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A modelling of an impacted structure based on constraint modes

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Abstract

The modelling of impacted structures may have a high numerical cost. For low velocity impact problem, single degree-of-freedom models may be used efficiently. Nevertheless, sometimes, these models are not suitable to describe the dynamic behaviour of the structure. This paper presents a new model of impacted structures which may be viewed as an extension of the single degree-of-freedom existing models: it allows for the dynamic behaviour of a structure and the numerical cost is low. The fundamental elements of this model are single degree-of-freedom systems called "anti-oscillators": the natural frequencies of these single degree-of-freedom systems correspond to the antiresonant frequencies of the structure. The applications given in this paper show that only a small number of anti-oscillators are required for accurate simulations of an impact event.

1. Introduction

Three-dimensional models are widely used to simulate the impact between two structures: such simulations require nonlinear calculation due to the contact between the structures. Comprehensive results can be obtained with these simulations but the numerical cost is very high.

For low-velocity impact, as considered in this paper, very simple analytical models may be used efficiently to determine the impact force. The structure is modelled as a single degree-of-freedom (dof) system connected to the impactor (a rigid mass) through a nonlinear interaction spring (see e.g. Refs. [1–3]). The stiffness of the single dof system is represented by the static stiffness combined with some other structural aspects (membrane stiffness, shear stiffness,...). Likewise, the stiffness of the interaction spring (stiffness k_{contact}) is derived from a contact law which can be determined experimentally [4]. The Hertz contact law is considered in this paper.

Nevertheless, it is not always possible to model a structure by a single dof system: that is the case, for instance, for a structure with a rigid body mode. Here, the static stiffness is equal to zero; hence the model should be a rigid mass. Moreover, sometimes the structural response may be complex, and the dynamic behaviour must be described with more than one eigenmode. A solution, proposed by Goldsmith [5] and Stronge [6] is to represent the structure in terms of a sum of the mode responses: a suitable truncation is then made to decrease the numerical cost of the simulations.

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The aim of this paper is to propose an alternative method to model an impacted structure, which allows the response to be described accurately with few dofs. These dofs are associated with a specific representation of the displacement field which is described in the next section; a model for an impacted structure is then established. Finally, some applications are presented to show that this new model is adequate to simulate impact events.

The location and the direction of the impact are supposed to be known. To derive analytical results, we considered a beam characterized by the Young's modulus E, the cross-section inertia I, the cross-section S, the length L and the density ρ.

2. Displacement field

The solution of a structural dynamics problem depends on the choice of a kinematic description which may be a mode superposition (available analytically for very few problems), an interpolation from nodal values in case of finite element (FE) analysis, etc. Nevertheless, some descriptions are more relevant than some others: a kinematic field which is near to the actual shape leads to more accurate results. That is why, in this paper, the chosen Rayleigh–Ritz model is based on:

- the "static mode" ϕ_{st} which is the shape caused by a static load F_{st} applied at the impact point in the direction of impact, such that the displacement at the impact location is equal to unity,
- the "constraint modes" which are the eigenmodes of the structure with an extra boundary condition: the displacement in the direction of the impact is zero at the impact location. So the circular eigenfrequencies associated with those modes are the circular antiresonant frequencies $\omega_{\mathrm{AR}\ i}$; the eigenshapes are the functions ϕ_i ($i \geqslant 1$). Note that the set $\{\phi_i,\ \omega_{\mathrm{AR}\ i}\}_{i=1..+\infty}$ depends on the impact point.

The notion of such modes is not new and they are widely used in the Craig and Bampton method [7]. The static mode corresponds to the displacement field used in the single dof models cited in the introduction. But the structure tends to react against the impact: if this reaction were infinite, no displacement would occurred at the impact location: that is why the constraint modes are introduced in our model. Nevertheless, the static mode is not necessarily orthogonal to the constraint modes. So to obtain an expansion of the displacement field in terms of an infinite sum of orthogonal functions, a "residual mode" ϕ_0 is then defined:

- ϕ_0 depends linearly on the static shape and on the set of constraint shapes $\{\phi_i\}_{i>0}$.
- ϕ_0 is orthogonal to the constraint shapes with respect to the mass operator.

Accordingly, $\{\phi_i(x)\}_{i=0..+\infty}$ is a set of orthogonal functions. The residual mode can be written as following:

$$\phi_0(x) = \phi_{\mathrm{st}}(x) - \sum_{i=1}^{+\infty} c_i \phi_i(x). \tag{1}$$

The coefficients c_i may easily be determined, owing to the orthogonal property between ϕ_0 and the constraint modes $\{\phi_i(x)\}_{i=1..+\infty}$:

$$M(\phi_0, \phi_i) = 0 \quad \text{for } i = 1.. + \infty, \tag{2}$$

where M is the mass operator, linked to the kinetic energy; for instance, for a beam (length L, density ρ, cross-section area S), M is given by

$$M(\phi_i, \phi_j) = \int_0^L \rho S \phi_i(x) \phi_j(x)\, \mathrm{d}x. \tag{3}$$

The coefficients $\{c_i\}_{i=1..+\infty}$ of expansion (1) are determined by using the orthogonal property expressed by Eq. (2):

$$c_i = \frac{M(\phi_{\mathrm{st}}, \phi_i)}{M(\phi_i, \phi_i)}. \tag{4}$$

The displacement field is then expanded in terms of an infinite sum of orthogonal functions:

$$w(\mathbf{x},t) = \sum_{i=0}^{+\infty} q_i(t)\,\phi_i(\mathbf{x}), \tag{5}$$

where $\mathbf{x}$ corresponds to the coordinates of a point M on the studied structure. A discretization is achieved by truncating expansion (5) to the rank N:

$$w(\mathbf{x},t) \simeq w^d(\mathbf{x},t) = \sum_{i=0}^{N} q_i(t)\,\phi_i(\mathbf{x}). \tag{6}$$

3. The anti-oscillators

The discretized displacement (6) may also be written as

$$w^d(\mathbf{x},t) = q_0(t)\phi_0(\mathbf{x}) + \sum_{i=1}^{N} q_i(t)\,\phi_i(\mathbf{x}) \tag{7}$$

$$= \lambda_0(t)\psi_0(\mathbf{x}) + \sum_{i=1}^{N}(\lambda_i(t) - \lambda_0(t))\psi_i(\mathbf{x}), \tag{8}$$

where

$$\lambda_0(t) = q_0(t) \quad \psi_0(\mathbf{x}) = \phi_{\mathrm{st}}(\mathbf{x}),$$

$$\lambda_i(t) = \frac{q_i(t)}{c_i} \quad \psi_i(\mathbf{x}) = c_i\phi_i(\mathbf{x}). \tag{9}$$

Actually, this change of variable is very interesting because it leads to represent a structure by the model depicted in Fig. 1: N single dof systems $(m_i, k_i)_{i=1..N}$ lying on a single dof system (m_0, k_0). The dofs are then the $N+1$ parameters $\{\lambda_i\}_{i=0..N}$. The single dof systems $(m_i, k_i)_{i=1..N}$ are referred to be "anti-oscillators": this will be explained in the following.

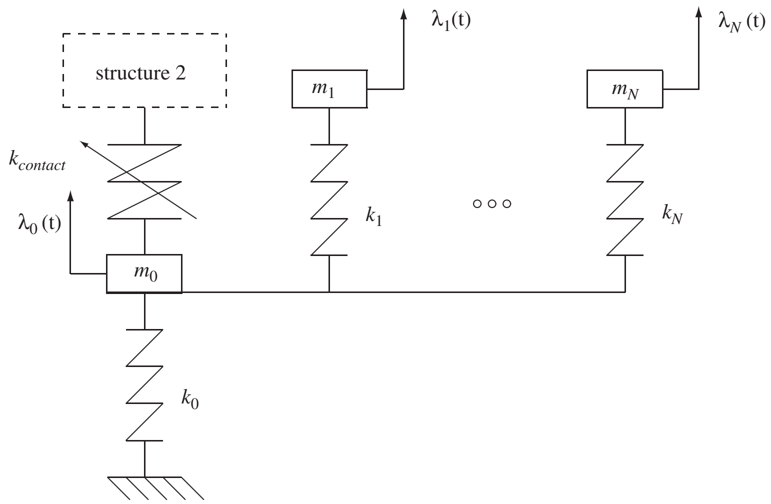

Fig. 1. Model of a structure impacted by another structure.

3.1. Masses $\{m_i\}_{i=0..N}$

The masses are derived from the kinetic energy $\mathscr{T}$ evaluated with the discretized displacement field (6):

$$\mathscr{T} = \frac{1}{2}M(\dot{w}^d, \dot{w}^d) = \frac{1}{2}\sum_{i=0}^{N}\sum_{j=0}^{N}\dot{q}_i M(\phi_i, \phi_j)\dot{q}_j. \tag{10}$$

Using the orthogonality properties of the eigenshapes ϕ_i with respect to the mass operator and definition (9) of the parameters λ_i, one can deduce:

$$\mathscr{T} = \frac{1}{2}\sum_{i=0}^{N} M(\phi_i, \phi_i)\dot{q}_i^2 = \frac{1}{2}M(\phi_0, \phi_0)\dot{\lambda}_0^2 + \frac{1}{2}\sum_{i=1}^{N} M(\phi_i, \phi_i)c_i^2\dot{\lambda}_i^2. \tag{11}$$

So, it is possible to identify the masses represented in Fig. 1:

- for $i>0$:

$$m_i = M(\phi_i, \phi_i)c_i^2 = \frac{M(\phi_{\mathrm{st}}, \phi_i)^2}{M(\phi_i, \phi_i)}, \tag{12}$$

- for $i = 0$:

$$m_0 = M(\phi_0, \phi_0). \tag{13}$$

The definition of the residual shape ϕ_0 given by Eq. (1), the orthogonality property of the ϕ_i and definition (4) of the coefficients c_i, give:

$$m_0 = M(\phi_{\mathrm{st}}, \phi_{\mathrm{st}}) - \sum_{i=1}^{N}\frac{M(\phi_{\mathrm{st}}, \phi_i)^2}{M(\phi_i, \phi_i)} \tag{14}$$

$$= m_{\mathrm{st}} - \sum_{i=1}^{N} m_i. \tag{15}$$

Eq. (15) shows that the mass m_0 is the difference between the "static mass" m_{st} and the sum of the masses $\{m_i\}_{i=1..N}$. Thus m_0 is called residual mass. This is the first difference with the traditional eigenmode analysis which requires a truncation that causes an underestimation of the kinetic energy involved in the impact. For the proposed description the mass is better taken into account owing to the residual mass.

It is worth noting that the masses are independent of the normalization of the eigenshapes $(\phi_i)_{i=1..N}$.

3.2. Stiffnesses $\{k_i\}_{i=0..N}$

The elastic strain energy $\mathscr{V}$ stored in the structure for the discretized displacement field (8) can be written as

$$\mathscr{V} = \frac{1}{2}K(w^d, w^d) \tag{16}$$

$$= \frac{1}{2}K(\psi_0, \psi_0)\lambda_0^2 + \frac{1}{2}\sum_{i=1}^{N}(\lambda_i - \lambda_0)^2 K(\psi_i, \psi_i) + \sum_{i=1}^{N}(\lambda_i - \lambda_0)\lambda_0 K(\psi_0, \psi_i), \tag{17}$$

where $K(\psi_i, \psi_j)$ is the stiffness operator which is related to the deformation energy; for example, for a beam, K is defined by

$$K(\psi_i, \psi_j) = \int_0^L EI\psi_i''(x)\,\psi_j''(x)\,\mathrm{d}x. \tag{18}$$

Note that the orthogonality property of the constraint modes with respect to the stiffness operator is used to obtain relation (17).

Moreover, in Appendix, the following orthogonality property is proved:

$$K(\psi_0, \psi_i) = 0 \quad \text{for } i = 1..N. \tag{19}$$

Hence, the stiffnesses $\{k_i\}_{i=0\cdots N}$ of the model depicted in Fig. 1 can be identified from the expression of the strain energy (17):

- $i = 0$: k_0 is the static stiffness:

$$k_0 = K(\psi_0, \psi_0) = K(\phi_{\text{st}}, \phi_{\text{st}}), \tag{20}$$

- $i = 1 \cdots N$:

$$k_i = K(\psi_i, \psi_i) = c_i^2 K(\phi_i, \phi_i). \tag{21}$$

  Moreover, a constraint mode $(\phi_i, \omega_{\text{AR }i})$ is an eigenmode of the structure with an extra boundary condition: the displacement at the impact location along the impact direction vanishes. Accordingly, the following relation holds:

$$K(\phi_i, \phi_i) = \omega_{\text{AR }i}^2 M(\phi_i, \phi_i), \tag{22}$$

  where the eigenfrequency $\omega_{\text{AR }i}$ of a constraint mode is an antiresonant frequency. Then we have a relation between k_i and m_i:

$$k_i = c_i^2 \omega_{\text{AR }i}^2 M(\phi_i, \phi_i) = \omega_{\text{AR }i}^2 m_i. \tag{23}$$

We can then conclude that the set of natural frequencies for the single dof systems ($i = 1..N$) is a subset of the antiresonant frequencies of the structure: that is why these systems are called anti-oscillators.

3.3. Rigid-body mode

The impacted systems with a rigid-body mode are very common. Such systems can also be modelled with anti-oscillators. To achieve such a modelling:

- the static mode is replaced by the rigid body mode;
- the residual mass is $m_0 = m_{\text{CR}} - \sum_{i=1}^{N} m_i$ with the "rigid-body" mass $m_{\text{CR}} = M(\phi_{\text{CR}}, \phi_{\text{CR}})$,
- $k_0 = 0$.

If the structure has more than one rigid body mode, this method still holds. Indeed, rigid-body modes can always be chosen such as:

- the displacement at the impact location in the impact direction for one of the rigid body modes is equal to one;
- the displacement in the direction of the impact at the impact location for the rest of the modes is equal to zero.

Hence, the first rigid-body mode replaces the static mode and the rest of the rigid-body modes are considered as constrained modes.

3.4. Comments

Fig. 1 shows that the mass matrix obtained with this discretization is a lumped-mass matrix. No special techniques have to be applied to the mass matrix to make it diagonal [8]. Hence this model permits to reduce the numerical cost for time simulation.

The nonlinear interaction between two structures can be described well by using a contact law which depends on the displacement at the impact location: that is why the FE method is suitable to describe an interaction whereas modal analysis needs to superpose the generalized coordinates at each time step. From this point of view, the anti-oscillator model is close to the FE method because the parameter λ_0 has a physical meaning: λ_0 is the displacement of the structure at the impact location.

It can be noted that this model may be considered as an extension of the single dof models described in the introduction of this paper. Indeed, if no anti-oscillators are considered, the structure is modelled by a mass (the residual mass) and the static stiffness: this corresponds to the classical single dof models. Moreover, one can note that a 1-dof model is obviously not accurate for the case where a structure with a rigid-body mode is studied: in that case, the structure would be represented as a mass only. This drawback is overcome with the new model thanks to the anti-oscillators.

4. Study case 1: structure with a rigid-body mode

One of the simplest structure to study analytically is a pinned–free beam which falls and hits a hemicylindrical medium at its free end. The anti-oscillators can easily be defined analytically, as explained in the following.

The studied beam is made up of aluminum and has a square cross-section: the characteristics are listed in Table 1. The hemicylindrical medium can be modelled as an elastic spring which has a nonlinear behaviour following a Hertz contact law. The stiffness is calculated from the curvature: $k_H = 1.55 \times 10^{10}\,\mathrm{N\,m^{-3/2}}$. The free end of the beam hits the spring with a linear velocity $v_0 = 3\,\mathrm{m\,s^{-1}}$: that means that the end of the beam is lifted up to 30 cm and then the end is dropped with no initial velocity. Then the beam underwent a rigid body rotation around the pinned end. The effects of transverse shear and rotary inertia are taken into account to deal with the "high" frequency eigenmodes correctly.

The aim of this example is to show that the anti-oscillator model is relevant to study the impact involving a structure with a rigid-body mode. The ability of this method is studied for:

- the determination of the response at any point of the structure,
- the description of the interaction.

The results will be compared to the ones obtained from either the analytical mode superposition or the FE method.

The anti-oscillator model requires to obtain the rigid mode and the constraint modes which can be determined analytically in this specific example. Indeed, the rigid-body mode is given by $\phi_{\mathrm{CR}}(x) = x/L$ and the constraint modes are the eigenmodes of a pinned–pinned beam since the free end has to be constrained to have no displacement. The constraint modes $(\omega_{\mathrm{AR}\ i}, \phi_i)$ of a pinned-pinned beam are well-known (see e.g. Ref. [9]):

1. $\phi_i(x) = C_i \sin(i\pi x/L)$;
2. $\omega_{\mathrm{AR}\ i}$ is a solution of the characteristic equation:

$$\Omega_i^2 \eta\alpha - \Omega_i((\eta + \alpha)i^2\pi^2 + 1) + i^4\pi^4 = 0,$$

Table 1
Beam characteristics

E (GPa)	ρ ($\mathrm{kg\,m^{-3}}$)	I ($\mathrm{cm^4}$)	S ($\mathrm{cm^2}$)	L (m)
67	2400	0.33	2	1

where

- $\Omega_i = \omega^2_{\mathrm{AR}\ i}(\rho S L^4/EI)$,
- $\alpha = (I/S)^2/L^2$: parameter for the effect of rotary inertia,
- $\eta = EI/k'SGL^2$: shear parameter (k' is the shear coefficient and G is the Coulomb's modulus).

The mass and stiffness parameters for the anti-oscillator model are then easily derived from relations (12), (15), (20), (23) and the definition of the mass operator for a beam (3):

$$m_i = \frac{2\rho S L}{i^2\pi^2}, \tag{24}$$

$$m_0 = \frac{\rho S L}{3} - \sum_{i=1}^{N} m_i, \tag{25}$$

$$k_i = \omega^2_{\mathrm{AR}\ i} m_i, \tag{26}$$

$$k_0 = 0. \tag{27}$$

It can be observed that the mass of the structure is described well, due to the residual mass: all the effective mass of the structure is considered for the simulation of the impact event, whatever the number of anti-oscillators.

Similarly, the eigenmodes of the actual structure may be obtained analytically and simulations can be carried out using the mode superposition. A FE model is also carried out, using Timoshenko beam elements (two nodes, two dofs per node and linear interpolation for displacement and rotation). The FE model results will be our references, so 100 elements are used: this discretization may be considered rather fine for this structure. Fig. 2 compares results from the different models: the different results are in a good agreement and the mode superposition and the AO model require a very small number of dofs to converge. Nevertheless, the estimation of the impact force leads to other results: much more dof are required whatever the method used. Indeed, the impact force is governed by the indentation, that is by a local behaviour; it is then interesting to see (Figs. 3 and 4) that 50 anti-oscillators are enough to obtain good results, whereas 90 eigenmodes are needed to obtain results of comparable accuracy. This fact points out that AO series converges faster than mode

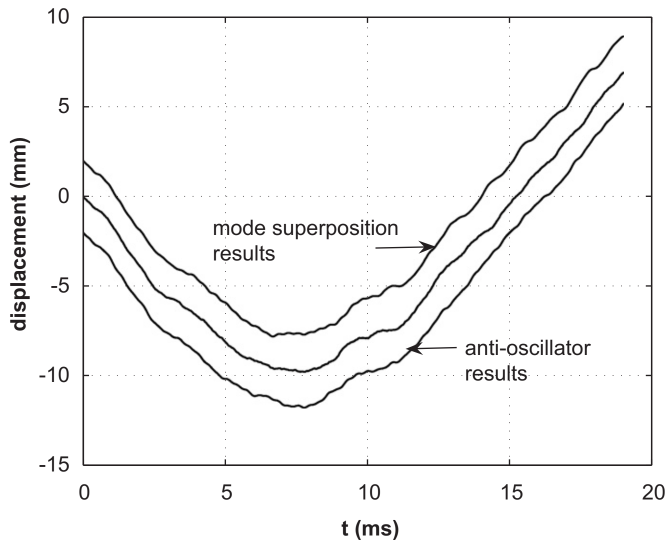

Fig. 2. Mid-span displacement: comparison between FE model (50 elements), a 9-AO modelling and a 10-mode superposition results (the results from AO analysis have been shifted up by 2 mm and the results from mode superposition have been shifted down by 2 mm to make the figure more readable).

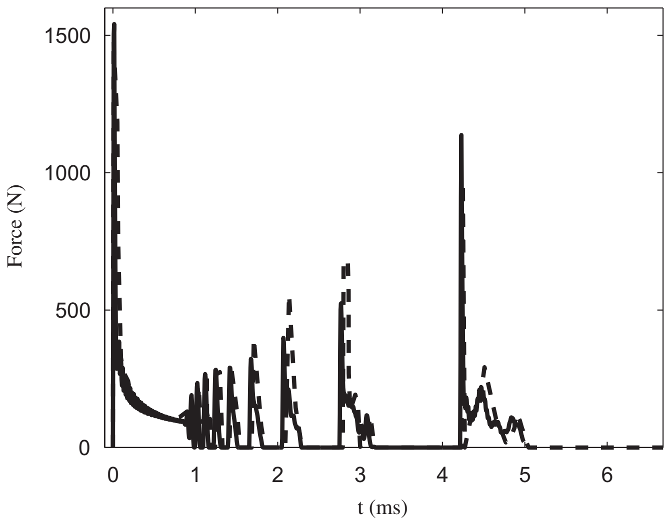

Fig. 3. Impact force: comparison between full FE model and analytical results with 50 anti-oscillators: - - FE model; – analytical results.

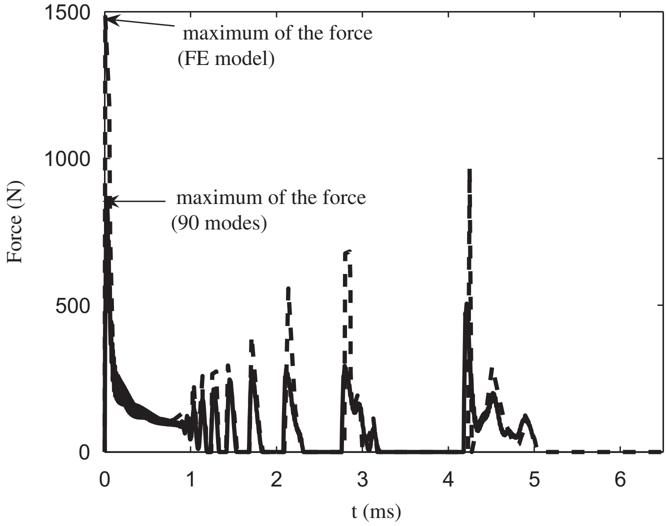

Fig. 4. Impact force: comparison between full FE model and analytical results with 90 eigenmodes: - - FE model; – analytical results.

superposition, for the considered case. That may be explained by remarking that the eigenshapes associated with the constraint modes (sine functions) describe very well the displacement field of the beam during the impact: indeed, the indentation is so small compared to the mid-span displacement, that the beam may be considered as a simply supported beam. In fact, it was this argument that leads to chose this kinematic field. On the contrary, the eigenshapes of the actual structure (pinned–free beam) are not as suitable to describe that the free end displacement almost vanishes during the impact.

5. Study case 2: simply supported beam-mass

In the previous section, analytical expressions for both eigenmodes and anti-oscillators were used. Generally such analysis is not possible and a FE model must be considered. Actually, this section aimed to show that the anti-oscillator method may be applied for practical applications: the method is applied to a structure defined from a FE model; it is showed that the characteristics of the model are easily obtained from the mass and the stiffness matrices: then the only difficulty is to obtain these matrices.

In the impact problem involving a sphere striking a pinned–pinned beam at mid-span, the analytical analysis is still possible but the determination of the anti-oscillators is more complicated. So a FE approach is used to determine the anti-oscillators characteristics and the eigenmodes of the actual structure as well. Hence, three different simulations will be compared in the following: a full FE model simulation, a mode superposition simulation and an anti-oscillator simulation. The eigenmodes and the "anti-eigenmodes" are extracted from the FE model. The characteristics of the beam (square cross-section) and the mass are listed in Table 2. In fact, the real interest of such simulations is to compare the accuracy of these two last models with respect to the rank of truncation: obviously, if there is no truncation, the results will be exactly the same as those obtained with the full FE model simulation.

Modelling with the finite element method: The used element is a Timoshenko beam element. Simulations are carried out with 50 elements.

Modelling with the eigenmodes: The eigenmodes are directly extracted from the mass and stiffness matrices of the FE model. In the simulations only limited eigenmodes are used, i.e. a truncation is done.

Modelling with the anti-oscillators: The anti-oscillators can be extracted from the FE model in a very simple manner. Indeed, a reduced mass matrix $\mathbf{M}_{\text{red}}$ and a reduced stiffness matrix $\mathbf{K}_{\text{red}}$ can be obtained by deleting the row and the column of the dof associated with the impact direction and location. The eigenmodes of these reduced matrices provide the constraint modes, i.e. the circular frequencies $\omega_{\text{AR}\ i}$ and the eigenvectors $\mathbf{V}_i$. The static mode is the displacement vector $\mathbf{V}_{\text{st}}$ produced by a static load acting at the impact location M_{impact} in the impact direction, such that there is a unit displacement at M_{impact}.

The characteristics of the anti-oscillators are hence determined with Eqs. (12), (15), (20) and (23):

$$m_{i>0} = \frac{(\mathbf{V}_{\text{st}}^t \mathbf{M} \mathbf{V}_i)^2}{\mathbf{V}_i^t \mathbf{M} \mathbf{V}_i}, \tag{28}$$

$$m_0 = \mathbf{V}_{\text{st}}^t \mathbf{M} \mathbf{V}_{\text{st}} - \sum_{i=1}^{N} m_i, \tag{29}$$

$$k_0 = \mathbf{V}_{\text{st}}^t \mathbf{K} \mathbf{V}_{\text{st}}, \tag{30}$$

$$k_{i>0} = \omega_{\text{AR}\ i}^2 m_i. \tag{31}$$

Fig. 5 shows the force obtained with 10 eigenmodes and Fig. 6 gives the force obtained with 9 anti-oscillators (i.e. 10 dof). Both results are compared to the force obtained with the full FE model. A very good agreement is found with only 10 dof with the AO model; that is not the case with the mode superposition: in fact 15 eigenmodes are required to obtain a good agreement. Similar simulations have been carried out with higher impact velocity (1, 10, $100\,\text{m}\,\text{s}^{-1}$) involving higher frequency components. In each case, the eigenmode simulations required 50% more dofs than anti-oscillator simulations to obtain accurate results.

6. Comments on the method

We must emphasize that this method requires an initial work very similar to the modal expansion approach: an eigenproblem must be solved. The AR-method is then notably interesting because, it requires a small number of dof to simulate an impact event, particularly to estimate the impact force. Then, even if CPU time is

Table 2
System characteristics

E (GPa)	ρ ($\text{kg}\,\text{m}^{-3}$)	S (cm^2)	I (cm^4)	L (m)	$m_{\text{proj}}/m_{\text{beam}}$	V_0 (m/s)
67	2400	2	0.33	1	1/2	0.01

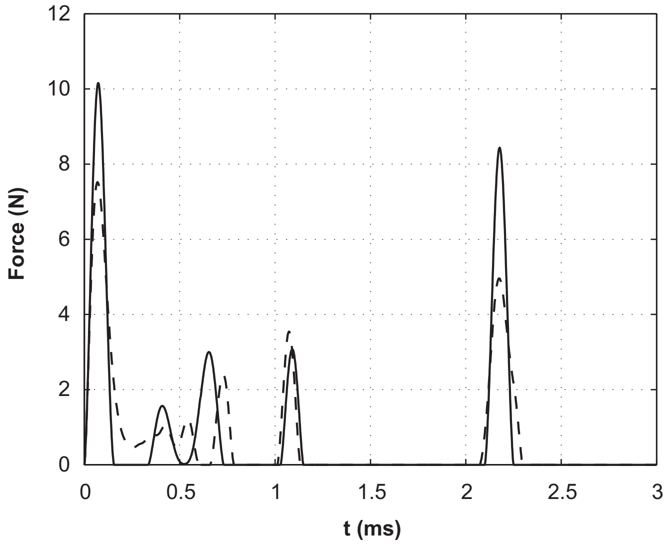

Fig. 5. Impact force: full finite element model (dashed line)—10-mode superposition (solid line).

lost when the eigenproblem is solved, a lot of CPU time may be earned during the simulations: that is particularly notable when the number of time steps is high.

This method has been illustrated on ‘‘academic examples’’, but the last example proved that it may be applied to industrial examples: only a FE model is required.

The modelling of impacted structures must describe:

- the quasi-static aspect when it exists: the AO model takes into account this aspect thanks to the static mode (i.e. the static stiffness),
- the dynamic aspect due to the inertia resistance which strives to cancel the movement: that is the role played by the anti-oscillators; in fact, the anti-oscillators act on the residual mass as a set of tuned-mass damper.

Accordingly the impact is controlled by the anti-oscillators. More precisely, the first few anti-oscillators are essential to describe the response of impacted structures, because they represent almost all the mass of the structure involved in the impact, as shown in Figs. 7 and 8. Indeed, for the two cases studied above, the mass of the first five anti-oscillators represents 89% of the rigid-body mass (pinned–free beam), and 86% of the static mass (simply supported beam). Moreover, the mass of the first anti-oscillator is more than the half of the mass of the structure. It shows that the first anti-oscillator has a strong influence on the impact behaviour. In particular, it can be observed from the simulations that the impact duration is evaluated well by the half of the period of the first anti-oscillator: in case of multiple impacts, the first impact duration is estimated well. Thus, for both studied cases, the impact duration is estimated from the displacement curves and the force curve, depicted in Figs. 9 and 10. The half of the natural period of the first anti-oscillator is also calculated in both cases. The results are listed in Table 3: from these results one can conclude that the impact duration is estimated well by the half of the natural period of the first anti-oscillator of the structure.

7. Conclusions

A new approach has been presented to model an impacted structure, that is based on a representation of the displacement field which is close to the shape of the structure during the impact event: the objective is to decrease the number of parameters required to describe the displacement field of the structure. This description leads to a representation of the structure as a set of single dof systems, called the anti-oscillators, laying on a mass. The dofs of this model are the displacement of the masses: hence, the mass matrix is directly diagonal. It is worth noting that the FE method gives a nondiagonal mass matrix and some manipulations

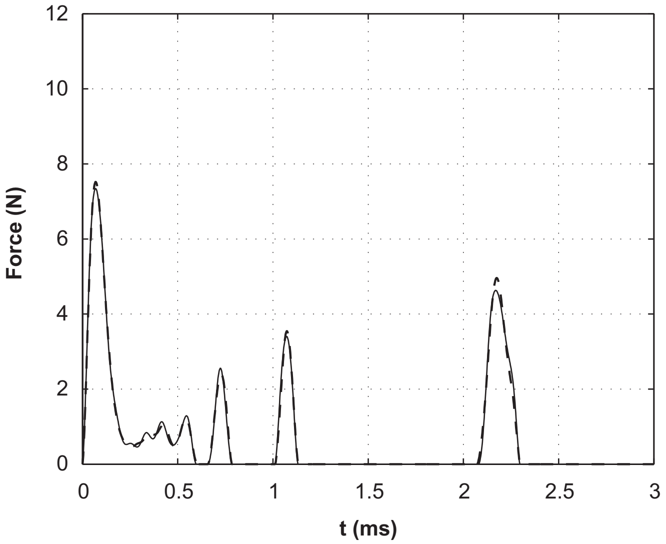

Fig. 6. Impact force: full finite element model (dashed line)—9-anti-oscillator model (solid line).

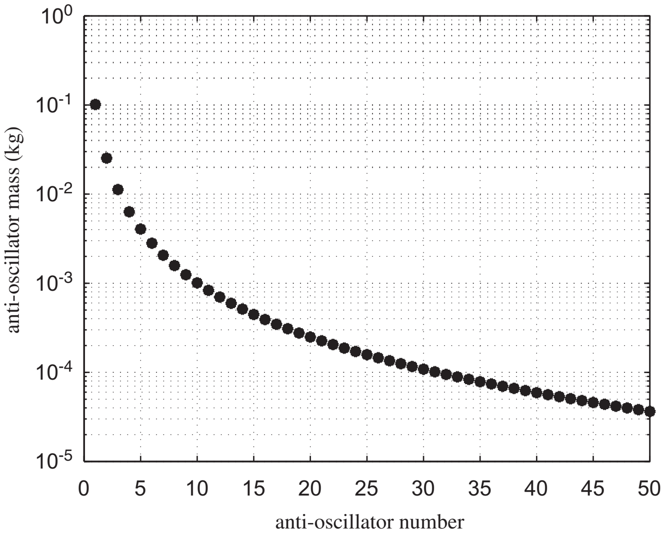

Fig. 7. Mass of the first 50 anti-oscillators: pinned–free beam.

must be done to make it diagonal in order to reduce the computational time for the simulations. It is important to note that this model can be applied to any structure, even to the ones with rigid body modes.

Moreover the anti-oscillator description has no lack of mass: the residual mass accounts for the difference of the mass between the masses of the anti-oscillators and the mass involved in the impact. Likewise, the static behaviour of the structure can be precisely simulated because the static stiffness is one of the parameters that intervene in the model. Moreover, the dof associated with the residual mass is the displacement at the point of impact of the structure. Thus, the interaction force is directly calculated with the displacement of the residual mass and the projectile.

The accuracy of the anti-oscillator model depends on the accuracy of the constraint modes. When these modes are extracted from a FE model, the refinement of the FE model is a key parameter. It is worth noting

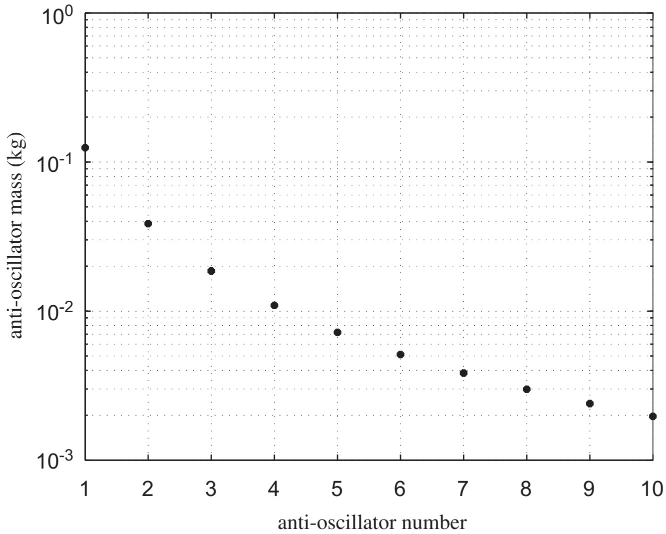

Fig. 8. Mass of the first 10 anti-oscillators: pinned–pinned beam.

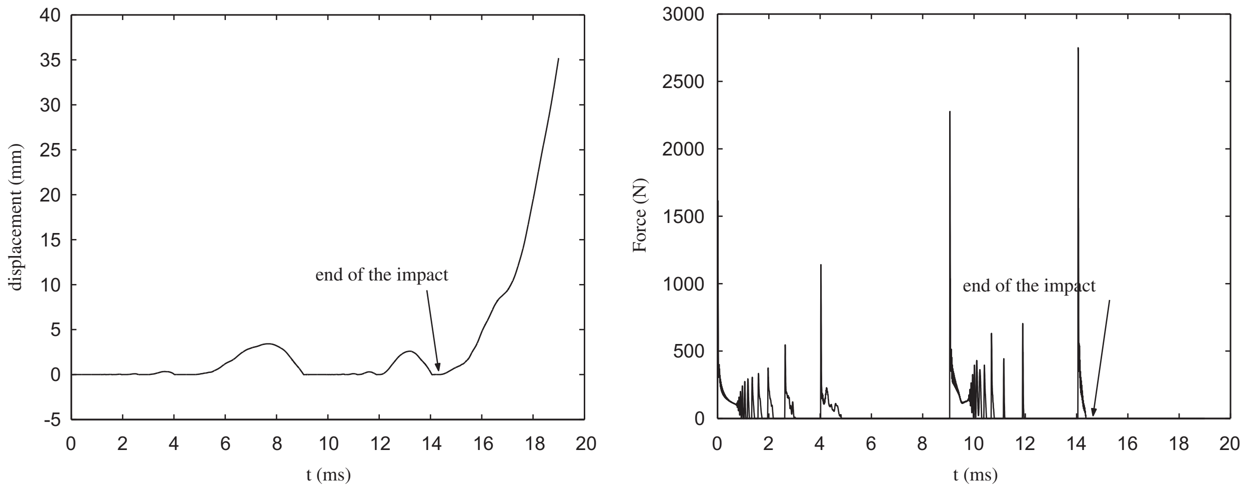

Fig. 9. Impact duration: pinned–free beam.

that the FE model must have a sufficient number of dofs to well describe the constraint eigenshapes. However, that does not imply that the anti-oscillator model must have a lot of dofs: few anti-oscillators are enough to simulate the impact behaviour of the structure very well. Likewise, the first anti-oscillator has a strong influence on the impact characteristics; indeed, it has the greatest mass of all the anti-oscillators. That is why it has a dominating role on the impact behaviour of the structure: it can be observed that the impact duration may be well estimated by the half natural period of the first anti-oscillator.

To summarize, the AR method presents the following advantages:

- easy and quick to be set up from a FE description of a structure: a eigenproblem has to be solved,
- the used displacement field describes very well the structure during an impact: that is why few anti-oscillator are required to accurately simulate the impacted structure: computational time may be earned for the impact simulations,
- the obtained model has a physical meaning that may be helpful to understand an impact.

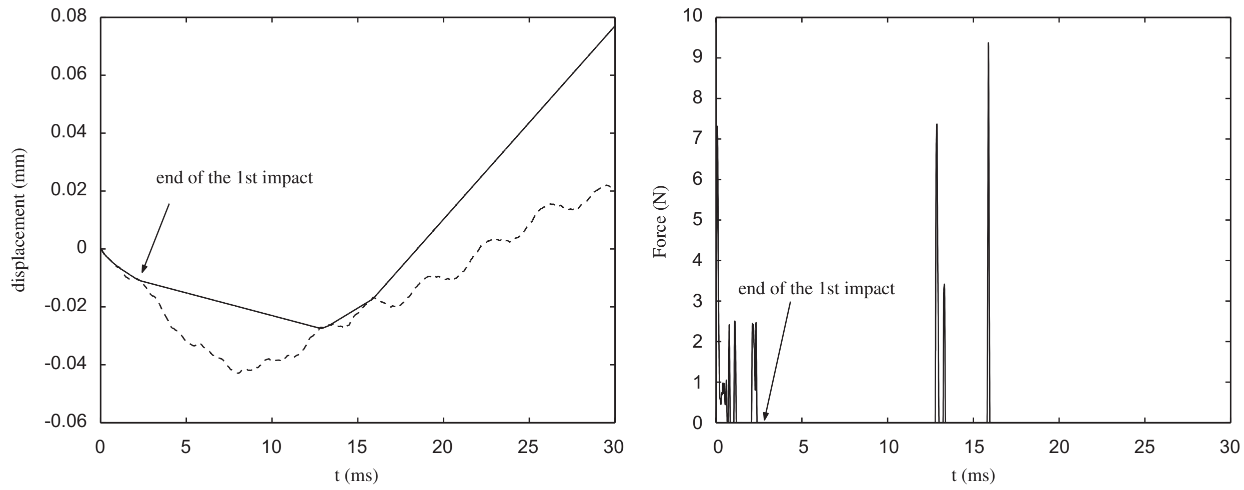

Fig. 10. Impact duration: pinned–pinned beam. - - beam; – mass.

Table 3
Comparison of impact duration with half-period of the first AO

	Impact duration (ms)	1/2 $T_{1\mathrm{AO}}$
Pinned–free beam	14.4	15.2
Pinned–pinned beam	2.4	2.5

Nevertheless, this model has a main drawback: the model characteristics depend on the point of impact. Hence, if multiple impact points must be considered in a design process, several static modes and several sets of constraint modes should be calculated and then increase the numerical cost.

Appendix

The following orthogonality property must be proved:

$$K(\psi_0, \psi_i) = 0 \quad \text{for } i = 1..N$$

In fact, this is equivalent to prove that:

$$K(\phi_{\mathrm{st}}, \phi_i) = 0 \quad \text{for } i = 1..N. \tag{32}$$

Let us consider:

- the stress field σ^{st} due to the static load F_{st}: by definition, the corresponding shape of the structure is ϕ_{st};
- $\varepsilon(\phi_i)$ $(i>0)$ is the strain field which is generated when the structure has a shape ϕ_i $(i>0)$ (i.e. eigenshape of a constraint mode).

The stiffness operator is then:

$$K(\phi_{\mathrm{st}}, \phi_i) = \int_{\mathrm{vol}} \sigma^{\mathrm{st}}_{kl} \varepsilon_{kl}(\phi_i)\, \mathrm{d}V \tag{33}$$

$$= \frac{1}{2} \int_{\mathrm{vol}} \sigma^{\mathrm{st}}_{kl} \left(\frac{\partial \phi_{ik}}{\partial x_l} + \frac{\partial \phi_{il}}{\partial x_k} \right) \mathrm{d}V \tag{34}$$

$$= \frac{1}{2}\int_{\text{vol}}\left(\frac{\partial\sigma_{kl}^{\text{st}}}{\partial x_l}\phi_{ik} + \frac{\partial\sigma_{kl}^{\text{st}}}{\partial x_k}\phi_{il}\right)\mathrm{d}V - \frac{1}{2}\int_S(\sigma_{kl}^{\text{st}}\phi_{ik}n_l + \sigma_{kl}^{\text{st}}\phi_{il}n_k)\,\mathrm{d}S \quad (35)$$

$$= -\frac{1}{2}\int_{\text{vol}}(f_k^b\phi_{ik} + f_l^b\phi_{il})\,\mathrm{d}V - \frac{1}{2}\int_S(\sigma_{kl}^{\text{st}}\phi_{ik}n_l + \sigma_{kl}^{\text{st}}\phi_{il}n_k)\,\mathrm{d}S \quad (36)$$

$$= -\int_{\text{vol}} f_k^b\phi_{ik}\,\mathrm{d}V - \frac{1}{2}\int_S(\sigma_{kl}^{\text{st}}\phi_{ik}n_l + \sigma_{kl}^{\text{st}}\phi_{il}n_k)\,\mathrm{d}S, \quad (37)$$

where f^b represents the body force. It is worth noting that σ^{st} are due to the only static load F_{st} applied at the impact location: in particular, the body force is considered to be zero. So, the first term of relation (37) cancels out. The surface integral vanishes as well. Indeed, the surface S can be expressed as a partition: $S = S_u \cup S_\sigma$ where S_u and S_σ are defined as:

- S_u corresponds to the kinematic boundary condition surface, where the displacement (i.e. ϕ_i) is equal to zero; so, the surface integral on the region S_u vanishes;
- S_σ corresponds to the static boundary condition surface, where no load is exerted except at the impact location M_{impact}; then $\sigma_{kl}^{\text{st}}n_l$ is zero except at M_{impact}; but $\phi_i(M_{\text{impact}})$ is zero because ϕ_i is a constraint mode; then the surface integral on S_σ vanishes as well.

Consequently, it has been proved that $K(\phi_{\text{st}}, \phi_i)$ vanishes for $i>0$.

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