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Evaluation of the Helmholtz boundary integral equation and its normal and tangential derivatives in two dimensions

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Abstract

This paper presents a means of evaluating singular integrals in the Helmholtz boundary integral equation and its normal and tangential derivatives in two dimensions. The subtraction–addition technique is applied to the singular integral equations to convert the singular integrals to either ordinary integrals with bounded integrands or modified singular integrals, including hypersingular integrals, with exact integration values. This regularization is performed before any discretization. The modified integral equations can be calculated by directly applying standard quadrature rules over the entire integration domain. Numerical computations involve evaluating the acoustic field associated with a radiating inverse elliptic cylinder. The velocity potential is obtained by applying the Burton–Miller method, which linearly combines the Helmholtz boundary integral equation with its normal derivative, to treat the fictitious characteristic frequencies. Further substituting the velocity potential into the regularized tangential derivative yields the surface tangential velocity. Comparing the numerical results with the analytical solutions verifies the effectiveness of the presented approach.

1. Introduction

The Helmholtz boundary integral equation has been extensively adopted for several decades for treating the exterior acoustic radiation and scattering by obstacles. Ciskowski and Brebbia [1] discussed this usage in detail. The main problem with this lies in the existence of nonunique solutions at the characteristic frequencies of the associated interior Dirichlet problem. Moreover, the characteristic frequencies of an arbitrarily shaped object are generally not known a priori, except when the associated interior problem has been solved. Nonuniqueness does not have any physical significance, and is a purely mathematical problem that arises from the breakdown of the boundary integral representation at the characteristic frequencies. Two major methods, appropriate for practical applications, have been applied to overcome this difficulty.

Schenck [2] proposed a combined Helmholtz integral equation formulation (CHIEF) that added some interior integral relationships to the surface Helmholtz integral equation. The subsequent overdetermined system of equations may then be solved by applying a least-squares technique. This method perhaps is the most widely used in engineering applications. However, selecting the optimum number and suitable positions

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of the interior points may become difficult as the wave frequency increases. This fact has motivated the proposition of numerous variants [3–6] effectively to treat interior points.

Burton and Miller [7] linearly combined the Helmholtz integral equation with its normal derivative to circumvent the nonuniqueness problem. This method possesses a more rigorous mathematical background than CHIEF. However, the normal derivative of the Helmholtz integral equation contains a hypersingular integral that involves a double normal derivative of the free-space Green's function. Most of the Burton–Miller-type approaches developed subsequently thus emphasize improving the numerical efficiency of the evaluation of the hypersingular integrals.

Much research has been conducted to describe hypersingular integrals in terms of Hadamard's finite part [8]. Based on Hadamard's finite-part interpretation, Chien et al. [9] employed some identities in the integral equation related to an interior Laplace problem to reduce the order of the kernel singularity. Liu and Rizzo [10] derived a weakly singular form of the hypersingular integral equations by subtracting a two-term Taylor series from the density function. Certain integral identities of static Green's function were used to assess the added-back terms. Ergin et al. [11] proposed the combination of a Burton–Miller-type time domain with a field integral equation for transient scattering from closed rigid bodies. Yang [12,13] expressed the unknown surface functions as a truncated Fourier–Chebyshev or –Legendre series. Some weakly singular integrals and the hypersingular integral were analytically evaluated based on the properties of Legendre functions. Harris and Chen [14] developed a high-order Galerkin method in terms of the singularity subtraction approach to reduce the hypersingular operator to a weakly singular one. In Ref. [14], the numerical procedures included two particular iterative solvers—the conjugate gradient normal method and the generalized minimal residual method. Gray et al. [15] employed the multiple polar coordinate transformations and analytic integration to evaluate directly Galerkin hypersingular integrals without recourse to Hadamard's finite part. Yan et al. [16] considered the normal derivative of solid angles on the surface. Sladek and Sladek [17] surveyed singular integral methods for both Galerkin and collocation formulations.

Tangential derivatives of the boundary integral equations are also useful in various engineering applications, for example, the calculation of the boundary stresses of elastostatic problems [18]. See Smirnov [19], Jaswon and Symm [20], and Colton and Kress [21] for the mathematical foundation. Meyer et al. [22] implicitly applied tangential operators to circumvent the hypersingularity in acoustic radiation problems. Bonnet and Guiggiani [23] evaluated the sensitivity to tangential perturbations of the singular points of boundary integrals that involve either weak or strong singularities. Both scalar potential and elastic problems were examined. Amini [24] analyzed the spectral properties of the single-layer Laplacian potential operator and its tangential derivatives on a circular boundary. This study expanded the unknown functions as Fourier series to yield some analytical results concerning the elements of discrete operators and their eigenvalues and eigenvectors. Jorge et al. [25] derived a self-regular formulation strategy in terms of Green's identity and its gradient form for Laplace's equation. Martínez-Castro and Gallego [26] derived an error estimator based on the tangential boundary integral equation residuals for Laplace and Helmholtz equations.

In the light of the above developments, this study converts the 2D Helmholtz boundary integral equation and its normal and tangential derivatives into a singularity-free form, facilitating numerical implementation. The rest of this study is organized as follows. Section 2 introduces the basic boundary integral equations. Section 3 describes the regularization method which does not require any approximations of the surface functions, such as a truncated Fourier–Chebyshev or –Legendre series adopted in Refs. [12,13], so as to provide wider applications. Section 4 numerically elucidates the effectiveness and accuracy of the proposed method for a pulsating inverse elliptic cylinder. Section 5 draws conclusions.

2. Basic equations

The exterior acoustic radiation problem in an unbounded ideal homogeneous medium is specified by the following wave equation:

$$\nabla^2 \phi(r,t) = \frac{1}{c^2}\frac{\partial^2 \phi(r,t)}{\partial t^2}, \tag{1}$$

where ∇^2 denotes the Laplacian operator in two dimensions, ϕ denotes the velocity potential that is twice continuously differentiable at point r and time t, and c represents the speed of sound in the medium at the equilibrium state. The velocity potential ϕ is related to the velocity $\boldsymbol{u}$:

$$\boldsymbol{u} = \nabla\phi. \tag{2}$$

For steady-state excitation with a time factor $\exp(-\mathrm{i}\omega t)$, Eq. (1) reduces to the Helmholtz wave equation

$$(\nabla^2 + k^2)\phi = 0, \tag{3}$$

where $\mathrm{i} = \sqrt{-1}$ denotes the imaginary unit, ω denotes the angular frequency, and $k = \omega/c$ represents the wavenumber. The velocity potential ϕ fulfills the Neumann boundary condition on a sufficiently smooth body surface ∂B,

$$\frac{\partial\phi}{\partial n_r} = f(r), \tag{4}$$

where $\partial/\partial n_r$ denotes differentiation along the outward normal direction at $r \in \partial B$ and $f(r)$ represents a specified function. The velocity potential ϕ should also meet the Sommerfeld radiation conditions at infinity

$$\phi = O(r^{-1/2}), \quad \frac{\partial\phi}{\partial r} - \mathrm{i}k\phi = O(r^{-1/2}) \tag{5}$$

as $r \to \infty$. The solution to the above boundary-value problem exists and is unique, as described in detail elsewhere [21].

According to the integral equation method, the Helmholtz integral relation takes the following form:

$$\varepsilon\phi(p) = \int_{\partial B} \phi(q) \frac{\partial G_k(p,q)}{\partial n_q} \, \mathrm{d}S(q) - \int_{\partial B} G_k(p,q) \frac{\partial\phi}{\partial n_q} \, \mathrm{d}S(q), \tag{6}$$

where $\phi(q)$ is Hölder continuous, $\phi(q) \in C^{0,\lambda}$, $0 < \lambda < 1$, extending over a simple closed Liapunov boundary ∂B, and

$$\varepsilon = \begin{cases} 1, & p \text{ exterior to } \partial B, \\ 1/2, & p \text{ on } \partial B, \\ 0, & p \text{ interior to } \partial B. \end{cases} \tag{7}$$

The free-space Green's function G_k in the 2D Helmholtz equation can be expressed as

$$G_k(p,q) = \frac{\mathrm{i}}{4} H_0^{(1)}(kR), \tag{8}$$

where $H_0^{(1)}(kR)$ denotes the Hankel function of the first kind and of order zero, and R represents the distance between the field point $p(x, y)$ and the source point $q(\xi, \eta)$. The first integral in Eq. (6) is regular and the second integral contains a weak logarithmic singularity when $p \in \partial B$ and $q \to p$.

Taking the normal derivative of Eq. (6) when $p \in \partial B$ leads to

$$\frac{1}{2}\frac{\partial\phi}{\partial n_p} = \text{⨎}_{\partial B} \phi(q) \frac{\partial^2 G_k(p,q)}{\partial n_p \partial n_q} \, \mathrm{d}S(q) - \int_{\partial B} \frac{\partial G_k(p,q)}{\partial n_p} \frac{\partial\phi}{\partial n_q} \, \mathrm{d}S(q), \tag{9}$$

where the first integral is interpreted as a Hadamard finite-part integral. The existence of Eq. (9) necessitates the continuity requirement $\phi \in C^{1,\lambda}$ [27]. Hypersingular integrals are unbounded; however, retaining only the finite parts enables the hypersingular integral equations to be used to solve various physical problems [17]. A unique behavior of hypersingular integrals is of particular concern; for example, the integral $\text{⨎}_{-1}^{1} (x+1)^{-2} \mathrm{d}x = -1/2$ contains a negative finite part although the integrand is always positive in the integration domain.

The tangential derivative of Eq. (6) when $p \in \partial B$ can be written as

$$\frac{1}{2}\frac{\partial\phi}{\partial \tau_p} = \int_{\partial B} \phi(q) \frac{\partial^2 G_k(p,q)}{\partial \tau_p \partial n_q} \, \mathrm{d}S(q) - \int_{\partial B} \frac{\partial G_k(p,q)}{\partial \tau_p} \frac{\partial\phi}{\partial n_q} \, \mathrm{d}S(q), \tag{10}$$

where $\partial/\partial\tau_p$ denotes differentiation along the unit tangential vector $\boldsymbol{\tau}(p)$. The first integral in Eq. (10) is regular and the second contains a strong singularity. Eq. (10) is not an integral equation because $\phi(q)$ is obtained from Eq. (6) or a linear combination of Eqs. (6) and (9), and $\partial\phi/\partial n_q$ is obtained from boundary condition (4). The singularity behavior of hypersingular and strongly singular boundary integral representations of potential gradient is discussed elsewhere [28,29].

The relaxation of the standard smoothness requirements is described elsewhere [30]. The weaker continuity conditions facilitate numerical calculations in practical applications. Notably, regularization methods markedly affect the numerical accuracy and efficiency. The following section introduces a method of treating singularities in Eqs. (6), (9) and (10).

3. Regularization

This section elucidates a regularization method for converting singular integrals into a form that allows the direct application of standard quadrature without recourse to another specific treatment. The regularization is implemented globally, rather than locally along each element as in conventional boundary element methods, so the regularization is implemented before boundary surfaces are discretized.

The boundary surface ∂B is assumed to be able to be described by the following vector function:

$$\boldsymbol{r}(\alpha) = x(\alpha)\boldsymbol{i} + y(\alpha)\boldsymbol{j}, \quad -1 \leqslant \alpha \leqslant 1, \tag{11}$$

where α denotes a parameter, and $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors that are parallel to the positive x and y axes, respectively. Accordingly, Eq. (6) for $p \in \partial B$ can be rewritten in the following parametric form:

$$\frac{\phi(\beta)}{2} = \int_{-1}^{1} \left[\phi(\alpha) \frac{\partial G_k(\alpha,\beta)}{\partial n_\alpha} - G_k(\alpha,\beta) \frac{\partial \phi}{\partial n_\alpha} \right] \frac{\mathrm{d}S}{\mathrm{d}\alpha}\, \mathrm{d}\alpha, \quad \beta \in [-1, 1], \tag{12}$$

where

$$\frac{\mathrm{d}S}{\mathrm{d}\alpha} = \left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\alpha} \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\alpha} \right)^{1/2} = \left[\left(\frac{\mathrm{d}x}{\mathrm{d}\alpha} \right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\alpha} \right)^2 \right]^{1/2} \equiv (x'^2(\alpha) + y'^2(\alpha))^{1/2}. \tag{13}$$

Applying the subtraction–addition technique to Eq. (12), we have

$$\begin{aligned} \frac{\phi(\beta)}{2} = & \int_{-1}^{1} \phi(\alpha) \frac{\partial G_k}{\partial n_\alpha} \frac{\mathrm{d}S}{\mathrm{d}\alpha}\, \mathrm{d}\alpha - \int_{-1}^{1} \left(G_k \frac{\partial \phi}{\partial n_\alpha} \frac{\mathrm{d}S}{\mathrm{d}\alpha} - G \frac{\partial \phi}{\partial n_\beta} \frac{\mathrm{d}S}{\mathrm{d}\beta} \right) \mathrm{d}\alpha \\ & + \frac{1}{2\pi} \frac{\partial \phi}{\partial n_\beta} \frac{\mathrm{d}S}{\mathrm{d}\beta} \int_{-1}^{1} \left[\ln R - \ln\left(\frac{\mathrm{d}S}{\mathrm{d}\beta} |\alpha - \beta| \right) \right] \mathrm{d}\alpha + \frac{1}{\pi} \frac{\partial \phi}{\partial n_\beta} \frac{\mathrm{d}S}{\mathrm{d}\beta} \ln \frac{\mathrm{d}S}{\mathrm{d}\beta} \\ & + \frac{1}{2\pi} \frac{\partial \phi}{\partial n_\beta} \frac{\mathrm{d}S}{\mathrm{d}\beta} \int_{-1}^{1} \ln |\alpha - \beta|\, \mathrm{d}\alpha, \end{aligned} \tag{14}$$

where the free-space Green's function G in the 2D Laplace equation can be expressed as

$$G = -\frac{\ln R}{2\pi}. \tag{15}$$

Applying Taylor's formula for small $|\alpha-\beta|$ yields

$$\lim_{\alpha \to \beta} \frac{\partial G_k}{\partial n_\alpha} = \lim_{\alpha \to \beta} \frac{\partial G}{\partial n_\alpha} = -\frac{\kappa(\beta)}{4\pi}, \tag{16}$$

$$\lim_{\alpha \to \beta} (G_k - G) = -\frac{1}{2\pi} \left(\gamma + \ln \frac{k}{2} \right) + \frac{\mathrm{i}}{4}, \tag{17}$$

$$\lim_{\alpha \to \beta} R \approx \frac{\mathrm{d}S}{\mathrm{d}\beta} |\alpha - \beta|, \tag{18}$$

where $\kappa(\beta)$ denotes the boundary curvature at point β:

$$\kappa(\beta) = \frac{x'(\beta)y''(\beta) - x''(\beta)y'(\beta)}{(x'^2(\beta) + y'^2(\beta))^{3/2}} \tag{19}$$

and γ represents Euler's constant [31]:

$$\gamma = \lim_{n\to\infty}\left(\sum_{m=1}^{n}\frac{1}{m} - \ln n\right) = 0.577215\ldots . \tag{20}$$

Eq. (16) is obtained from calculus, as in, for example, [19]. Yang [32] derived Eq. (17). Eqs. (16)–(18) clearly demonstrate that the first three integrals of Eq. (14) are bounded. The final integral of Eq. (14) can be exactly evaluated as follows [33]:

$$\int_{-1}^{1} \ln|\alpha - \beta|\,\mathrm{d}\alpha = \begin{cases} (1+\beta)\ln(1+\beta) + (1-\beta)\ln(1-\beta) - 2, & \beta \neq \pm 1, \\ 2\ln 2 - 2, & \beta = \pm 1. \end{cases} \tag{21}$$

The modified equation, Eq. (14), is therefore a singularity-free boundary integral equation that yields the unknown potential function ϕ.

The regularization technique presented above can be used to rewrite Eq. (9) as follows:

$$\begin{aligned}
\frac{1}{2}\frac{\partial\phi}{\partial n_\beta} = {} & \int_{-1}^{1}\left[\phi(\alpha)\frac{\partial^2 G_k}{\partial n_\alpha \partial n_\beta}\frac{\mathrm{d}S}{\mathrm{d}\alpha} - \phi(\beta)\left(\frac{\partial^2 G}{\partial n_\alpha \partial n_\beta} + \frac{k^2 G}{2}\right)\frac{\mathrm{d}S}{\mathrm{d}\beta}\right]\mathrm{d}\alpha \\
& - \frac{\phi(\beta)}{2\pi}\frac{\mathrm{d}S}{\mathrm{d}\beta}\int_{-1}^{1}\left(\frac{\partial^2 \ln R}{\partial n_\alpha \partial n_\beta} + \frac{1}{R^2}\right)\mathrm{d}\alpha \\
& + \frac{\phi(\beta)}{2\pi}\frac{\mathrm{d}S}{\mathrm{d}\beta}\int_{-1}^{1}\left[\frac{1}{R^2} - \frac{1}{(\alpha-\beta)^2}\frac{1}{(\mathrm{d}S/\mathrm{d}\beta)^2}\right. \\
& \left. + \frac{1}{\alpha-\beta}\frac{x'(\beta)x''(\beta) + y'(\beta)y''(\beta)}{(x'^2(\beta) + y'^2(\beta))^{3/2}}\frac{1}{\mathrm{d}S/\mathrm{d}\beta}\right]\mathrm{d}\alpha \\
& + \frac{\phi(\beta)}{2\pi}\frac{1}{\mathrm{d}S/\mathrm{d}\beta}\,⨎_{-1}^{1}\frac{1}{(\alpha-\beta)^2}\mathrm{d}\alpha \\
& - \frac{\phi(\beta)}{2\pi}\frac{x'(\beta)x''(\beta) + y'(\beta)y''(\beta)}{(x'^2(\beta) + y'^2(\beta))^{3/2}}\int_{-1}^{1}\frac{1}{\alpha-\beta}\mathrm{d}\alpha \\
& - \frac{k^2\phi(\beta)}{4\pi}\frac{\mathrm{d}S}{\mathrm{d}\beta}\int_{-1}^{1}\left[\ln R - \ln\left(\frac{\mathrm{d}S}{\mathrm{d}\beta}|\alpha-\beta|\right)\right]\mathrm{d}\alpha - \frac{k^2\phi(\beta)}{2\pi}\frac{\mathrm{d}S}{\mathrm{d}\beta}\ln\frac{\mathrm{d}S}{\mathrm{d}\beta} \\
& - \frac{k^2\phi(\beta)}{4\pi}\frac{\mathrm{d}S}{\mathrm{d}\beta}\int_{-1}^{1}\ln|\alpha-\beta|\,\mathrm{d}\alpha - \int_{-1}^{1}\frac{\partial G_k}{\partial n_\beta}\frac{\partial\phi}{\partial n_\alpha}\frac{\mathrm{d}S}{\mathrm{d}\alpha}\mathrm{d}\alpha,
\end{aligned} \tag{22}$$

where

$$\lim_{\alpha\to\beta}\left[\frac{\partial^2 G_k}{\partial n_\alpha \partial n_\beta} - \frac{\partial^2 G}{\partial n_\alpha \partial n_\beta} - \frac{k^2 G}{2}\right] = \frac{k^2}{4\pi}\left(\frac{1}{2} - \ln\frac{k}{2} - \gamma\right) + i\frac{k^2}{8}, \tag{23}$$

$$\lim_{\alpha\to\beta}\left(\frac{\partial^2 \ln R}{\partial n_\alpha \partial n_\beta} + \frac{1}{R^2}\right) = 0, \tag{24}$$

$$\lim_{\alpha\to\beta}\left[\frac{1}{R^2}-\frac{1}{(\alpha-\beta)^2}\frac{1}{(\mathrm{d}S/\mathrm{d}\beta)^2}+\frac{1}{\alpha-\beta}\frac{x'(\beta)x''(\beta)+y'(\beta)y''(\beta)}{(x'^2(\beta)+y'^2(\beta))^{3/2}}\frac{1}{\mathrm{d}S/\mathrm{d}\beta}\right]$$
$$=\frac{3}{4}\frac{x''^2(\beta)+y''^2(\beta)}{(x'^2(\beta)+y'^2(\beta))^2}-\frac{1}{3}\frac{x'(\beta)x'''(\beta)+y'(\beta)y'''(\beta)}{(x'^2(\beta)+y'^2(\beta))^2}-\kappa^2(\beta), \tag{25}$$

$$⨍_{-1}^{1}\frac{1}{(\alpha-\beta)^2}\,\mathrm{d}\alpha=-\frac{1}{1-\beta}-\frac{1}{1+\beta}, \tag{26}$$

$$\int_{-1}^{1}\frac{1}{\alpha-\beta}\,\mathrm{d}\alpha=\ln\frac{1-\beta}{1+\beta}. \tag{27}$$

The last three integrals in Eq. (22) are given by Eqs. (18), (21) and (16), respectively. The derivation of Eq. (23) has been presented elsewhere [32] and that of Eq. (25) is presented in Appendix A. Kaya and Erdogan [34] discussed Hadamard's finite-part integrals, and obtained Eqs. (26) and (27). The integral equation, Eq. (22), is therefore regular, because all integrals are either bounded or explicitly determined.

Finally, the singularity-free form of Eq. (10) can be written as

$$\frac{1}{2}\frac{\partial\phi}{\partial\tau_\beta}=\int_{-1}^{1}\phi(\alpha)\frac{\partial^2 G_k}{\partial\tau_\beta\partial n_\alpha}\frac{\mathrm{d}S}{\mathrm{d}\alpha}\,\mathrm{d}\alpha-\int_{-1}^{1}\left(\frac{\partial G_k}{\partial\tau_\beta}\frac{\partial\phi}{\partial n_\alpha}-\frac{1}{2\pi}\frac{1}{\alpha-\beta}\frac{1}{\mathrm{d}S/\mathrm{d}\alpha}\frac{\partial\phi}{\partial n_\beta}\right)\frac{\mathrm{d}S}{\mathrm{d}\alpha}\,\mathrm{d}\alpha$$
$$-\frac{1}{2\pi}\frac{\partial\phi}{\partial n_\beta}\int_{-1}^{1}\frac{1}{\alpha-\beta}\,\mathrm{d}\alpha, \tag{28}$$

where

$$\lim_{\alpha\to\beta}\frac{\partial^2 G_k}{\partial\tau_\beta\partial n_\alpha}=\frac{\kappa(\beta)}{4\pi}\frac{x'(\beta)x''(\beta)+y'(\beta)y''(\beta)}{(x'^2(\beta)+y'^2(\beta))^{3/2}}-\frac{1}{12\pi}\frac{x'(\beta)y'''(\beta)-x'''(\beta)y'(\beta)}{(x'^2(\beta)+y'^2(\beta))^2}, \tag{29}$$

$$\lim_{\alpha\to\beta}\left(\frac{\partial G_k}{\partial\tau_\beta}-\frac{1}{2\pi}\frac{1}{\alpha-\beta}\frac{1}{\mathrm{d}S/\mathrm{d}\alpha}\right)=-\frac{1}{4\pi}\frac{x'(\beta)x''(\beta)+y'(\beta)y''(\beta)}{(x'^2(\beta)+y'^2(\beta))^{3/2}} \tag{30}$$

and the final integral is obtained from Eq. (27). Eqs. (29) and (30) are derived in Appendix A.

The regularized equations (14), (22) and (28) can be computed by directly applying the standard quadrature. The Burton–Miller method linearly combines Eqs. (14) and (22), yielding the unique surface function ϕ. Further substituting ϕ and the boundary condition $\partial\phi/\partial n$ into Eq. (28) yields the surface tangential velocity $\partial\phi/\partial\tau$.

4. Illustrative problems

Consider a pulsating inverse elliptic cylinder with minor axis a and major axis b (Fig. 1). The inverse elliptic-cylinder coordinates $(\eta,\ \theta)$ are related to rectangular Cartesian coordinates $(x,\ y)$ by the following transformation:

$$x=\frac{c\cosh\eta\cos\theta}{\cosh^2\eta-\sin^2\theta},\quad y=\frac{c\sinh\eta\sin\theta}{\cosh^2\eta-\sin^2\theta}, \tag{31}$$

where $c=a\cosh\eta$, $\eta\geqslant 0$ and $0\leqslant\theta\leqslant 2\pi$. The radiation problem was simulated by placing a point source of unit strength, $\phi_s=\mathrm{i}H_0^{(1)}(kR)/4$, at the center of the cylinder. Such a radiator was selected to determine accurately the limiting values, such as those given by Eqs. (16), (17), (23), (25), (29) and (30) in Section 3. The 60-point trapezoidal rule was applied along the θ axis. The subsequent system of linear algebraic equations was solved using the LU decomposition method. Notably, the nodal, integration and collocation points that are conventionally used in boundary element methods all imply the same thing when the proposed singularity-free integral equations are implemented.

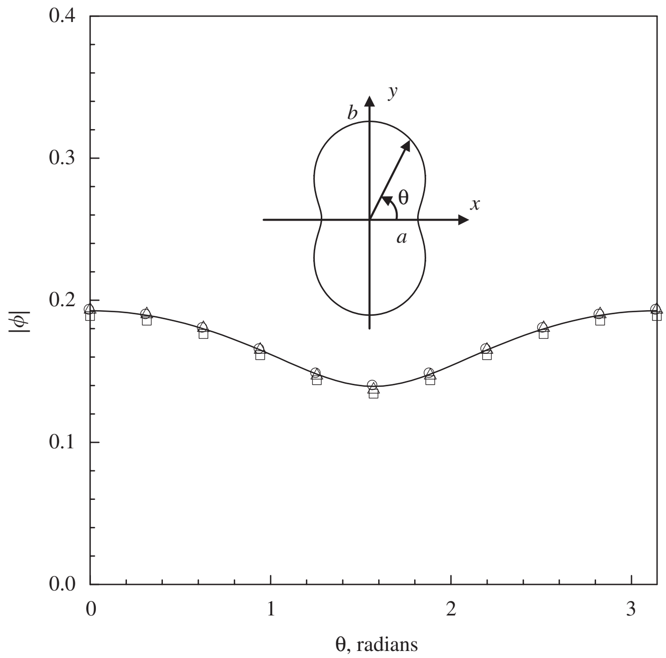

Fig. 1. Amplitude of surface field ϕ for a pulsating inverse elliptic cylinder of a:$b = 1$:2 with the normalized wavenumber $ka = 1$. —, the analytical solution; ○, the Helmholtz integral equation (14); △, Eq. (14) + i/ka × Eq. (22); □, Chien's method. The 60-point trapezoidal rule was applied along the θ-axis.

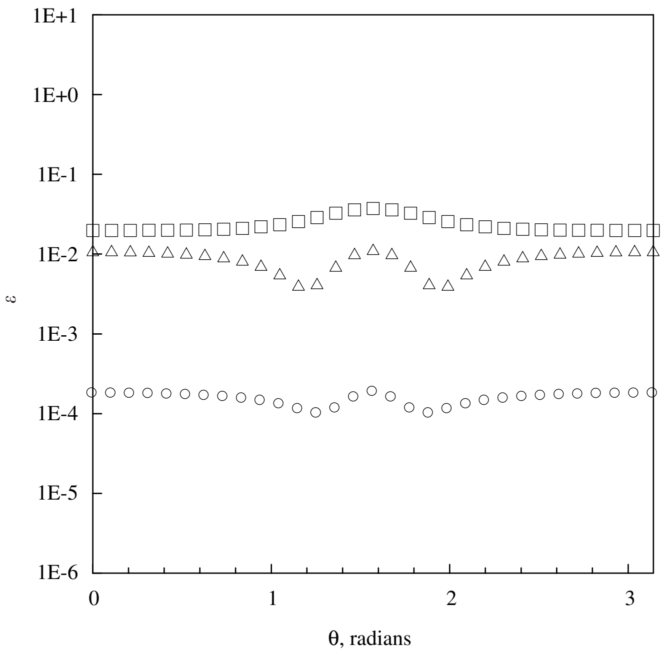

Fig. 2. Relative error ε of computed amplitude of surface field ϕ in Fig. 1. Other definitions are given in Fig. 1.

Fig. 1 displays the surface function ϕ on a 1:2 inverse elliptic cylinder for the normalized wavenumber $ka = 1$. According to Fig. 1, the numerical results from Eq. (14) and the composite equation, Eq. (14) + i/ka × Eq. (22), correlate well with the analytical solution. Fig. 2 plots the relative error ε, defined as |(computed result—analytical solution)/maximum of analytical solution|, and the maximum ε in Eq. (14) is

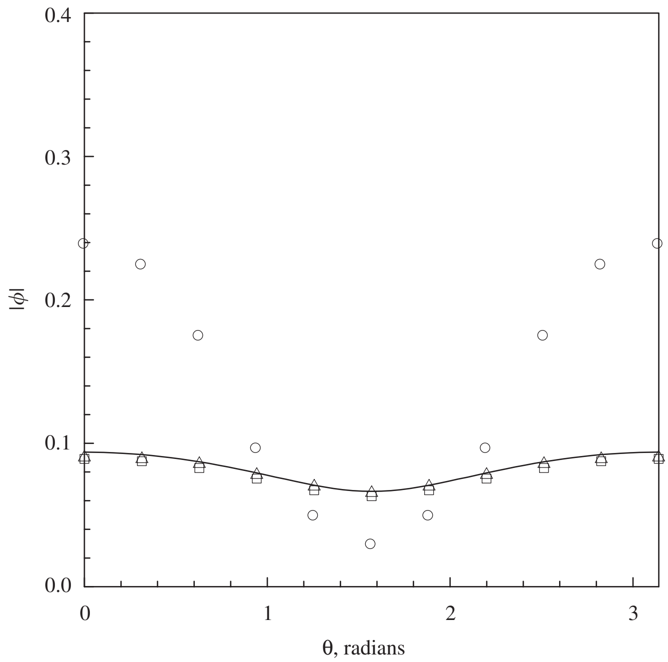

Fig. 3. Amplitude of surface function ϕ for a pulsating inverse elliptic cylinder of a:$b = 1$:2 with the normalized characteristic wavenumber $ka = 4.5$. —, the analytical solution; ○, the Helmholtz integral equation (14); △, Eq. (14) + i/ka × Eq. (22); □, Chien's method. The 60-point trapezoidal rule was applied along the θ-axis.

around 2×10^{-4} while that in the composite equation is around 10^{-2}. The great accuracy verifies the effectiveness of Eqs. (14) and (22). Notably, the order of the singularity of the Helmholtz boundary integral is $O(\ln R)$, and that of its normal derivative is $O(R^{-2})$. This fact explains why the numerical result obtained using Eq. (14) is more accurate than that obtained using the composite equation. Restated, although regularization facilitates the evaluation of singular integrals, the original singularity characteristic is retained.

Another Burton–Miller-type method developed by Chien et al. [9] was adopted to compare the accuracy and efficiency. Chien et al. applied certain identities in the interior Laplace problem to reduce the order of the hypersingularity. The singularity-free form of Chien's formulation, in Ref. [32], was calculated herein, using numerical procedures that are similar to those of the proposed method. The corresponding numerical implementation necessitates evaluating two more surface functions—the source distribution that equalizes the potential of the boundary surfaces and the normal derivative of the velocity potential in the interior Laplace problem. Figs. 1 and 2 indicate that both Burton–Miller-type methods have roughly the same accuracy. The proposed method, however, is slightly more accurate than Chien's method, because the hypersingular integrals in the proposed method are directly evaluated, whereas those in Chien's method are indirectly evaluated. Notably, the computational time of the proposed method is approximately 20% less than that of Chien's method, which requires two more matrices to be computed, as stated earlier. The proposed method's efficiency is expected to be higher for larger-scale problems because more computational time is required to evaluate matrices.

Fig. 3 plots the surface function ϕ for the normalized characteristic wavenumber $ka = 4.5$. In this case, the Helmholtz integral Eq. (14) alone cannot yield surface function. In this figure, the results obtained using the composite equation, Eq. (14) + i/ka × Eq. (22), closely correspond to the analytical solution. Fig. 4 displays the relative error of Fig. 3. Figs. 3 and 4 also plot numerical results obtained using Chien's method. These figures, again, reveal that the proposed method is more accurate than Chien's method.

Finally, substituting the calculated surface function ϕ into Eq. (28) yields the surface tangential velocity $\partial\phi/\partial\tau$. Fig. 5 plots numerical results at both $ka = 1$ and 4.5. Fig. 6 displays the relative errors. Fig. 6 demonstrates that doubling the number of integration points substantially increases the accuracy. Comparing the numerical results with the analytical solutions verifies the effectiveness of Eq. (28).

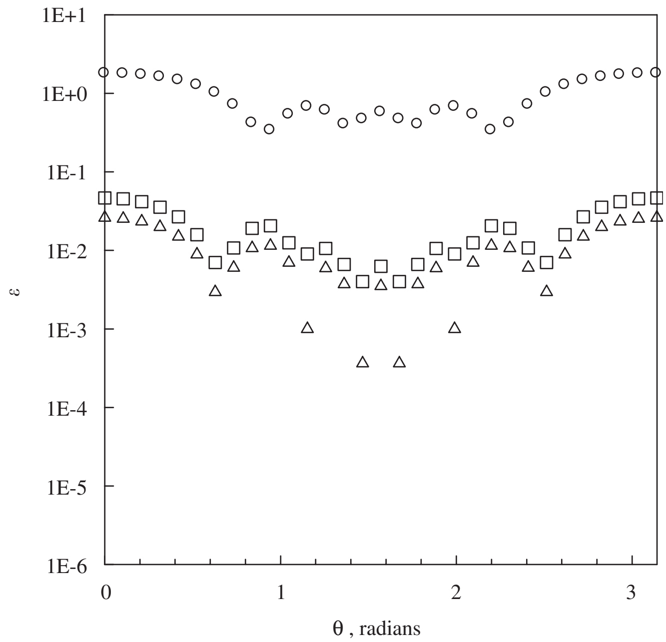

Fig. 4. Relative error ε of computed amplitude of surface field ϕ in Fig. 3. Other definitions are given in Fig. 3.

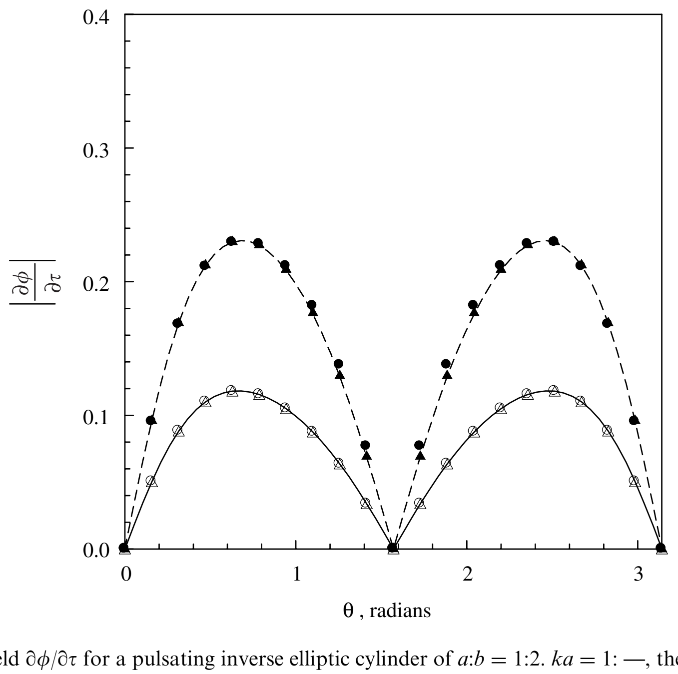

Fig. 5. Amplitude of surface field $\partial\phi/\partial\tau$ for a pulsating inverse elliptic cylinder of a:b = 1:2. $ka = 1$: —, the analytical solution; tangential derivative equation (28), ○ 60 integration points, △, 120 integration points. $ka = 4.5$: - - -, the analytical solution; tangential derivative equation (28), ●, 60 integration points, ▲, 120 integration points.

5. Conclusions

The foregoing illustrative examples demonstrate the implementation of the proposed singularity-free equations. The pressure and velocity fields are evaluated without any problem of nonuniqueness. A comparison

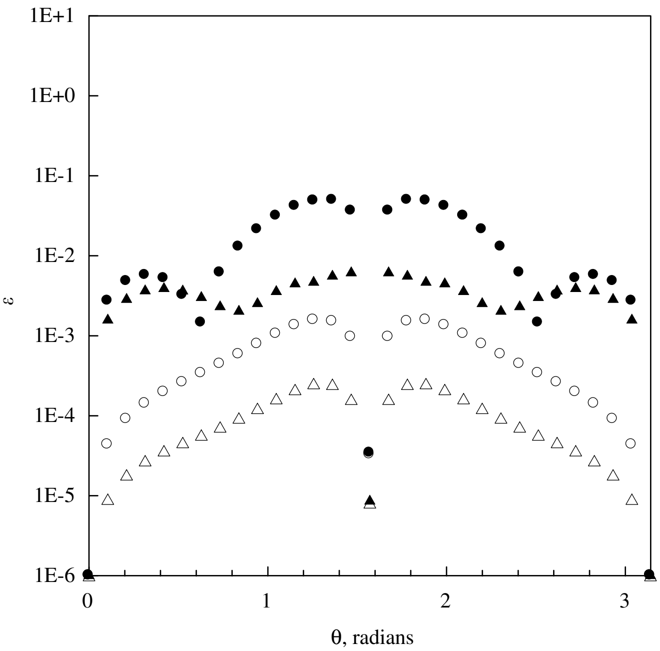

Fig. 6. Relative error ε of computed amplitude of surface field $\partial\phi/\partial\tau$ in Fig. 5. Other definitions are given in Fig. 5.

with Chien's method [9] confirms the accuracy and efficiency of the proposed method. The surface variables of an ideal fluid prescribe the outer boundary conditions when the boundary layers of a viscous fluid are considered [35]. The presented formulation is applicable for the acoustical ($k>0$) and potential flow ($k=0$) problems.

This study develops a global method: the regularization is performed before any discretization. The local method, or the standard boundary element method, usually requires, for example, a polar coordinate transformation on each element, and then evaluates the transformed integrals by applying standard quadrature. The global method is generally more accurate and efficient than the local method. See, for example Ref. [36] for a more detailed discussion.

This study converts the Helmholtz integral equation and its normal and tangential derivatives into a form that contains either singular integrals with analytical integration values, or regular integrals with smooth integrands, which can be explicitly determined when the integration point coincides with the field point. This approach is extremely attractive because it allows the integration rules to be applied directly to evaluate the integrals in the integral equations. The singularity-free form of the tangential derivative is probably the first one to appear in the literature.

The two-dimensional (2D) formulation is applicable for three-dimensional (3D) axisymmetric bodies [37] and slender bodies using the strip theory [38]. Extending to other 3D acoustical problems is conceptually straightforward; however, the implementation may be technically difficult [39]. Combining the modified integral equations with the boundary element methods warrants attention to enable the method to be applied to practical problems.

In summary, we outline the main features of the approach below:

(1) No series or other approximations are required in the process of regularization. This implies the accuracy of the numerical results.
(2) The formulation is expressed in a form without any types of singularities, including weak singularity. This implies the completeness of the regularization.
(3) The formulation can be calculated by directly applying quadrature rules over the entire integration domain. Other specific techniques, such as the logarithmic Gaussian quadrature, are not required. This implies the easiness of the numerical implementation.

(4) The formulation allows the use of the boundary-element-free method when the boundary surface can be mathematically or numerically described, as the illustrative examples show. The convenient incorporation with other numerical methods implies the universality of the formulation.

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Appendix A

A.1. Derivation of Eq. (25)

Let points $p(x(\beta), y(\beta))$ and $q(x(\alpha), y(\alpha))$ be on a sufficiently smooth boundary surface ∂B. Applying Taylor's formula leads to

$$x(\alpha) = x(\beta) + x'(\beta)(\alpha-\beta) + x''(\beta)\frac{(\alpha-\beta)^2}{2!} + x'''(\beta)\frac{(\alpha-\beta)^3}{3!} + \ldots, \tag{A.1}$$

$$y(\alpha) = y(\beta) + y'(\beta)(\alpha-\beta) + y''(\beta)\frac{(\alpha-\beta)^2}{2!} + y'''(\beta)\frac{(\alpha-\beta)^3}{3!} + \ldots. \tag{A.2}$$

The distance R between points p and q is

$$R^2 = [x(\alpha) - x(\beta)]^2 + [y(\alpha) - y(\beta)]^2. \tag{A.3}$$

Substituting Eqs. (A.1) and (A.2) into Eq. (A.3) leads to

$$\begin{aligned} R^2 = {} & (\alpha-\beta)^2[x'^2(\beta) + y'^2(\beta)] + (\alpha-\beta)^3[x'(\beta)x''(\beta) + y'(\beta)y''(\beta)] \\ & + (\alpha-\beta)^4\left[\frac{x''^2(\beta) + y''^2(\beta)}{4} + \frac{x'(\beta)x'''(\beta) + y'(\beta)y'''(\beta)}{3}\right] \\ & + (\alpha-\beta)^5\,\frac{x''(\beta)x'''(\beta) + y''(\beta)y'''(\beta)}{6} + \ldots. \end{aligned} \tag{A.4}$$

Furthermore, from Eq. (A.4), we have

$$\begin{aligned} \frac{1}{R^2} = {} & \frac{1}{(\alpha-\beta)^2(x'^2+y'^2)} \\ & \times \frac{1}{1 + (\alpha-\beta)(x'x''+y'y'')/(x'^2+y'^2) + (\alpha-\beta)^2((1/4)(x''^2+y''^2)/(x'^2+y'^2) + (1/3)(x'x'''+y'y''')/(x'^2+y'^2) + \ldots} \\ = {} & \frac{1}{(\alpha-\beta)^2(x'^2+y'^2)}\left\{1 - (\alpha-\beta)\frac{x'x''+y'y''}{x'^2+y'^2}\right. \\ & \left. + (\alpha-\beta)^2\left[-\frac{1}{4}\frac{x''^2+y''^2}{x'^2+y'^2} - \frac{1}{3}\frac{x'x'''+y'y'''}{x'^2+y'^2} + \frac{(x'x''+y'y'')^2}{(x'^2+y'^2)^2}\right] + \ldots\right\} \\ = {} & \frac{1}{(\alpha-\beta)^2}\frac{1}{(x'^2+y'^2)} - \frac{1}{(\alpha-\beta)}\frac{x'x''+y'y''}{(x'^2+y'^2)^2} \\ & - \frac{1}{4}\frac{x''^2+y''^2}{(x'^2+y'^2)^2} - \frac{1}{3}\frac{x'x'''+y'y'''}{(x'^2+y'^2)^2} + \frac{(x'x''+y'y'')^2}{(x'^2+y'^2)^3} + O(|\alpha-\beta|). \end{aligned} \tag{A.5}$$

According to Eq. (A.5), we obtain

$$\lim_{\alpha\to\beta}\left[\frac{1}{R^2}-\frac{1}{(\alpha-\beta)^2}\frac{1}{(\mathrm{d}S/\mathrm{d}\beta)^2}+\frac{1}{\alpha-\beta}\frac{x'x''+y'y''}{(x'^2+y'^2)^{3/2}}\frac{1}{\mathrm{d}S/\mathrm{d}\beta}\right]$$
$$=\frac{3}{4}\frac{x''^2+y''^2}{(x'^2+y'^2)^2}-\frac{1}{3}\frac{x'x'''+y'y'''}{(x'^2+y'^2)^2}-\kappa^2, \tag{A.6}$$

where

$$\frac{\mathrm{d}S}{\mathrm{d}\beta}=(x'^2+y'^2)^{1/2} \tag{A.7}$$

and κ is the boundary curvature that can be written as

$$\kappa=\frac{x'y''-x''y'}{(x'^2+y'^2)^{3/2}}. \tag{A.8}$$

A.2. Derivation of Eqs. (29) and (30)

The free-space Green's function G_k of the 2D Helmholtz equation can be expressed as

$$G_k=\frac{\mathrm{i}}{4}H_0^{(1)}(kR)=-\frac{1}{4}Y_0(kR)+\frac{\mathrm{i}}{4}J_0(kR), \tag{A.9}$$

where Y_0 is the Bessel function of the second kind

$$Y_0(kR)=\frac{2}{\pi}\left[\ln\left(\frac{kR}{2}\right)+\gamma\right]J_0(kR)+\frac{2}{\pi}\left[\frac{1}{(1!)^2}\frac{1}{4}(kR)^2-\left(1+\frac{1}{2}\right)\frac{1}{(2!)^2}\frac{1}{4^2}(kR)^4\right.$$
$$\left.+\left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{1}{(3!)^2}\frac{1}{4^3}(kR)^6-\dots\right] \tag{A.10}$$

and J_0 is the Bessel function of the first kind

$$J_0(kR)=1-\frac{1}{(1!)^2}\frac{1}{4}(kR)^2+\frac{1}{(2!)^2}\frac{1}{4^2}(kR)^4-\frac{1}{(3!)^2}\frac{1}{4^3}(kR)^6+\dots. \tag{A.11}$$

The tangential derivatives of Y_0 and J_0 with respect to β can be written as

$$\frac{\partial Y_0(kR)}{\partial\tau_\beta}=\frac{2}{\pi}\frac{1}{R}\frac{\partial R}{\partial\tau_\beta}J_0(kR)+\frac{2}{\pi}\left[\ln\left(\frac{kR}{2}\right)+\gamma\right]\frac{\partial J_0(kR)}{\partial\tau_\beta}+\dots$$
$$=-\frac{2}{\pi}\frac{1}{(\alpha-\beta)(x'^2+y'^2)^{1/2}}+\frac{1}{\pi}\frac{x'x''+y'y''}{(x'^2+y'^2)^{3/2}}+O(R), \tag{A.12}$$

$$\frac{\partial J_0(kR)}{\partial\tau_\beta}=-\frac{1}{2}k^2R\frac{\partial R}{\partial\tau_\beta}+\dots=O(R), \tag{A.13}$$

where

$$\frac{\partial R}{\partial\tau_\beta}=-\frac{R}{(\alpha-\beta)(x'^2+x'^2)^{1/2}}+O(R). \tag{A.14}$$

Thus, we obtain

$$\lim_{\alpha\to\beta}\frac{\partial G_k}{\partial\tau_\beta}=\lim_{\alpha\to\beta}\left(-\frac{1}{4}\frac{\partial Y_0(kR)}{\partial\tau_\beta}+\frac{i}{4}\frac{\partial J_0(kR)}{\partial\tau_\beta}\right)$$
$$\approx\frac{1}{2\pi}\frac{1}{(\alpha-\beta)}\frac{1}{(x'^2+y'^2)^{1/2}}-\frac{1}{4\pi}\frac{x'x''+y'y''}{(x'^2+y'^2)^{3/2}}. \tag{A.15}$$

The normal derivatives of Y_0 and J_0 with respect to α can be written as

$$\frac{\partial Y_0(kR)}{\partial n_\alpha} = \frac{2}{\pi}\frac{1}{R}\frac{\partial R}{\partial n_\alpha} J_0(kR) + \frac{2}{\pi}\left[\ln\left(\frac{kR}{2}\right) + \gamma\right]\left(-\frac{k^2}{2}R\frac{\partial R}{\partial n_\alpha} + \frac{k^4}{16}R^3\frac{\partial R}{\partial n_\alpha} + \dots\right) + \frac{k^2}{\pi}R\frac{\partial R}{\partial n_\alpha} + \dots, \tag{A.16}$$

$$\frac{\partial J_0(kR)}{\partial n_\alpha} = -\frac{1}{2}k^2 R\frac{\partial R}{\partial n_\alpha} + \frac{1}{16}k^4 R^3 \frac{\partial R}{\partial n_\alpha} + \dots. \tag{A.17}$$

The tangential derivatives of Eqs. (A.16) and (A.17) with respect to β can be written as

$$\begin{aligned}\frac{\partial^2 Y_0(kR)}{\partial \tau_\beta \partial n_\alpha} &= -\frac{2}{\pi}\frac{1}{R^2}\frac{\partial R}{\partial \tau_\beta}\frac{\partial R}{\partial n_\alpha} J_0(kR) + \frac{2}{\pi}\frac{1}{R}\frac{\partial^2 R}{\partial \tau_\beta \partial n_\alpha} J_0(kR) + \cdots \\ &= -\frac{\kappa}{\pi}\frac{x'x'' + y'y''}{(x'^2 + y'^2)^{3/2}} + \frac{1}{3\pi}\frac{x'y''' - x'''y'}{(x'^2 + y'^2)^2} + O(R),\end{aligned} \tag{A.18}$$

$$\frac{\partial^2 J_0(kR)}{\partial \tau_\beta \partial n_\alpha} = -\frac{k^2}{2}\frac{\partial R}{\partial \tau_\beta}\frac{\partial R}{\partial n_\alpha} - \frac{k^2}{2}R\frac{\partial^2 R}{\partial \tau_\beta \partial n_\alpha} + \cdots = O(R), \tag{A.19}$$

where

$$\frac{\partial^2 R}{\partial \tau_\beta \partial n_\alpha} = -\frac{\kappa}{2}\frac{R}{(\alpha - \beta)(x'^2 + y'^2)^{1/2}} + O(R). \tag{A.20}$$

From Eqs. (A.18) to (A.20), we obtain

$$\begin{aligned}\lim_{\alpha \to \beta}\frac{\partial^2 G_k}{\partial \tau_\beta \partial n_\alpha} &= \lim_{\alpha \to \beta}\left(-\frac{1}{4}\frac{\partial^2 Y_0(kR)}{\partial \tau_\beta \partial n_\alpha} + \frac{i}{4}\frac{\partial^2 J_0(kR)}{\partial \tau_\beta \partial n_\alpha}\right) \\ &= \frac{\kappa}{4\pi}\frac{x'x'' + y'y''}{(x'^2 + y'^2)^{3/2}} - \frac{1}{12\pi}\frac{x'y''' - x'''y'}{(x'^2 + y'^2)^2}.\end{aligned} \tag{A.21}$$

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