Available online at www.sciencedirect.com

ScienceDirect

Journal of Sound and Vibration 301 (2007) 994–1006

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

Torsional vibration of a cracked rod by variational formulation and numerical analysis

T.G. Chondros*, G.N. Labeas

Mechanical Engineering and Aeronautics Department, University of Patras, 26500 Patras, Greece

Received 2 May 2006; received in revised form 1 November 2006; accepted 13 November 2006
Available online 16 January 2007

Abstract

The torsional vibration of a circumferentially cracked cylindrical shaft is studied through an "exact" analytical solution and a numerical finite element (FE) analysis. The Hu–Washizu–Barr variational formulation is used to develop the differential equation and the boundary conditions of the cracked rod. The equations of motion for a uniform cracked rod in torsional vibration are derived and solved, and the Rayleigh quotient is used to further approximate the natural frequencies of the cracked rod. Results for the problem of the torsional vibration of a cylindrical shaft with a peripheral crack are provided through an analytical solution based on variational formulation to derive the equation of motion and a numerical analysis utilizing a parametric three-dimensional (3D) solid FE model of the cracked rod. The crack is modelled as a continuous flexibility based on fracture mechanics principles. The variational formulation results are compared with the FE alternative. The sensitivity of the FE discretization with respect to the analytical results is assessed.

1. Introduction

The investigation of cracked rotors behaviour in torsional vibration and the development of crack detection methods for rotating shafts were initiated at about 1970. In a literature survey on the dynamics of cracked rotors by Wauer [1] it is stated that obviously the first work was done by the General Electric Company. Due to the turbine rotor failures at Southern California Edison's Mohave station in 1970 and 1971 [2], industry's attention was focused on problems of turbine-generator shaft failures, caused by transmission system operation and system faults leading to fatigue cracks. Metallurgical examination revealed that the failure was due to fatigue-propagated cracks in the rotors. Also, circumferential cracks often appear in a variety of machinery such as gas and steam turbines and aircraft engines, especially in welded rotors. Identification of the crack and its depth in service is of paramount importance for the system planners to modify operating practices before large amounts of shaft fatigue life have been consumed.

The fundamental frequency vibration problem has been investigated experimentally and analytically by Dimarogonas [2,3], to assess the possibility of crack detection without interrupting the operation of the machine. For a stepped rotor, the transfer matrix technique was used to compute the change in critical speed

*Corresponding author. Tel.: +30 2610997264; fax: +30 2610997744.
E-mail address: chondros@mech.upatras.gr (T.G. Chondros).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2006.11.004

of a shaft due to the crack. The results confirmed that for small crack depths the change in critical speed is proportional to $(a/D)^2$, where a is the crack depth and D is the shaft diameter, but it was concluded that the measurement of the change in critical speed is not an efficient way to monitor rotor cracks. In order to estimate the remaining of operating shaft fatigue life, it is required that the cumulative fatigue damage associated with various types of system disturbances is determined. This is necessary because cracks initiate on a microscopic level and are not observable until the shaft has suffered extensive damage. Continued exposure to system disturbances could result in crack growth and subsequent shaft fatigue failure. A case of a generator rotor, in which a transverse fatigue crack could grow to a very large extent for several years until its extension reached 60% of the rotor section area, without being detected, is presented in Ref. [4].

Since the early 1980s, there has been a substantial body of academic research on the monitoring and early warning of cracked rotors. A state of the art review on the vibration of cracked structures [2] provides a detailed description of the papers that followed the initial investigations reported in Ref. [1] including analytical, numerical and experimental investigations. Gounaris and Papadopoulos [5] applied a new method in rotating cracked shafts to identify the depth and the location of a transverse surface crack. They reported that when a crack exists in a structure, such as a beam or rod, the excitation in one-direction causes coupled vibrations in other directions, which can be used to identify the crack. A local compliance matrix of suitable degrees of freedom was used to model the transverse crack in a shaft of circular cross-section, based on available expressions of the stress intensity factors and the associated expressions for the strain energy release rates. Then, the shaft was modelled as a rotating Timoshenko beam including the gyroscopic effect and the axial vibration due to coupling.

A broad review of the state of the art in fault diagnosis of rotating shafts and rotors is included in Ref. [6]. It was pointed out that mass unbalance, bent shafts and cracked shafts should be given special treatment. Various existing methods for crack detection in rotating shafts are compared in Ref. [7]. Among those methods, the lumped crack flexibility is shown to gain wider acceptance from researchers due to the availability of various crack models in the literature. The effect of a notch on the structure is usually simulated by a local bending moment or a reduced section, with magnitudes estimated experimentally or analytically, or with fine-mesh finite element (FE) techniques.

Although there is an extensive literature on the vibration of cracked shafts, attention is restricted to theoretical methods for estimating the effect of a crack on overall dynamics. Such estimates were made numerically or by introducing the crack local flexibility concept. Many researchers have used the vibration response to detect cracks in a structure. These detection schemes are based on the fact that the presence of a crack in a structure leads to stiffness reduction resulting to lower natural frequencies. A significant amount of research "empirical and theoretical" has been conducted in predicting the presence and location of a crack using the vibration behaviour of the structure. Researchers used beam elements of various dimensions and concentrated masses along the shaft axis under the FE method, to model stepped shafts and turbine discs. However, using these approaches, it becomes difficult to model the crack location, since every relocation into a shaft segment of different geometry requires the development of a new cracked beam element [6,7].

The influence of a peripheral crack upon the torsional dynamic behaviour of a rod was introduced by Dimarogonas and Massouros [8]. The assumption of a torsional linear spring model for the peripheral crack led to the result that the introduction of a crack results in lower torsional natural frequencies because of the added flexibility. Experimental results were in close agreement with the analysis. The results showed that the change in dynamic response due to the crack was high enough to allow the detection of the crack and estimate its location or magnitude for moderate and deep cracks. Analysis of a cracked rotor using theory of shafts with dissimilar moments of inertia and a complete flexibility matrix of the cross-section containing the crack was derived in Refs. [9,10], respectively.

Flaws in rotating shafts and service damage produce local change in stiffness altering their dynamic response. This approach suffers from various limitations mainly from the fact that the modification of the stress field induced by the crack or flaw is decaying with the distance from it. Several models dealing with the stress field due to the crack have been proposed [11–13]. A circular shaft with a transverse crack, in general, can be modelled in the vicinity of the crack by way of a local flexibility (compliance) matrix, connecting longitudinal, bending and shear forces and displacements [14]. If torsion is added, a 6×6 compliance matrix may result. This matrix, which in absence of a crack is a diagonal matrix, has off diagonal terms, which indicate coupling of the respective forces and displacements, and therefore coupling of the respective motions. In particular, coupling exists for a cracked shaft between torsion and shear [10,15,16]. The restriction on crack geometry with the assumption of symmetric cracks, or a peripheral symmetric crack avoids the coupling of torsional and flexural motion, which follows a non-symmetric crack configuration.

Christides and Barr [17] extended Barr's theory for the uncracked Euler–Bernoulli beam and developed the theory for the vibration of the cracked beam based on the general variational principle and the independent assumptions about the displacement, the momentum, the strain and the stress fields of the cracked beam. Christides and Barr [17,18] made a rigorous step towards the development of a cracked Euler–Bernoulli beam theory by deriving the differential equation and associated boundary conditions for a uniform beam containing one or more pairs of symmetric cracks. The reduction to one spatial dimension was achieved by using integrations over the cross-section after certain stress, strain, displacement and momentum fields were chosen.

Further, the modification of the stress field induced by the crack was incorporated through a local empirical function, the so-called "crack function", to model the stress concentration near the crack tip. This crack function assumed an exponential decay with the distance from the crack, and included a parameter that could be evaluated by experiments. This theory was further extended [13] for the torsional vibration of the cracked shaft based on the general variational principle. The equations of motion for a uniform shaft in torsional vibration were derived. The shaft had one or more pairs of transverse symmetrically disposed open-edge cracks along its length, to avoid coupling of torsional and bending vibration following a non-symmetric crack configuration. The cracks were regarded as always open to avoid the nonlinearity associated with the compressive stresses over a closing crack face.

The presence of a crack in a rotor produces vibration of the second and higher harmonics of the rotating frequency [2,12]. However, the amplitudes of those harmonics can be measured only if the frequency of one of the harmonics closely matches one of the natural frequencies of the shaft. While the signature analysis can easily predict the presence of a crack, it is not an easy task to locate the crack using the signature graph. The formulation of the analytical model consists either of a simple model with analysis results obtainable in closed form, hence giving a good qualitative understanding of the effect of the crack on the vibrational characteristics, or to more sophisticated models requiring demanding computational techniques, such as FE analysis.

For a quick determination of the lower natural frequency of the system, the application of the Rayleigh's principle [12] provides a fast and rather accurate method, for the computation of the fundamental frequency. This approximate energy approach states that in a conservative system, the total energy remains constant, and when the system vibrates at a natural mode, it has harmonic motion at the corresponding natural frequency. This frequency of vibration has a stationary value in the neighbourhood of a natural mode that is calculated using the Rayleigh's quotient. This method was initially introduced by Christides and Barr [13] for the investigation of the changes of the fundamental frequency of a cracked shaft with increasing crack depth with the application of the variational principle. A similar approach with application of the Rayleigh's quotient and fracture mechanics methods for the investigation of the changes of the fundamental frequency of a cracked rod was derived in Ref. [19].

The FE method [12] has been a standard structural analysis tool leading to reliable numerical results; however, it has the drawback of generality loss and the difficulty to extract the general features of the rotor behaviour. The analysis of cracked rotor systems is very complex and many simplifying assumptions have to be made in the mathematical modelling process. The key issues in developing a proper modelling technique of a cracked rotor is to model the crack more accurately, and furthermore identify the variation in the stress field over one revolution due to the opening and closing of the crack (crack breathing) and the stress–strain field complexity in the region of the developing crack. Also, the modification of the stress field induced by the crack is decaying with the distance from the crack or flaw and a direct method relating flaw position and size with stiffness change is not easy to be developed.

In this paper, a continuous model for the cracked rod is introduced for vibration analysis and crack identification The cracked rod model satisfies the Euler–Bernoulli theory [12,13,20], i.e., the planar cross-sections of the undeformed rod remain plane after deformation, perpendicular to any axis along the length of the rod and retain their original size and shape after deformation, and the vibration of the cracked rod as a

1-D structural member is described by a linear hyperbolic partial differential equation of second order with respect to time. The rod has one circumferential open edge crack along its length and instead of the introduction of an empirical function for the modification of the stress field in the vicinity of the crack, it will be evaluated here with fracture mechanics methods. The application of the Rayleigh's principle is also used for the computation of the fundamental frequency. A parametric FE model using 3-D solid elements was employed for the numerical prediction of the dynamic response of the cracked rod based on the commercial code ANSYS [21]. The model can be modified accordingly in order to analyse different rod geometrical configurations and varying crack locations and depth. Numerical results from the continuous cracked rod torsional vibration theory are compared with the approximate energy method approach, the FE method and experimental data from the literature.

2. Torsional vibration of a continuous cracked rod using the variational theorem and the crack disturbance factor

2.1. The variational theorem

The Hu–Washizu–Barr [18,22,23] variational principle in linear elasticity allows independent variation of displacement, strain, and stress for the construction of approximate equations of equilibrium in elastostatic problems. Application of the Hu–Washizu variational principle requires plausible assumptions about the momentum, strain and stress field to be chosen. Christides and Barr extended the Hu–Washizu principle further by incorporating the perturbation in the stress and strain distributions of the beam due to the presence of the crack through local functions which have their maximum value at the cracked section and decay exponentially from the crack location. The equations of motion of a cracked beam-like structure are derived through the Hu–Washiszu–Bar [17,18] variational principle in linear elasticity in which displacement, strain and stress can be independently varied.

A continuous rod with an open peripheral surface crack is shown in Fig. 1. The x-axis of the rod is taken along its straight centre-line while y and z are the principal axes of the cross-section. Let the displacement components be denoted by u_i, the strain components by γ_{ij} and the stress components by σ_{ij} with $i, j = 1, 2, 3$ referring to Cartesian axes x, y, z. Normal engineering notation will be used here with $u_1 = u$, $u_2 = v$, $u_3 = w$.

Let p_i be the momentum such that $T_m = 1/2\ \rho\delta_{ij}p_ip_j$ will be the kinetic energy density (δ_{ij} is the Kronecker's delta). For arbitrary independent variations δu_i, $\delta\gamma_{ij}$, $\delta\sigma_{ij}$, and δp_i, the extended Hu–Washizu variational principle was introduced by Barr [18] in the form

$$\int_V \Big\{ \left[\sigma_{ij,j} + F_i - \rho\dot{p}_i\right]\delta u_i + \left[\sigma_{ij} - W_{,\gamma_{ij}}\right]\delta\gamma_{ij} + \left[\gamma_{ij} - \left(1 - \frac{1}{2}\delta_{ij}\right)(u_{i,j} + u_{j,i})\right]\delta\sigma_{ij} + \left[\rho\dot{u}_i - T_{m,p_i}\right]\delta p_i \Big\}\,\mathrm{d}V + \int_{S_g}\left[\bar{g}_i - g_i\right]\delta u_i\,\mathrm{d}S + \int_{S_u}[u_i - \bar{u}_i]\delta g_i\,\mathrm{d}S = 0, \tag{1}$$

where $W(\gamma_{ij})$ is the strain energy density function, ρ is the density of the material. F_i, g_i and u_i are, respectively, the body forces, the surface traction and the surface displacement. Moreover, V is the total volume of the solid and S_g and S_u are its external surfaces. The overbar denotes the prescribed values of the surface traction and

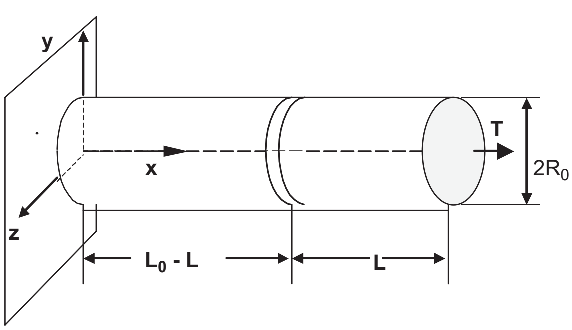

Fig. 1. Geometry of a circular cracked rod.

the surface displacement. The prescribed surface tractions g_i are applied over the surface S_g and the prescribed displacements u_i are over S_u. Together S_g and S_u make up the total surface of the solid. The differentiation with respect to time ($\partial/\partial t$) is indicated by a dot. Commas in the subscripts indicate differentiation with respect to Cartesian axes.

To derive the governing equation and applicable boundary conditions for the torsional vibration of a cracked shaft through the variational theorem (Eq. (1)), the x-axis is taken along the center line of the bar, and the yz plane is the plane of the cross-section. The subscripts i, j of the stresses σ_{ij} take the values 1, 2, 3 corresponding to the Cartesian coordinates x, y and z, respectively.

The stress and strain fields of the cracked shaft can be obtained by adding the disturbance functions to the stress and strain distributions of the undamaged shaft. Since the distribution of each stress (and the equivalent strain) component is unique, the most general situation will be considered here in which a different disturbance function is added to each component. Thus, the disturbance in the direct components σ_{xx} and γ_{xx} is introduced through a function $f_1(x,y,z)$, the disturbance in the shear components σ_{xy} and γ_{xy} is introduced through a function $f_2(x,y,z)$ and the disturbance in the shear components σ_{xz} and γ_{xz} is introduced through a function $f_3(x,y,z)$. The three crack functions f_1, f_2 and f_3 are all, at present, unknown. These functions have their maximum value at the crack tip and decay with the distance from the cracked section. As mentioned earlier, Christides and Bar [13,17] used an empirical exponential function to describe the stress disturbance due to a crack. In this work, the stress disturbance function will be found using fracture mechanics results.

It is further assumed that the presence of the cracks does not alter in any way the displacement fields of the rod. Finally, the direct strains γ_{yy} and γ_{zz} will be taken as non-zero although the corresponding direct stresses will be assumed zero. For a uniform rod the following assumptions [13] are made

$$\begin{aligned}
&u = \phi(y,z)\theta'(x,t), \quad \upsilon = -z\theta(x,t), \quad w = y\theta(x,t),\\
&p_x = 0, \quad p_y = -zP(x,t), \quad p_z = yP(x,t),\\
&\gamma_{xx} = \left[\phi + f_1(x,y,z)\right] S_1(x,t), \quad \gamma_{yy} = \gamma_{zz} = -v\gamma_{xx},\\
&\gamma_{xy} = \left[\frac{\partial\phi}{\partial y} - z + f_2(x,y,z)\right] S_2(x,t),\\
&\gamma_{xz} = \left[\frac{\partial\phi}{\partial z} + y + f_3(x,y,z)\right] S_2(x,t), \quad \gamma_{yz} = 0,\\
&\sigma_{xx} = \left[\phi + f_1(x,y,z)\right] T_1(x,t), \quad \sigma_{yy} = \sigma_{zz} = 0,\\
&\sigma_{xy} = \left[\frac{\partial\phi}{\partial y} - z + f_2(x,y,z)\right] T_2(x,t),\\
&\sigma_{xz} = \left[\frac{\partial\phi}{\partial z} + y + f_3(x,y,z)\right] T_2(x,t),\\
&\sigma_{yz} = 0,\\
&X_x = X_y = X_z = 0
\end{aligned} \tag{2}$$

where $P(x,t)$ is the velocity function, $S_1(x,t)$, $S_2(x,t)$ strain functions, $T_1(x,t)$ $T_2(x,t)$ stress functions and X_i the body forces. In these equations θ, P, S_1, S_2, T_1 and T_2 are all unknown functions of the axial coordinate x and the time t. The warping function $\phi(y,z)$ associated with the strain and displacement fields due to the rotation of the cross-section along with warping of the section out of its plane, is approximated by that of static torsion and it should satisfy two important conditions [24]. First, it is a harmonic function, i.e., it satisfies the condition $\partial^2\phi/\partial y^2 + \partial^2\phi/\partial z^2 = 0$ everywhere in the cross-section. This condition also implies that in case of a doubly symmetrical cross-section, e.g., a circle, an ellipse or a rectangle cross-section, the function $\phi(y,z)$ is odd in both the y and the z coordinates. Second, the shear stresses σ_{xy} and σ_{xz} of the undamaged shaft (which are functions of ϕ) are such that the condition $\sigma_{xy}v_y + \sigma_{xz}v_z = 0$ is satisfied on the lateral surfaces of the shaft, in which v_y and v_z are the directional cosines. This condition implies that the resultant shearing stress on the boundary is directed along the tangent to the boundary. Also, the rod with the symmetric circumferential crack twisted about its central axis does not present any warping and therefore it is feasible to obtain stress disturbance functions through fracture mechanics. In this case, the warping function will vanish [13,24], that is $\phi = 0$.

2.2. The equation of motion

Independent assumptions about the displacement, the momentum, the strain and the stress fields of the cracked rod are considered for the derivation of the equations of motion for a uniform rod in torsional vibration, where the rod has one circumferential open edge crack along its length. The stress and strain fields of the cracked rod can be obtained by adding disturbance functions to the stress and strain distributions of the undamaged rod. The most general situation will be considered here, in which a different disturbance function is added to each component of the stress functions of the undamaged rod. Thus, the disturbance in the normal components σ_{xx} and γ_{xx} is introduced through a function $f_1(x,y,z) = f_1$, the disturbance in the shear components σ_{xy} and γ_{xy} is introduced through a function $f_2(x,y,z) = f_2$ and the disturbance in the shear components σ_{xz} and γ_{xz} is introduced through a function $f_3(x,y,z) = f_3$.

Christides and Barr [13] derived the general equation for torsional vibration of a cracked rod considered as a 1-D continuous model in the form of the following fourth-order differential equation

$$\eta_1 E\theta^{iv} + 2\eta_1' E\theta''' + \left(\eta_1'' E - \eta_2 G\right)\theta'' - \eta_2 G\theta' + \rho J\ddot{\theta} = 0, \tag{3}$$

where E is the Young's modulus, $G = E/[2(1+\nu)]$, ν is the Poisson's ratio, θ is the angular displacement, θ', θ'', θ''', θ^{iv}, θ^{v}, the derivatives with respect to the coordinate x, $\ddot{\theta}$ the derivative with respect to time, ρ is the material's mass density, J is the polar moment of inertia, $\eta_1 = (R+F)\ Q_1(x)$, $\eta_2 = (L+J+D+M)Q_2(x)$, $Q_1(x) = (R+F)/(R+2F+B)$, $Q_2(x) = (L+J+D+M)/(L+J+C+2D+2M)$ and the integrals B, C, D, F, J, K, L, M, and R are defined over the rod cross-section A as follows

$$\begin{aligned}
&B(x) = \int_A (f_1^2)\mathrm{d}A, \quad C(x) = \int_A (f_2^2 + f_3^2)\mathrm{d}A, \quad D(x) = \int_A \{(\partial\phi/\partial y)f_2 + (\partial\phi/\partial z)f_3\}\,\mathrm{d}A,\\
&F(x) = \int_A (\phi f_1)\mathrm{d}A, \quad M(x) = \int_A \{yf_3 - zf_2\}\mathrm{d}A, \quad J = \int_A \{y^2 + z^2\}\,\mathrm{d}A,\\
&K = \int_A \{(\partial\phi/\partial y)^2 + (\partial\phi/\partial z)^2\}\,\mathrm{d}A, \quad L = \int_A \{(\partial\phi/\partial z)y - (\partial\phi/\partial y)z\}\,\mathrm{d}A, \quad R = \int_A (\phi^2)\,\mathrm{d}A.
\end{aligned} \tag{4}$$

In Eq. (4) the functions F, B, C, D and M are integrals involving the crack disturbance functions f_1, f_2, f_3, and the functions K, L, R are integrals involving the warping function ϕ over the intact section.

If the angular displacement θ is taken in the normal mode form $\theta(x,t) = W(x)\cos\omega^* t$, where $W(x)$ is an assumed shape function and ω^* the fundamental frequency of the cracked rod, Eq. (1) becomes [13]

$$\left(\eta_1 EW''\right)'' - \left(\eta_2 GW'\right)' = \omega^{*2}\rho JW, \tag{5}$$

where primes indicate differentiation with respect to the coordinate x.

The above differential equation of motion (5) is of the fourth order; therefore, two boundary conditions must be satisfied at each end of the rod. The boundary conditions appropriate to this equation are obtained by equating the boundary terms to zero in the case of prescribed external forces and prescribed displacements, respectively. A cantilever rod with its fixed end at $x = 0$, has at that point displacements u, v and w all prescribed as zero, while at the free end $x = L_0$ has the forces prescribed as zero.

Application of the above boundary conditions at the boundary terms described in detail by Christides and Barr [13] yields the displacements, stresses, strains and velocity momentum terms for the cracked rod from Eq. (2). Assuming that the axial stress σ_{xx} has no effect in the torsional vibration of cracked shafts with circular cross-section, the crack disturbance function $f_1 = 0$. Since only torsional vibration is considered here, it is expected that $f_2 = 0$, and functions $\eta_1(x)$ and $\eta_2(x)$, become

$$\begin{aligned}
&\eta_1(x) = 0,\\
&\eta_2(x) = \frac{(J+M)^2}{(J+C+2M)}.
\end{aligned} \tag{6}$$

The variation of the natural frequencies and in particular the fundamental frequency with increasing crack depth is of interest in the present analysis for the purpose of diagnosing and monitoring cracks. From Eqs. (4) and (6), and the boundary terms described in Ref. [13] for the torsional vibration of a cracked rod of length L_0 and radius R_0, the solution of Eq. (5) is found with the aid of Mathematica and the fundamental frequency ratio ω^*/ω is shown in Fig. 2, for different crack depths. Here, ω is the frequency of torsional vibration for the uncracked rod and ω^* the frequency of torsional vibration for the cracked rod. In Fig. 2, experimental results taken from Dimarogonas and Massouros [8] are compared with the solution of Eq. (5). The analytic solution found in Ref. [8] and the analytic solution from Wauer [11], both with lumped crack flexibility are also shown in Fig. 2. These results agree very well.

2.3. The crack disturbance function

For a circular rod of length L_0 and cross-sectional radius R_0 in pure torsion with a symmetric ring-shaped crack, as shown in Fig. 1, the stress disturbance function due to the presence of the crack is set in the form

$$f_3(x) = \frac{D_0\sqrt{a}F_{\mathrm{III}}(\alpha)\beta}{\sqrt{2r}}, \tag{7}$$

where D_0 is a constant to be found with fracture mechanics principles, a is the crack depth, r is the distance from the crack tip, α is the ratio a/R_0, β is the ratio R_0/L_0 and

$$F_{\mathrm{III}}(\alpha) = \frac{3}{8}\sqrt{1-\alpha}\begin{bmatrix} 1+\frac{1}{2}(1-\alpha)+\frac{3}{8}(1-\alpha)^2+ \\ \frac{5}{16}(1-\alpha)^3+\frac{35}{128}(1-\alpha)^4 \\ +0.208(1-\alpha)^5 \end{bmatrix}. \tag{8}$$

For a ring-shaped crack of depth *a,* the shear stress distribution for the tearing mode III will be denoted as σ_{yz} given in Ref. [25] as

$$\sigma_{yz} = \frac{K_{\mathrm{III}}}{\sqrt{2\pi r}}\cos^2\frac{\theta}{2}, \tag{9}$$

where K_{III} is the stress intensity factor for the tearing mode III and r, θ the distance and orientation from the crack tip, respectively.

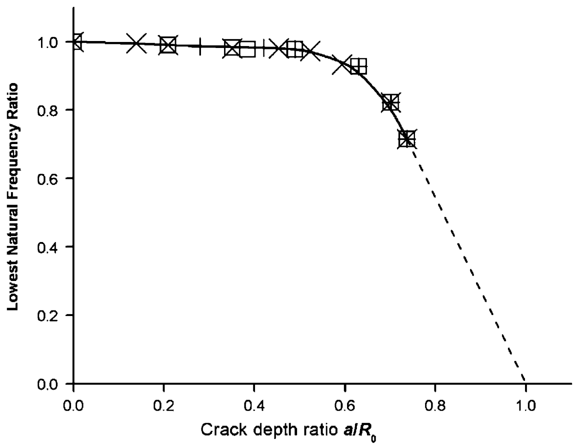

Fig. 2. Lowest natural frequency shifting ratio ω^*/ω vs. crack depth ratio; comparison of analytical and experimental results: (———) solution of Eq. (5); (+) Dimarogonas and Massouros, experimental; (□) Dimarogonas and Massouros, analytical; and (×) Wauer's solution.

For a rod having a circumferential crack under torque T [25–27], K_{III} is given as

$$K_{\mathrm{III}} = \sigma_{Tn}\sqrt{\pi a}F_{\mathrm{III}}(\alpha), \tag{10}$$

where $\sigma_{Tn} = 2T/\pi(R_0–a)^3$

In order to calculate the scale factor D_0 in Eq. (7), a circular rod loaded with a torque T is considered with a ring shaped crack of depth a, as shown in Fig. 1. Under general loading, the additional twist angle θ^* in the direction of the torque T, due to the presence of a crack, will be computed using Castigliano's theorem and the generalization of the Paris equation [28–30]. D_0 will be computed so that the relative twist of the two ends will be equal to the one computed with fracture mechanics methods.

If U_T is the strain energy due to a crack, Castigliano's theorem demands that the additional twist θ^* due to the initial torque T is

$$\theta^* = \partial U_T/\partial T. \tag{11}$$

The strain energy U_T and the strain energy density J_S have the form [25–31], respectively

$$U_T = 2\pi\int_0^a (R-a)J_S\,\mathrm{d}a \tag{12}$$

and

$$J_S = \frac{(1+v)\left(1-v^2\right)}{E}K_{\mathrm{III}}^2. \tag{13}$$

From Eqs. (7), (11), (12), and (13) the additional twist θ^* is calculated as

$$\theta^* = \frac{4\pi(1-v^2)\sigma_{Tn}}{G}\left[\frac{4a-R_0}{R_0-a}+\frac{(R_0-a)^3}{12R_0^3}\right]. \tag{14}$$

On the other hand, according to the rod deformation analysis theory the general form for the twist angle of the rod is [32,33]

$$\theta^* = TL_0/GJ, \tag{15}$$

where T is the applied torque, L_0 the length of the rod, $G = E/[2(1+\nu)]$, E is the Young's modulus, v is the Poisson's ratio, and J is the polar moment of inertia.

Assuming that the stress disturbance function $f_3(x)$ in Eq. (4) acts directly on the torque T, Eq. (7), the additional twist due to the crack located at $x = L$ takes the form

$$\theta^* = D_0\frac{\sqrt{a}\sigma_{Tn}\pi L_0(1-a)^3R_0^3F_{\mathrm{III}}(\alpha)\beta}{2GJ}. \tag{16}$$

Eqs. (11) and (15) yield the constant D_0 required for the derivation of the crack disturbance function, as

$$D_0 = \frac{4\sqrt{2}\pi\left(1-v^2\right)\sqrt{L_0-L}\beta}{L_0\sqrt{a}F_{\mathrm{III}}(\alpha)}\left[\frac{4\alpha-1}{(1-\alpha)^4}+\frac{1}{12}\right]. \tag{17}$$

2.4. The Rayleigh's quotient

Alternatively, an approximate energy method approach, the Rayleigh's quotient method as used in Refs. [13,19], can be employed to estimate the fundamental frequency ω^* of the cracked rod for various crack depths.

In Eq. (5) denoting $L(W) = (\eta_1 EW'')''-(\eta_1 GW')'$ and $N(W) = \rho JW$, the Rayleigh's quotient associated with the differential equation of motion takes the form [13]

$$Q_R(W) = \frac{\int WL(W)\,\mathrm{d}x}{\int WN(W)\,\mathrm{d}x}, \tag{18}$$

where the two integrals are over the length of the rod.

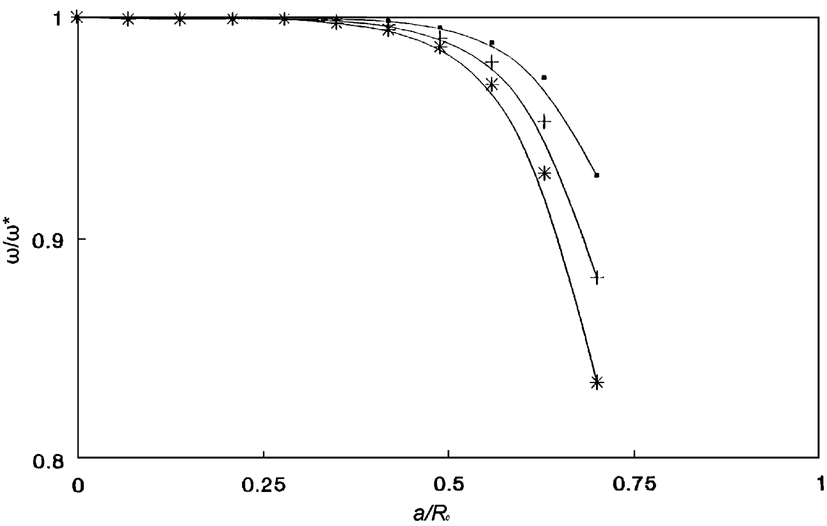

Fig. 3. Lowest natural frequency shifting ratio ω/ω^* for increasing crack depth of the rod in torsional vibration. Solution of Eq. (20): (●) $R_0/L_0 = .0071$; (+) $R_0/L_0 = .0102$; and (*) $R_0/L_0 = .0133$.

Therefore, $Q_R(W)$ provides an estimate for ω^* from an assumed function $W(x)$ which, for the fundamental mode, will exceed the true value. The numerator of Eq. (18) can be integrated by parts resulting in boundary terms which are taken to be zero for an appropriate choice of $W(x)$. The Rayleigh's quotient then, Eq. (18) provides an approximation for the fundamental frequency ω^* of the cracked rod as

$$\omega^{*2} = \frac{\int_0^{L_0} \left\{ \eta_1 E(W'')^2 + \eta_2 G(W')^2 \right\} \mathrm{d}x}{\rho J \int_0^{L_0} W^2 \, \mathrm{d}x}. \tag{19}$$

From the experimental work available in the literature [8,11,13], it is demonstrated that the natural torsional frequencies of the cracked rod are rather insensitive to small depths of the cracks. Only when the total crack depth is more than 50% of the rod diameter, there is an appreciable drop in the value of the natural frequencies. This suggests that the shape function $W(x)$ of the cracked rod might be approximated reasonably well by the shape function of the undamaged rod. From Eq. (19) an approximate solution of the fundamental frequency ω^* for the cracked rod with a peripheral crack at mid-span is calculated, and will be used for the estimation of the fundamental frequency shifting ratio as

$$\text{fundamental frequency shifting ratio} = \omega^*/\omega. \tag{20}$$

the relative frequency change of the cracked versus the uncracked rod, for different crack depths and R/L_0 ratios as shown in Fig. 3. The more slender the rod is, the smaller the frequency shifting ratio drop is.

3. FE analysis of the vibrating cracked rod

The FE formulation of an Euler–Bernoulli cracked beam using 3-D solid elements, leads to a system of linear algebraic equations of the form [12]

$$[\mathbf{M}]^S\{\ddot{\mathbf{q}}\}^S + [\mathbf{C}]^S\{\dot{\mathbf{q}}\}^S + [\mathbf{K}]^S\{\mathbf{q}\}^S = \{\mathbf{f}\}^S, \tag{21}$$

where $[\mathbf{M}]^S$, $[\mathbf{C}]^S$ and $[\mathbf{K}]^S$ are the mass, damping and stiffness matrices for the vibrating system and $\mathbf{q}(t)$ the response of the vibrating cracked rod of Fig. 1 in a stationary coordinate system. The damping part of Eq. (21) is neglected, since undamped vibration is considered here. For the solution of Eq. (21), the developed linearized 3-D FE model of the cracked rod is shown in Fig. 4.

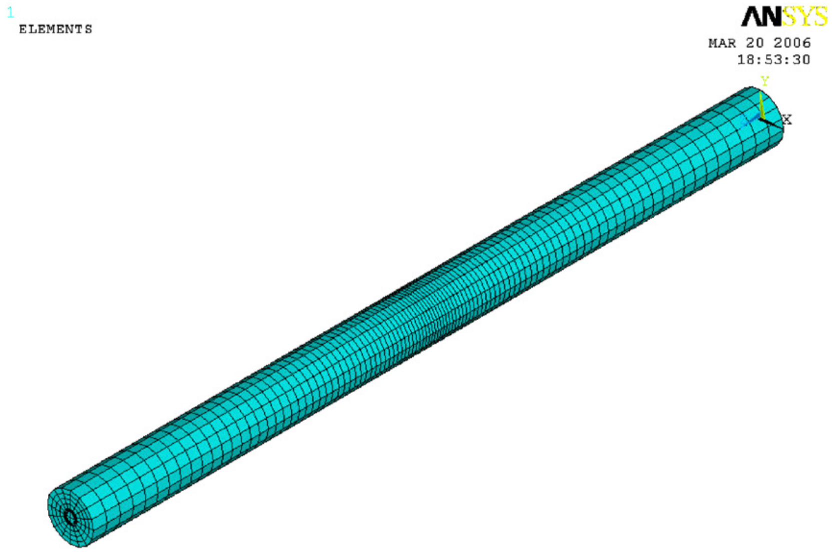

Fig. 4. Three-dimensional FE model of the cracked rod.

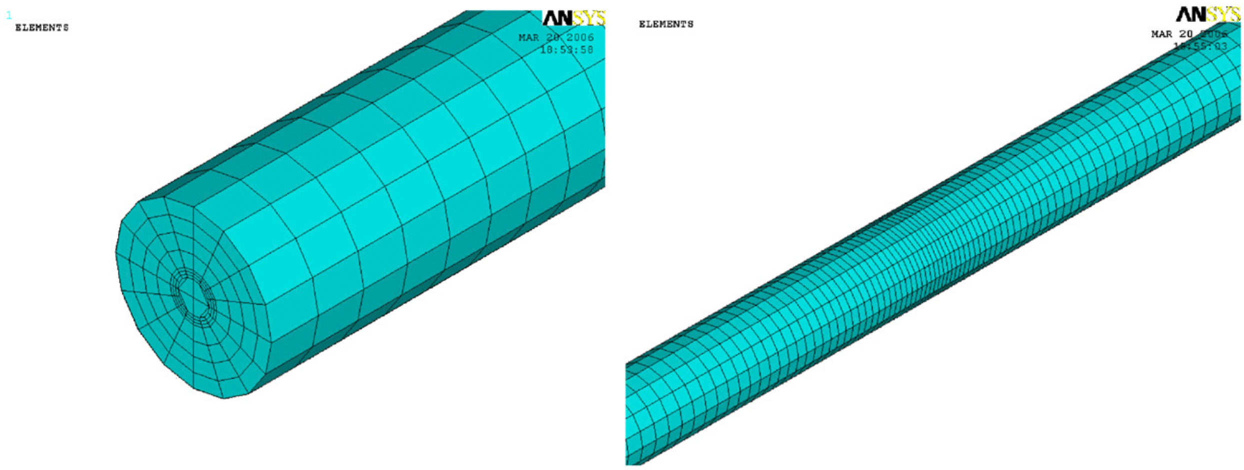

Fig. 5. Details of the FE discretization at the area of the circumferential crack.

The FE mesh of the considered crack rod is developed using the FE software ANSYS [21] and the non-singular 8-node brick element 'solid 45', which has three degrees of freedom per node, i.e., the displacements in the x, y and z directions. The circumferential crack is modelled by assuming the corresponding nodes of the two crack surfaces to deform independently. The crack surfaces are modelled using double nodes which are identical in location but topologically belong to the two different crack faces. Contact elements are not used in the present model, therefore, contact or friction between the crack faces is not taken into account. Details of the FE mesh generated around the crack faces are presented in Fig. 5. It may be observed that the elements length is reduced in the axial direction towards the crack area, as this is the most deformable and most stressed area of the model, requiring a better refinement.

The solution of the modal eigenvalue problem, using the developed FE model, has revealed the extensional, bending and twisting vibration modes, as well as, their interactions. In present, only the torsional natural frequencies are investigated. The sensitivity of the FE mesh with respect to the numerical results for various

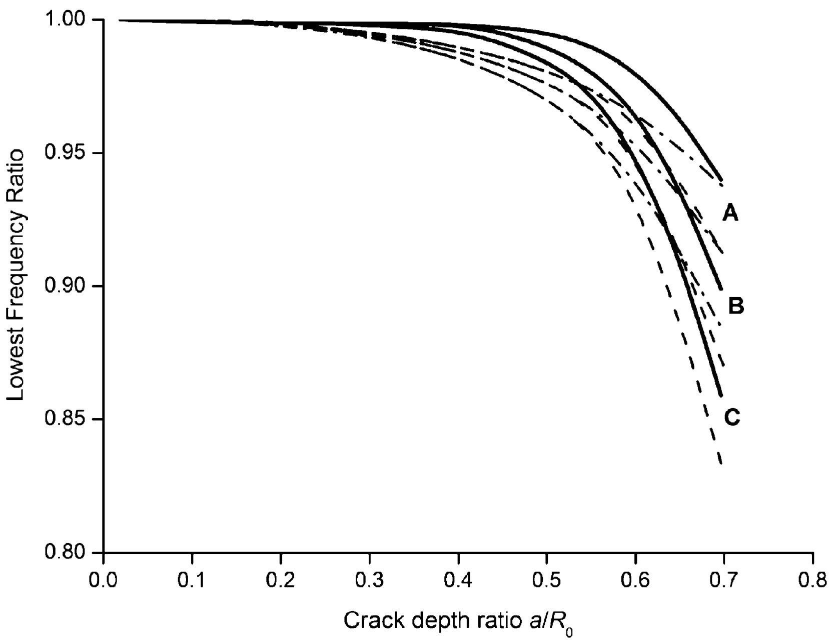

Fig. 6. Lowest natural frequency drop for increasing crack depth of the rod in torsional vibration and varying mesh size in the FE analysis. Rod radius to length ratios: (A) $R_0/L_0 = .0071$; (B) $R_0/L_0 = .0102$ s and (C) $R_0/L_0 = .0133$. (———) Solution of Eq. (5); (–· –· –) FE solution (normal mesh) and (––––) FE solution (dense mesh).

crack depth ratios has been studied. The FE mesh has been refined successively, from rough to very dense mesh and results of the corresponding frequency drop are shown in Fig. 6.

In Fig. 7 the results for the fundamental frequency shifting ratio of the FE model with a dense mesh are compared with the results from the solution of Eqs. (5) and (20) for different R_0/L_0 ratios and varying crack depth. The crack is considered at mid-span of the cantilever rod shown in Fig. 1. By comparing the FE results to the corresponding results from the application of the continuous cracked rod torsional vibration theory (Eq. (5)), a very good correlation may be observed.

4. Conclusions

In this paper, the modelling and the formulation of the governing dynamic equations for cracked rods in torsional vibration have been studied. The Hu–Washizu–Barr variational formulation was used to develop the differential equation and the boundary conditions of the cracked rod [13,18,20,22]. The method is based on the general variational principle and the independent assumptions about the displacement, the momentum, the strain and the stress fields of the cracked rod, as well as, the equations of motion for a uniform rod in torsional vibration with one circumferential open edge crack along its length, were derived.

The differential equation for the torsional vibration of the cracked rod, Eq. (5) was solved with the aid of Mathematica. The Rayleigh's quotient [12,13,19], an approximate energy method was used to estimate the fundamental frequency ω^* of the cracked rod yielding the fundamental frequency shifting ratio ω^*/ω, the relative frequency change of the cracked versus the uncracked rod, for different crack depths. Results were compared with available analytical and experimental data.

The crack is introduced as a stress disturbance function, the stress field is determined by fracture mechanics methods, thus a priori assumption for the extent of the stress field due to the crack is not required. The crack was regarded as always open to avoid the nonlinearities associated with the compressive stresses over a closing crack face.

In addition, a parametric 3-D FE model using 3-D solid elements was employed for the analysis of the cracked rod behaviour. This model based on the commercial code ANSYS [21] was used for the numerical

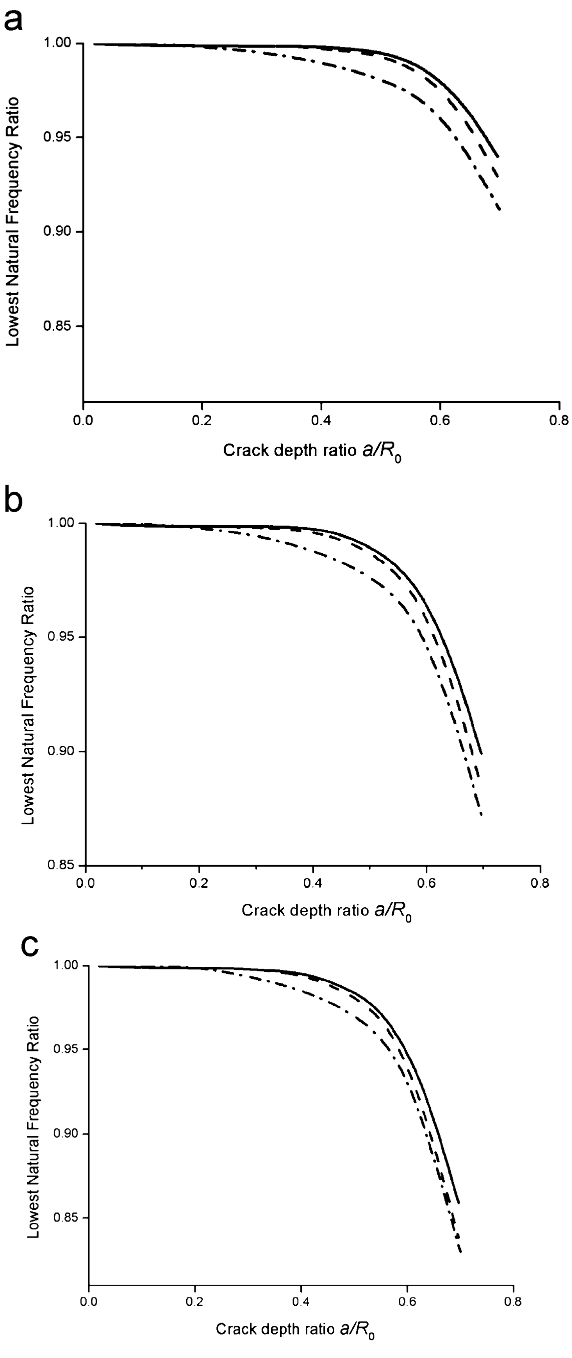

Fig. 7. Lowest natural frequency shifting ratios vs. crack depth ratio. Comparison of analytical and numerical results. Rod radi us to length ratios: (a) $R_0/L_0 = .0071$; (b) $R_0/L_0 = .0102$ and (c) $R_0/L_0 = .0133$. (———) Eq. (5); (– – – –) Eq. (20) and (– · – · –) FE dense mesh.

prediction of the dynamic response of the cracked rod. The model can be modified accordingly in order to analyse different rod geometrical configurations and varying crack locations and depth.

Numerical results from the continuous cracked rod torsional vibration theory are compared with the approximate energy method approach, the FE method and experimental data from the literature. The 3-D solid FE model results with appropriate assumptions regarding the rod and crack geometry, provide good agreement with the variational formulation methods for the cracked rod analysis. Careful observation of the behaviour of these damage models can lead to extension of their utility for defects of practical engineering importance in the area of vibration and fault detection of cylindrical shafts and rotors.

References

[1] J. Wauer, On the dynamics of cracked rotors: a literature survey, *Applied Mechanics Review* 43 (1) (1990) 13–17.
[2] A.D. Dimarogonas, Vibration of cracked structures: a state of the art review, *Engineering Fracture Mechanics* 55 (5) (1996) 831–857.
[3] C.A. Papadopoulos, A.D. Dimarogonas, Coupled vibration of cracked shafts, *Journal of Vibration and Acoustics* 114 (1992) 461–467.
[4] V. Bicego, E. Lucon a, C. Rinaldi a, R. Crudeli, Failure analysis of a generator rotor with a deep crack detected during operation: fractographic and fracture mechanics approach, *Nuclear Engineering and Design* 188 (1999) 173–183.
[5] G.D. Gounaris, C.A. Papadopoulos, Crack identification in rotating shafts by coupled response measurements, *Engineering Fracture Mechanics* 69 (2002) 339–352.
[6] S. Edwards, A.W. Lees, M.I. Friswell, Fault diagnosis of rotating machinery, *The Shock and Vibration Digest* 30 (1) (1998) 4–13.
[7] H. Keiner, M.S. Gadala, Comparison of different modelling techniques to simulate the vibration of a cracked rotor, *Journal of Sound and Vibration* 254 (5) (2002) 1012–1024.
[8] A.D. Dimarogonas, G. Massouros, Torsional vibration of a shaft with a circumferential crack, *Engineering Fracture Mechanics* 15 (3–4) (1981) 439–444.
[9] A.D. Dimarogonas, C.A. Papadopoulos, Vibration of cracked shafts in bending, *Journal of Sound and Vibration* 91 (1983) 583–593.
[10] C.A. Papadopoulos, A.D. Dimarogonas, Coupled longitudinal and bending vibrations of a rotating shaft with an open crack, *Journal of Sound and Vibration* 117 (1987) 81–93.
[11] J. Wauer, Modelling and formulation of equation of motion for cracked rotating shafts, *International Journal of Solids and Structures* 26 (4) (1990) 901–914.
[12] A.D. Dimarogonas, *Vibration for Engineers*, second ed., Prentice-Hall, Englewood Cliffs, NJ, 1996.
[13] S. Christides, A.D.S. Barr, Torsional vibration of cracked beams of non-circular cross-section, *International Journal of Mechanical Sciences* 28 (7) (1986) 473–490.
[14] A.D. Dimarogonas, S.A. Paipetis, *Analytical Methods in Rotor Dynamics*, Applied Science, London, 1983.
[15] C.A. Papadopoulos, A.D. Dimarogonas, Coupled vibration of cracked shafts, *Journal of Vibration and Acoustics* 114 (1992) 461–467.
[16] A.K. Darpe, K. Gupta, A. Chawla, Coupled bending, longitudinal and torsional vibration of a cracked rotor, *Journal of Sound and Vibration* 269 (2004) 33–60.
[17] S. Christides, A.D.S. Barr, One-dimensional theory of cracked Bernoulli–Euler beams, *International Journal of Mechanical Sciences* 26 (11/12) (1984) 639–648.
[18] A.D.S. Barr, An extension of the Hu–Washizu variational principle in linear elasticity for dynamic problems, *Transactions ASME Journal of Applied Mechanics* 33 (2) (1966) 465.
[19] T.G. Chondros, Variational formulation of a rod under torsional vibration for crack identification, *Theoretical and Applied Fracture Mechanics* 44 (2005) 95–104.
[20] B.K. Donaldson, *Analysis of Aircraft Structures An Introduction*, McGraw-Hill, New York, 1993.
[21] ANSYS, Inc. 2003, *ANSYS Version 7.1*.
[22] H.C. Hu, On some variational principles in the theory of elasticity and plasticity, *Scientia Sinica* 4 (1955) 33–55.
[23] K. Washizu, *On the variational principles of elasticity and plasticity*, Technical Report 25–18, Contract No. N5-07833, Massachusetts Institute of Technology, Cambridge, MA, USA, 1955.
[24] A.E.H. Love, *The Mathematical Theory of Elasticity*, fourth ed., Cambridge University Press, Cambridge, 1952.
[25] H. Tada, P. Paris, G.S. Sih, *1985 The Stress Analysis of Cracks Handbook*, Del Research Corporation, Hellertown, PA, USA, 1973.
[26] G.R. Irwin, Analysis of stresses and strains near the end of a crack traversing a plate, *Journal of Applied Mechanics* 24 (1957) 361–364.
[27] G.R. Irwin, *Fracture in Handbuch der Physik*, Vol. 6, Springer, Heidelberg, 1958, pp. 551–590.
[28] G.C. Sih, Strain energy density factor applied to mixed mode problems, *Engineering Fracture Mechanics* 10 (1974) 305–321.
[29] G.C. Sih, Some basic problems in fracture mechanics and new concepts, *Engineering Fracture Mechanics* 5 (1973) 365–377.
[30] G.C. Sih, B. McDonald, Fracture mechanics applied to engineering problems, strain energy density fracture criterion, *Engineering Fracture Mechanics* 6 (1974) 493–507.
[31] G.C. Sih, B.M. Barthelemy, Mixed mode fatigue crack growth predictions, *Engineering Fracture Mechanics* 13 (1980) 439–451.
[32] A.D. Dimarogonas, *Machine Design A CAD Approach*, Wiley, New York, 2001.
[33] F.P. Beer, E.R. Johnston, *Mechanics and Materials*, McGraw-Hill, New York, 1981.