives the value of the first six eigenfrequencies of the "plate coupled to a water-filled cavity" system, with respect to the number of

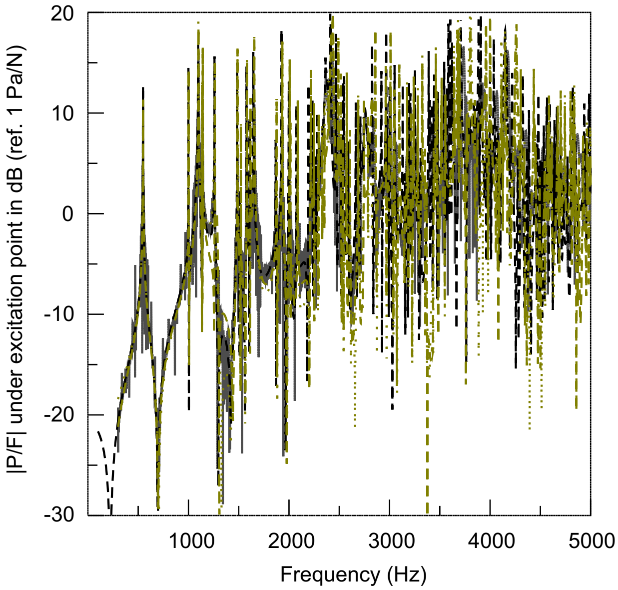

Fig. 4. Comparison of acoustic pressure levels inside fluid for point under the excitation point when cavity contains air. ---- Analytical approach, — MF computation, ······ $(\mathbf{u}, p)$ formulation, ---- $(\mathbf{u}, \phi)$ formulation.

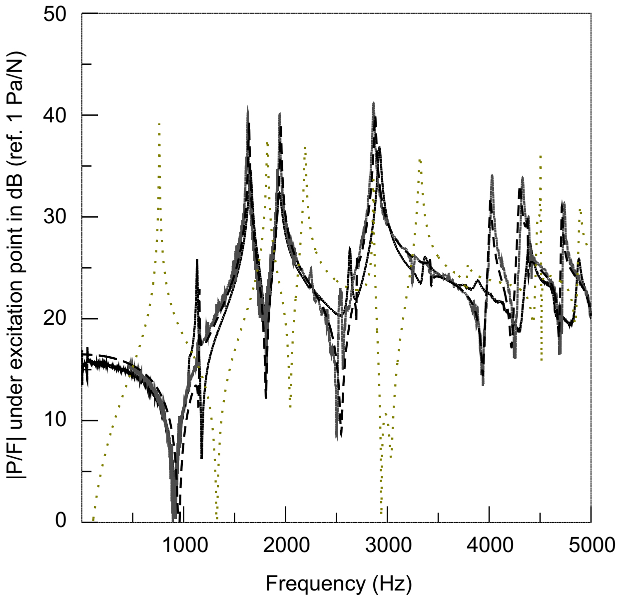

Fig. 5. Comparison of acoustic pressure levels inside fluid for point under the excitation point when cavity contains water. — Measurement, ---- Analytical approach, — MF computation, ······ $(\mathbf{u}, p)$ formulation.

acoustic modes introduced in the coupled system, and to the expression used for the static displacement potential ϕ_0.

Fig. 6 shows the convergence of these frequencies (toward measured and analytical frequencies) in a graphic form. We have calculated the relative frequency difference between the six computed frequencies and the measured and analytical frequencies (which are given in Table 1), with respect to the variation of the number of acoustic modes.

The four curves represent the arithmetic average of the errors (in percent), calculated for each of the six frequencies.

- The two curves in *grey* represent the mean relative error of the frequencies, when the classical expression (9) is used for ϕ_0.
- The two curves in *black* represent the mean relative error of the frequencies, when the exact solution $(\phi_0^0 + \phi_0^1)$ is used for ϕ_0.

As one can see in Table 2 and from Fig. 6, the convergence of the eigenfrequencies of the coupled system contained in the frequency band [0, 5000 Hz] is very slow when using the classical expression of the static displacement potential ϕ_0. This convergence has not yet been obtained for 1500 acoustic modes. For this number of modes, the residual error on the computed eigenfrequencies compared to the analytical and measured frequencies is about 2.4%, although this error decreases when the number of acoustic modes increases. Introduction of the exact solution $(\phi_0^0 + \phi_0^1)$ for the static displacement potential obviously improves the convergence of the eigenfrequencies. The convergence is obtained rapidly with a small number of acoustic modes (10–15 modes) and the residual error is about 0.22%.

4.7.1.2. Influence of ϕ_0^1 term. Table 3 shows the influence of ϕ_0^1 term in the exact solution of the static potential on the convergence of the first six eigenfrequencies of the "plate coupled to a water-filled cavity" system, with respect to the number of acoustic modes introduced in the coupled system.

As it can be observed in Table 3 where ϕ_0^1 term is not used, the convergence of the first six eigenfrequencies of the coupled system is obtained for iso-volumic modes only; modes where one of the two wavenumbers m or n is odd ($m = 3$ or $n = 3$). For these modes, the term $\int_S \varphi_\beta \, ds$, contained in $\mathbf{M}_{ad}$ matrix (defined by Eq. (A.9)) and $\mathbf{K}_{ad}$ matrix, is perfectly equal to zero. Therefore, ϕ_0^1 term has no influence on these modes.

The modes, where m and n are even ($m \geqslant 2$ and $n \geqslant 2$), are the "non iso-volumic modes", for which the term $\int_S \varphi_\beta \, ds$ is different from zero.

Table 2
Convergence of eigenfrequencies of "plate coupled to a water-filled cavity" system

Number of acoustic modes	Computed frequencies (Hz)					
	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5	Mode 6
$(\mathbf{u}, \phi)$ *formulation using classical static potential* ϕ_0						
5	1298.	2032.	2601.	3138.	3784.	4087.
10	1298.	1953.	2328.	3138.	3527.	4087.
20	1248.	1912.	2279.	2992.	3379.	3909.
50	1218.	1820.	2187.	2895.	3290.	3808.
100	1200.	1763.	2108.	2843.	3159.	3719.
200	1187.	1730.	2066.	2799.	3076.	3611.
500	1177.	1701.	2028.	2769.	3011.	3550.
1000	1172.	1688.	2011.	2754.	2979.	3521.
1500	1170.	1683.	2005.	2747.	2965.	3507.
$(\mathbf{u}, \phi)$ *formulation using exact static potential* $(\phi_0^0 + \phi_0^1)$						
5	1155.	1650.	1976.	2716.	2920.	3457.
10	1155.	1647.	1961.	2715.	2899.	3456.
20	1154.	1646.	1959.	2709.	2890.	3439.
50	1154.	1645.	1958.	2705.	2887.	3432.
100	1154.	1644.	1957.	2704.	2884.	3429.
200	1154.	1644.	1956.	2704.	2883.	3427.
500	1154.	1644.	1956.	2703.	2883.	3426.
1000	1154.	1644.	1956.	2703.	2883.	3426.
1500	1154.	1644.	1956.	2703.	2883.	3426.

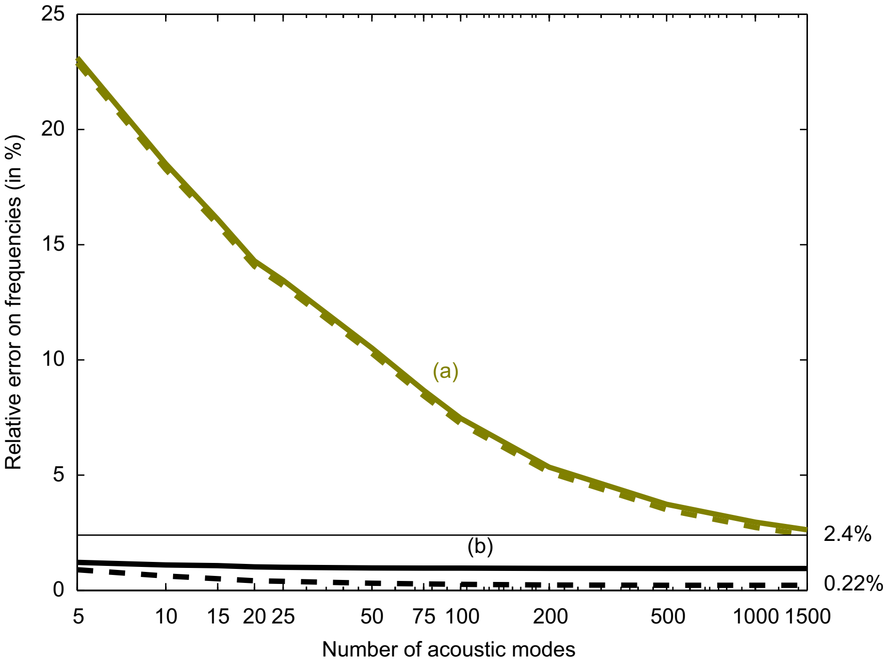

Fig. 6. Influence of the number of acoustic modes and of the expression of static potential on convergence of the first eigenfrequencies of the coupled system: "plate coupled to a water-filled cavity". (a) Classical static potential ϕ_0: average of first six frequencies compared to — measured frequencies and ---- *analytical* frequencies (b) Exact static potential $(\phi_0^0 + \phi_0^1)$: average of first six frequencies compared to — *measured* frequencies and ---- *analytical* frequencies.

Table 3
Values of first six eigenfrequencies of "plate coupled to a water-filled cavity" system

$(\mathbf{u}, \phi)$ *formulation using only incompressible potential* ϕ_0^0 (*without* ϕ_0^1) Number of acoustic modes	Computed frequencies (Hz)					
	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5	Mode 6
Wavenumbers (m, n)	(2, 2)	(3, 2)	(2, 3)	(3, 3)	(4, 2)	(2, 4)
5	**1465.**	1650.	1976.	**2775.**	2920.	**3702.**
10	**1465.**	1647.	1961.	**2775.**	2900.	**3702.**
20	**1464.**	1646.	1960.	**2766.**	2890.	**3688.**
50	**1462.**	1645.	1958.	**2762.**	2887.	**3669.**
100	**1462.**	1645.	1956.	**2760.**	2885.	**3665.**
200	**1462.**	1644.	1956.	**2759.**	2883.	**3662.**
500	**1462.**	1644.	1956.	**2759.**	2883.	**3661.**
1000	**1462.**	1644.	1956.	**2758.**	2883.	**3660.**
1500	**1462.**	1644.	1956.	**2758.**	2883.	**3660.**

Therefore, one can see the very great importance of the complementary term ϕ_0^1 on the obtention of the convergence of the 'wet' eigenfrequencies coming from the "non iso-volumic modes".

4.7.1.3. Influence of number of structural modes introduced. Table 4 shows the convergence of the first six eigenfrequencies of the "plate coupled to a water-filled cavity" system, with respect to the number of acoustic modes introduced in the coupled system and also with respect to the number of structural modes in vacuo used.

As it can be seen in Table 4, there is 'no influence' of the number of structural modes in vacuo on the convergence of these eigenfrequencies; even though this number is large (i.e. 100 modes). The convergence of the 'wet' eigenfrequencies only depends on the number of acoustic modes introduced in the system.

4.7.2. Comparisons for $(\mathbf{u}, \phi)$ *formulation with respect to the choice of the static displacement potential and to the number of acoustic modes*

In this section, we will see the influence of the choice of the static displacement potential expression, and the number of acoustic modes introduced in the modal simulations on the vibroacoustic response trends of the coupled system. The results presented below are obtained by three simulations using the modal approach whose varying parameters are:

(1) Use of the classical term ϕ_0 and 49 acoustic modes: curves of results are plotted in *dotted grey lines* (case 1).
(2) Use of the classical term ϕ_0 and 1500 acoustic modes: curves of results are plotted in *dashed grey lines* (case 2).

Table 4
Convergence of eigenfrequencies of "plate coupled to a water-filled cavity" system with respect to the number N_s of used structural modes and the number N_F of used acoustic modes

$(\mathbf{u}, \phi)$ *formulation using classical static potential* ϕ_0		Computed frequencies (Hz)					
N_F	N_s	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5	Mode 6
5	25	1298.	2032.	2601.	3138.	3784.	4087.
5	36	1298.	2031.	2601.	3138.	3784.	4087.
5	49	1298.	2031.	2601.	3137.	3784.	4087.
10	25	1298.	1953.	2328.	3138.	3527.	4087.
10	36	1298.	1953.	2328.	3138.	3527.	4087.
10	49	1298.	1953.	2328.	3137.	3527.	4087.
10	100	1298.	1952.	2328.	3137.	3527.	4087.
20	25	1248.	1912.	2279.	2992.	3379.	3909.
20	36	1248.	1911.	2279.	2992.	3377.	3909.
20	49	1248.	1911.	2279.	2990.	3377.	3909.
50	25	1218.	1820.	2187.	2895.	3290.	3808.
50	36	1218.	1819.	2186.	2895.	3289.	3808.
50	49	1218.	1819.	2186.	2895.	3289.	3808.
100	25	1200.	1763.	2108.	2843.	3159.	3719.
100	36	1200.	1762.	2106.	2843.	3157.	3719.
100	49	1200.	1762.	2106.	2841.	3157.	3719.
100	100	1200.	1762.	2106.	2841.	3157.	3719.
200	25	1187.	1730.	2066.	2799.	3076.	3611.
200	36	1187.	1730.	2065.	2799.	3075.	3611.
200	49	1187.	1730.	2065.	2798.	3075.	3611.
500	25	1177.	1701.	2028.	2769.	3011.	3550.
500	36	1177.	1701.	2028.	2769.	3010.	3550.
500	49	1177.	1701.	2028.	2767.	3010.	3550.
1000	25	1172.	1688.	2011.	2754.	2979.	3521.
1000	36	1172.	1688.	2011.	2754.	2978.	3521.
1000	49	1172.	1688.	2011.	2753.	2978.	3520.
1000	100	1172.	1688.	2011.	2752.	2978.	3520.
1500	25	1170.	1683.	2005.	2747.	2965.	3507.
1500	36	1170.	1682.	2004.	2747.	2964.	3507.
1500	49	1170.	1682.	2004.	2745.	2964.	3507.

(3) Use of the exact solution $(\phi_0^0 + \phi_0^1)$ and 49 acoustic modes: curves of results are plotted in *mixed grey dash–dotted lines* (case 3).

All these results are only compared to the analytical solution.

Figs. 7 and 8 show the comparison of the internal pressure within the cavity for two points. Convergence in form and in evaluation of resonant frequencies is perfectly obtained for case 3. Convergence in form is also obtained for case 2 and only for evaluation of the first resonant frequency. The values of other resonant frequencies are not yet obtained for 1500 acoustic modes. For case 1, neither convergence in form nor convergence in values of resonant frequencies are obtained, therefore case 1 will be eliminated in the following comparisons. The number of acoustic modes introduced for this case is too small to ensure a correct convergence of the modal approach.

4.7.3. *Overall comparisons with other approaches*

This section is dedicated to the overall comparisons on the vibroacoustic response of the coupled system between experiments and the different approaches used. The three other selected points where measurements of the FRF were performed are included in these comparisons, and results coming from case 1 are eliminated. Figs. 9–11 show the comparisons.

One can see in Figs. 9–11 the perfect agreement with measurements (and with both analytical approach and MF numerical computation) of the $(\mathbf{u}, \phi)$ modal method, when it uses the exact solution $(\phi_0^0 + \phi_0^1)$ of the static displacement potential with a small number of acoustic modes (only 49 modes).

These comparisons show the great importance of getting a very precise computation of the static displacement potential if we want to obtain perfect convergence of the modal method for dense slightly compressible internal fluids.

5. Conclusion

A validation on measurements and by comparison with other approaches (analytical and numerical) of a modal approach using a $(\mathbf{u}, \phi)$ formulation has been performed for a vibroacoustic system composed of an

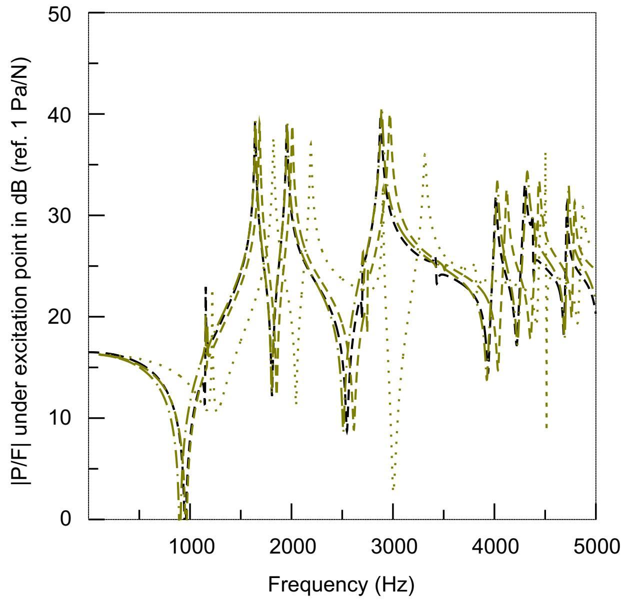

Fig. 7. Comparison of acoustic pressure levels inside fluid for point under the excitation point. $(\mathbf{u}, \phi)$ formulation with variation of number of acoustic modes and of expression of the static displacement potential. ---- Analytical approach, ······ $(\mathbf{u}, \phi)$ formulation using classical ϕ_0 and 49 acoustic modes, --- $(\mathbf{u}, \phi)$ formulation using classical ϕ_0 and 1500 acoustic modes, $- \cdot -$ $(\mathbf{u}, \phi)$ formulation using $(\phi_0^0 + \phi_0^1)$ and 49 acoustic modes.

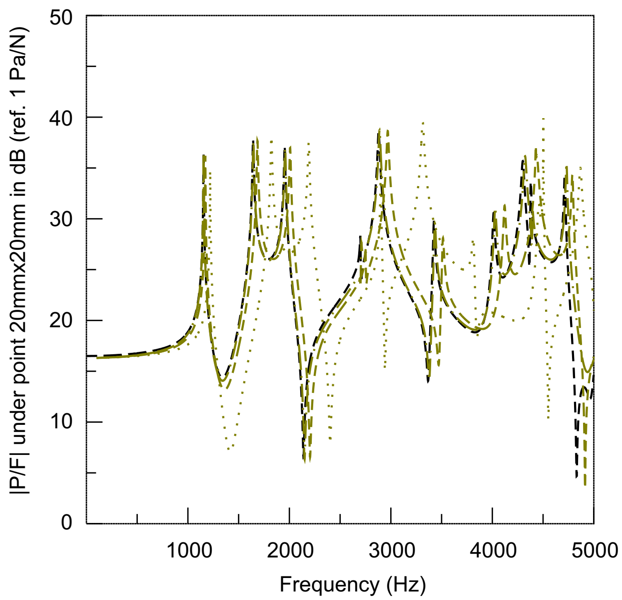

Fig. 8. Comparison of acoustic pressure levels inside fluid for point under the structural point 20 mm × 20 mm. $(\mathbf{u}, \phi)$ formulation with variation of number of acoustic modes and of expression of the static displacement potential. - - - - Analytical approach, ······ $(\mathbf{u}, \phi)$ formulation using classical ϕ_0 and 49 acoustic modes, - - - - $(\mathbf{u}, \phi)$ formulation using classical ϕ_0 and 1500 acoustic modes, $- \cdot -$ $(\mathbf{u}, \phi)$ formulation using $(\phi_0^0 + \phi_0^1)$ and 49 acoustic modes.

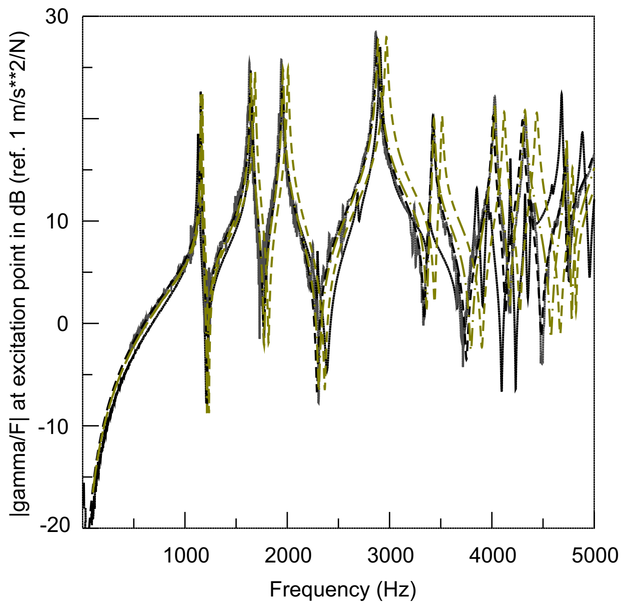

Fig. 9. Comparison of vibratory levels at the excitation point. — Measurement, - - - - Analytical approach, — MF computation, - - - $(\mathbf{u}, \phi)$ formulation using classical ϕ_0 and 1500 acoustic modes, $- \cdot -$ $(\mathbf{u}, \phi)$ formulation using $(\phi_0^0 + \phi_0^1)$ and 49 acoustic modes.

elastic rectangular plate coupled to a water-filled parallelepipedic cavity. The frequency band [0, 5000 Hz] has been analysed. Within this band, the studied vibroacoustic system has modal behaviour. Therefore, a modal approach is well-adapted to deal with it. For this system, the application of the modal approach is essentially analytical (eigenmodes, eigenfrequencies, algebraic integrals and quadratures), which avoids numerical approximations and which makes the validation very interesting from a qualitative point of view. The most

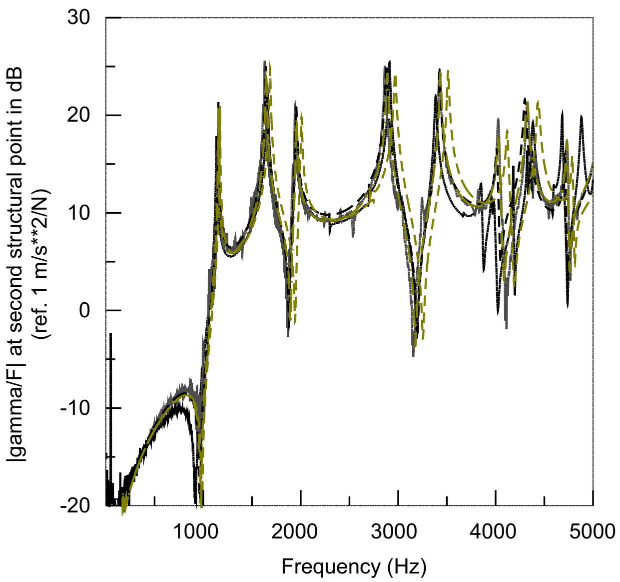

Fig. 10. Comparison of vibratory levels at the second structural point. — Measurement, ---- Analytical approach, — MF computation, --- $(\mathbf{u}, \phi)$ formulation using classical ϕ_0 and 1500 acoustic modes, $- \cdot -$ $(\mathbf{u}, \phi)$ formulation using $(\phi_0^0 + \phi_0^1)$ and 49 acoustic modes.

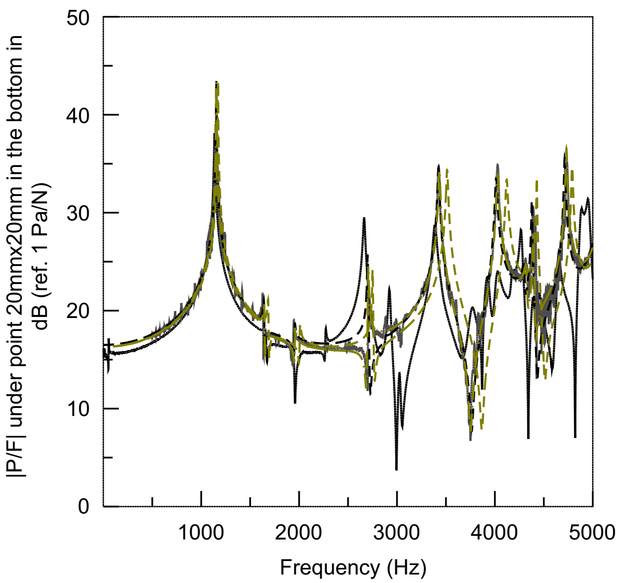

Fig. 11. Comparison of acoustic pressure levels inside fluid for point under the structural point 20 mm × 20 mm in the bottom of box. — Measurement, ---- Analytical approach, — MF computation, --- $(\mathbf{u}, \phi)$ formulation using classical ϕ_0 and 1500 acoustic modes, $- \cdot -$ $(\mathbf{u}, \phi)$ formulation using $(\phi_0^0 + \phi_0^1)$ and 49 acoustic modes.

interesting part of the modal method is that it relies on the use of two decoupled modal bases: a structural basis for the structure in vacuo and an acoustic basis for the rigid-walled cavity. Extraction of these two modal bases is essential for complex systems because they can be obtained separately and it is relatively easy to do it with a standard finite element code dedicated to dynamics and internal acoustics.

We have shown first that the standard $(\mathbf{u}, p)$ and proposed $(\mathbf{u}, \phi)$ formulations are equivalent for light fluids. But a $(\mathbf{u}, \phi)$ formulation using both static pressure and static displacement potential terms is however necessary to deal with dense slightly compressible fluids, such as liquids. Construction of the classical static displacement potential term ϕ_0 is possible for complex vibroacoustic systems, but as we have shown (in this study), the convergence toward the correct resonant frequencies of the coupled system is very slow and requires the introduction of a large number of acoustic modes.

Finally, we have shown that, for the studied test case, the use of the exact expression of the static displacement potential obviously accelerates the convergence of the method. In that case, a small number of acoustic modes is needed. But this particular solution for the static displacement potential is specific to this test case and it cannot be generalized to other systems.

As a consequence of this study, the introduction of the static displacement potential in a $(\mathbf{u}, \phi)$ formulation, calculated in a very precise manner (analytically or by adequate finite element procedures, ...), is of prime importance for the convergence of a modal method for a complex internal vibroacoustic system when cavity contains a liquid.

Otherwise, in order to improve the convergence of the method without constructing the classical term of the static displacement potential, some possible options are to use directly the 'wet' structural eigenfrequencies and the 'wet' structural modes of the "structure coupled with an incompressible fluid".

- A first way is to introduce into the matrix reduced system an empirical expression of the "fluid added mass matrix", such as the one used in Ref. [7].
- A second possibility is to use integral methods for "structures coupled to external unbounded fluids", such as the method developed in Ref. [17], which directly provides a correct evaluation of the 'wet' structural eigenfrequencies from the data of the 'dry' eigenfrequencies and from the construction (by the method) of the "equivalent fluid added mass matrix". But this latter approach is relatively expensive in its use.

Appendix A. Formulation of the reduced modal system for a rectangular plate coupled to a parallelepipedic cavity

A.1. Analytical expression of the static displacement potential

For a rectangular plate coupled to a parallelepipedic cavity, the static displacement potential ϕ_0 can be split into two terms, such that:

$$\phi_0 = \phi_0^0 + \phi_0^1, \tag{A.1}$$

where ϕ_0^0 represents the potential of incompressible fluid, for which an exact analytical expression can be constructed explicitly, and ϕ_0^1 is a complementary solution for the "non iso-volumic modes".

The two functions ϕ_0^1 and ϕ_0^0 are the solutions of the following acoustic problem:

$$\begin{cases} \Delta\phi_0^1 = \dfrac{1}{V_F}\int_S \mathbf{u}.n_F \,\mathrm{d}s \quad \Delta\phi_0^0 = 0 \quad \text{in } \Omega_F, \\ \dfrac{\partial\phi_0^1}{\partial n} = \dfrac{\partial\phi_0^0}{\partial n} = \dfrac{1}{S}\int_S \mathbf{u}.n_F \,\mathrm{d}s \quad \text{on plate area } S, \\ \dfrac{\partial\phi_0^1}{\partial n} = \dfrac{\partial\phi_0^0}{\partial n} = 0 \quad \text{on rigid faces } (\Sigma_F - S), \\ l(\phi_0^1) = l(\phi_0^0) = 0. \end{cases} \tag{A.2}$$

A.2. Expression of the complementary solution ϕ_0^1

ϕ_0^1, satisfying Eqs. (A.2) and projected on the structural modal basis: φ_β, for $\beta = \{1, \ldots, N_s\}$, has a very simple polynomial solution:

$$\phi_0^1(z) = \frac{1}{2V_F}\left[z^2 - \frac{L_z^2}{3}\right]\sum_{\beta=1}^{N_s}\left(\int_S \varphi_\beta \,\mathrm{d}s\right)u_\beta, \tag{A.3}$$

where z is the axis of the cavity normal to the plate (see Fig. 2).

A.3. Exact expression of the incompressible potential ϕ_0^0

The incompressible potential ϕ_0^0, solution of Eqs. (A.2) for a parallelepipedic cavity whose geometry and orientation of axes are defined in Fig. 2, takes the form:

$$\phi_0^0(x,y,z) = \sum_{(p,q)\neq(0,0)} \Phi_{pq}\cos(k_p x)\cos(k_q y)\cosh\left(z\sqrt{k_p^2+k_q^2}\right). \tag{A.4}$$

The solution, projected on the structural modal basis: φ_β, for $\beta=\{1,\ldots,N_s\}$, is

$$\phi_0^0(x,y,z) = \sum_{\beta=1}^{N_s} \phi_{0,\beta}^0(x,y,z)u_\beta, \tag{A.5}$$

where

$$\phi_{0,\beta}^0(x,y,z) = \sum_{(p,q)\neq(0,0)} \Phi_{pq,\beta}\cos(k_p x)\cos(k_q y)\cosh\left(z\sqrt{k_p^2+k_q^2}\right), \tag{A.6}$$

and

$$\Phi_{pq,\beta} = \frac{\int_S \varphi_\beta \ \cos(k_p x)\cos(k_q y)\,\mathrm{d}s}{L_x^p L_y^q\sqrt{k_p^2+k_q^2}\ \sinh\left(L_z\sqrt{k_p^2+k_q^2}\right)},$$

$$k_p = \frac{p\pi}{L_x}, \quad k_q = \frac{q\pi}{L_y},$$

$$L_x^p = \begin{cases} L_x & \text{if } p=0 \\ \dfrac{L_x}{2} & \text{if } p\neq 0 \end{cases}, \quad L_y^q = \begin{cases} L_y & \text{if } q=0 \\ \dfrac{L_y}{2} & \text{if } q\neq 0 \end{cases}.$$

The final expression of $\phi_0^0(x,y,z)$ is then:

$$\phi_0^0(x,y,z) = \sum_{\beta=1}^{N_s} u_\beta \sum_{(p,q)\neq(0,0)} \frac{\int_S \varphi_\beta \cos(k_p x)\cos(k_q y)\,\mathrm{d}s}{L_x^p L_y^q\sqrt{k_p^2+k_q^2}\ \sinh\left(L_z\sqrt{k_p^2+k_q^2}\right)} \times \cos(k_p x)\cos(k_q y)\cosh\left(z\sqrt{k_p^2+k_q^2}\right). \tag{A.7}$$

A.4. New expression of the variational formulation projected on modal bases

When using $\partial\mathbf{u}=\varphi_\gamma$ and $\partial\phi=\psi_\gamma$ as test functions and solution (A.1) of ϕ_0, the variational formulation (5) of the coupled problem (3)–(4), projected on structural and acoustic modal bases is:

$$\begin{cases} \lambda_\gamma^s\mu_\gamma^s u_\gamma + \dfrac{\rho_F c_F^2}{V_F}\sum\limits_{\beta=1}^{N_s}\left(\int_S \varphi_\gamma\,\mathrm{d}s\right)\left(\int_S \varphi_\beta\,\mathrm{d}s\right)u_\beta \\ \quad = \omega^2\mu_\gamma^s u_\gamma + \omega^2\sum\limits_{\beta=1}^{N_s}\rho_F\left[\dfrac{L_z}{3S}\left(\int_S \varphi_\gamma\,\mathrm{d}s\right)\left(\int_S \varphi_\beta\,\mathrm{d}s\right) + V_{\beta\gamma}\right]u_\beta \\ \qquad +\omega^2\sum\limits_{\beta=1}^{N_F} C_{\gamma\beta}\phi_\beta + F_\gamma \quad \forall\gamma=\{1,\ldots,N_s\}, \quad \text{(a)} \\ \lambda_\gamma^F\mu_\gamma^F\phi_\gamma = \omega^2\left\{\mu_\gamma^F\phi_\gamma + \sum\limits_{\beta=1}^{N_s} C_{\beta\gamma}u_\beta\right\} \quad \forall\gamma=\{1,\ldots,N_F\}. \quad \text{(b)} \end{cases} \tag{A.8}$$

For the problem (A.8) above, the matrix reduced system to be solved is the initial linear system (13) in which the added mass matrix $\mathbf{M}_{\mathrm{ad}}$ of size $N_s \times N_s$ is now defined by the following expression:

$$\mathbf{M}_{\mathrm{ad}} = [M_{\mathrm{ad}}]_{\alpha\beta} = \rho_F \left[\frac{L_z}{3S} \left(\int_S \varphi_\alpha \, \mathrm{d}s \right) \left(\int_S \varphi_\beta \, \mathrm{d}s \right) + V_{\alpha\beta} \right], \tag{A.9}$$

where

$$V_{\alpha\beta} = \sum_{(p,q) \neq (0,0)} \int_S \varphi_\alpha \cos(k_p x) \cos(k_q y) \, \mathrm{d}s \times \int_S \varphi_\beta \cos(k_p x) \cos(k_q y) \, \mathrm{d}s \times \frac{\coth\left(L_z \sqrt{k_p^2 + k_q^2} \right)}{L_x^p L_y^q \sqrt{k_p^2 + k_q^2}}. \tag{A.10}$$

A.5. New solutions of the coupled problem

Problem (A.8) is the particular variational formulation of the coupling between an elastic rectangular plate and a parallelepipedic cavity containing an internal fluid. From the computation of vectors $\mathbf{U}$ and $\mathbf{\Phi}$ of the matrix reduced system (13), the final solutions of $\mathbf{u}$ and p are given by Eq. (6) and Eq. (2) where ϕ is expressed as follows:

$$\phi = \phi_0^0 + \phi_0^1 + \sum_{\alpha=1}^{N_F} \phi_\alpha \psi_\alpha, \tag{A.11}$$

with: ϕ_0^1 given by Eq. (A.3) and ϕ_0^0 by Eq. (A.7).

Appendix B. Analytical eigenforms and eigenfrequencies of a rigid-walled parallelepipedic cavity

B.1. Mode shapes

Acoustic eigenmodes can be expressed as
$\psi_{pqr}(x,y,z) = \psi_p(x) \times \psi_q(y) \times \psi_r(z) \quad \forall \{p,q,r\} \geqslant 0$ or as:

$$\psi_{pqr}(x,y,z) = \cos\left(\frac{p\pi}{L_x} x \right) \times \cos\left(\frac{q\pi}{L_y} y \right) \times \cos\left(\frac{r\pi}{L_z} z \right). \tag{B.1}$$

B.2. Eigenfrequencies

Acoustic eigenfrequencies are defined by

$$f_{pqr} = \frac{c_F}{2} \sqrt{\frac{p^2}{L_x^2} + \frac{q^2}{L_y^2} + \frac{r^2}{L_z^2}} \quad \text{(in Hz)}. \tag{B.2}$$

(p,q,r) are the wavenumbers and L_x, L_y and L_z the dimensions of the cavity in the directions x, y, z.

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