Available online at www.sciencedirect.com
ScienceDirect
Journal of Sound and Vibration 302 (2007) 1–17
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Vibration of non-homogeneous circular Mindlin plates with variable thickness

U.S. Gupta*, R. Lal, Seema Sharma

Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee 247 667, Uttaranchal, India

Received 14 October 2005; received in revised form 27 April 2006; accepted 10 July 2006
Available online 24 January 2007

Abstract

Free vibration analysis of non-homogeneous circular plates of variable thickness has been presented using the first-order shear deformation plate theory of Mindlin. Chebyshev collocation technique has been employed to obtain the first three natural frequencies and mode shapes for clamped, simply supported and free plate with four-digit exactitude. The effect of structural parameters such as taper, non-homogeneity and density and that of edge conditions on the natural frequencies has been investigated. Comparison of results for some special cases with published results obtained from other approximate methods has been presented which shows an excellent agreement.

1. Introduction

Circular and annular plates with variable thickness are widely used in many engineering structures and machines. By appropriate variation of plate thickness, these plates can have significantly greater efficiency for vibration as compared to the plates of uniform thickness and also provide the advantage of reduction in weight and size. A considerable amount of information up to 1985 for various types of thickness variations and edge conditions has been provided in Leissa's monographs [1] and subsequent review articles [2–7]. Studies of circular plates with variable thickness after 1985 have been carried out by a number of researchers and are reported in Refs. [8–17] to mention the prominent ones.

It has been observed that the research work is mainly focused on thin plates, which are analyzed on the basis of Kirchhoff thin plate theory also known as classical plate theory (CPT). It is well known that CPT overpredicts the Eigen-frequencies in all the modes and the error increases with the increase in plate thickness. To improve the accuracy, use of theories that include shear deformation and rotatory inertia becomes significant. Several thick plate theories have been proposed in the past five decades (i.e. Mindlin and Deresiewicz [18], Levinson [19], Reddy [20], Leung [21]). A literature survey by Liew et al. [22] presents an excellent review of earlier research work/investigations on thick plate vibrations. Mindlin's plate theory has been found to improve the accuracy of CPT for moderately thick plates.

*Corresponding author. Tel.: +91 1332 285147(O), +91 1332 285174(R); fax: +91 1332 273560.
E-mail address: usgptfma@iitr.ernet.in (U.S. Gupta).

0022-460X/$ - see front matter © 2006 Published by Elsevier Ltd.
doi:10.1016/j.jsv.2006.07.005

The object of the present work is to investigate the natural frequencies of non-linearly tapered circular plates of moderate thickness taking into account the non-homogeneity, which arises due to variations in Young's modulus and density of the plate material. Very few models representing the behavior of non-homogeneity have been proposed in the literature. The earliest model was proposed by Bose [23] in which the Young's modulus and density vary with radius vector, i.e. $(E, \rho) = (E_0, \rho_0)r$. In 1969, Biswas [24] considered the exponential variations for torsional rigidity and density of the material. Later on, in a series of papers, Tomar et al. [25–28] have analyzed the dynamic behavior of isotropic plates of variable thickness of different geometries, wherein the non-homogeneity of the material of the plate arises due to the variation in Young's modulus and density along one direction as $(E, \rho) = (E_0, \rho_0)e^{\beta x}$, where E_0, ρ_0 are constants and β is the non-homogeneity parameter. In all the Refs. [23–28], the Poisson's ratio is assumed to remain constant. However, the consideration of both E and ρ varying exponentially with the same parameter β does not seem to have any justification as has also been pointed out by Rao et al. [29]. In the present study, a new model has been considered in which Young's modulus and density are assumed to vary exponentially in radial direction in different manner (i.e. $E = E_0 e^{\mu x}$, $\rho = \rho_0 e^{\eta x}$), which is more realistic, while Poisson's ratio has been assumed to be constant.

The consideration of non-homogeneity, variable thickness together with the inclusion of transverse shear and rotatory inertia leads to a set of coupled differential equations with variable coefficients, whose analytical solution is not feasible. An approximate solution has been obtained employing Chebyshev collocation technique, which has the advantage of minimax property. The first three natural frequencies have been computed for various values of plate parameters for three different edge conditions. Mode shapes for specified plate parameters have been presented.

2. Mathematical formulation

According to Mindlin's plate theory [18], the differential equations which govern the axisymmetric motion of non-homogeneous circular plates of radius a, thickness $h(r)$, density $\rho(r)$ referred to a cylindrical polar coordinate system (r, θ, z) (Fig. 1A) have been derived using Hamilton's principle and are given as follows:

$$\frac{\partial M_r}{\partial r} + \frac{M_r - M_\theta}{r} - Q_r - \frac{\rho h^3}{12}\frac{\partial^2 \psi_r}{\partial t^2} = 0, \tag{1}$$

$$\frac{1}{r}Q_r + \frac{\partial Q_r}{\partial r} - \rho h \frac{\partial^2 w}{\partial t^2} = 0, \tag{2}$$

where t is the time, w the transverse deflection, ψ_r the angle of rotation in the rz-plane and M_r, M_θ and Q_r are the moment and shear resultants per unit length.

For non-homogeneous plate material,

$$M_r = D\left(\frac{\partial \psi_r}{\partial r} + \frac{\upsilon}{r}\psi_r\right),$$

$$M_\theta = D\left(\frac{\psi_r}{r} + \upsilon\frac{\partial \psi_r}{\partial r}\right),$$

$$Q_r = \kappa G h\left(\psi_r + \frac{\partial w}{\partial r}\right). \tag{3}$$

where $D(\equiv D(r)) = E(r)h^3/12(1-\upsilon^2)$ is the flexural rigidity, $\kappa(=\pi^2/12)$ an averaging shear coefficient and $E(r), G(\equiv G(r)), \upsilon$ are the elastic constants.

Introducing the non-dimensional variables

$$R = r/a, \quad H = h/a, \quad \overline{w} = w/a, \quad T = t\sqrt{E_0/\rho_0 a^2(1-\upsilon^2)}, \tag{4}$$

together with quadratic thickness variation, i.e.

$$H = h_0(1 + \alpha R + \beta R^2) \quad \text{such that} \quad |\alpha| \leqslant 1,\ |\beta| \leqslant 1 \text{ and } \alpha + \beta > -1 \tag{5}$$

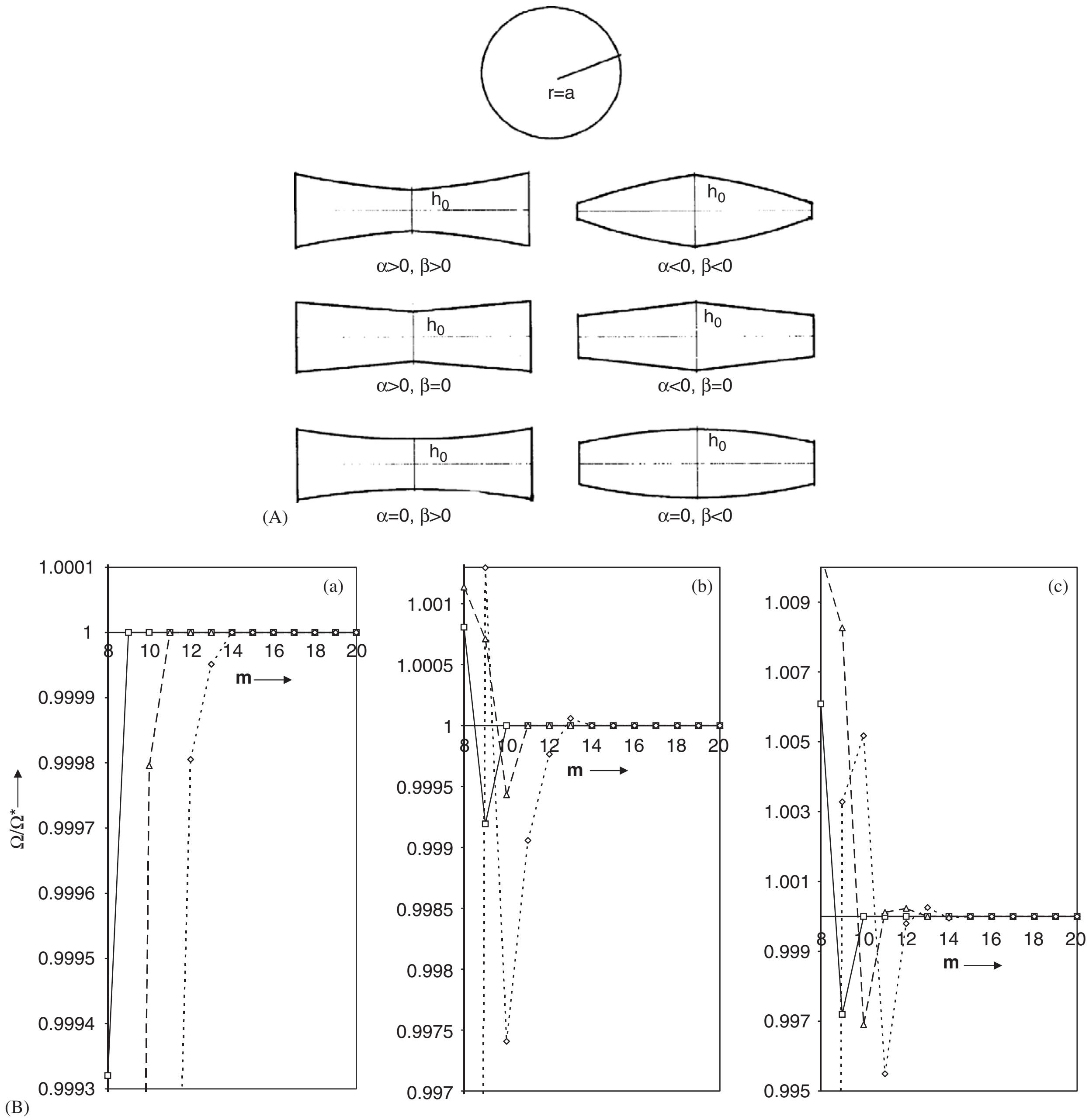

Fig. 1. (A) Geometry and cross-section of the tapered circular plate for quadratic thickness variation, i.e. $h = h_0(1+\alpha x+\beta x^2)$. (B) Convergence of the normalized frequency parameter, Ω/Ω^*, for the first three modes of vibration with grid refinement for $h_0 = 0.05$, $\eta = -0.5$, $\mu = 1.0$, $\alpha = 0.0$, $\beta = 0.5$ for (a) clamped (b) simply supported and (c) free plate. (—□—) fundamental mode; (– –△– –), second mode; (······◇······), third mode. Ω^*—the corresponding results using 20 collocation points.

and assuming the exponential variation for the non-homogeneity of material as follows:

$$E = E_0 e^{\mu R}, \quad \rho = \rho_0 e^{\eta R}, \tag{6}$$

Eqs. (1) and (2) reduce to

$$A_1 \frac{dW}{dR} + A_2 \frac{d^2\psi}{dR^2} + A_3 \frac{d\psi}{dR} + (A_4 + A_5\Omega^2)\psi = 0, \tag{7}$$

$$B_1 \frac{d^2 W}{dR^2} + B_2 \frac{dW}{dR} + B_3 \Omega^2 W + B_4 \frac{d\psi}{dR} + B_5 \psi = 0, \quad (8)$$

where $\overline{w}(R,T) = W(R)e^{i\Omega T}$, $\psi_r(R,T) = \psi(R)e^{i\Omega T}$ (for harmonic vibrations), Ω is the frequency parameter, μ and η the non-homogeneity parameters, α and β the taper parameters, h_0, ρ_0 and E_0 are the thickness, density and Young's modulus, respectively, at the center of the plate,

$$A_1 = -6\kappa(1-\upsilon)R^2, \quad A_2 = H^2R^2, \quad A_3 = (\mu H^2 + 3HH')R^2 + H^2 R,$$
$$A_4 = (\mu H^2 + 3HH')R\upsilon - H^2 - 6\kappa(1-\upsilon)R^2, \quad A_5 = R^2H^2 e^{(\eta-\mu)R},$$
$$B_1 = HR, \quad B_2 = H(1+\mu R) + RH', \quad B_3 = \frac{2}{\kappa(1-\upsilon)} RH e^{(\eta-\mu)R},$$
$$B_4 = B_1, \quad B_5 = B_2, \quad \text{and} \quad \Omega = \omega\sqrt{\frac{\rho_0 a^2(1-\upsilon^2)}{E_0}},$$

where ω is the circular frequency in radians per second.

Coupled differential equation (7) and (8), together with the edge conditions at the edge $R=1$ and regularity condition at the center of plate $R=0$, constitute a boundary value problem, which has been solved by Chebyshev collocation technique.

Table 1
Value of frequency parameter Ω for clamped plate

Mode	α	β	$\mu=-0.5$			$\mu=0.0$			$\mu=1.0$		
			Ω_c	Ω_s		Ω_c	Ω_s		Ω_c	Ω_s	
			$h_0=0.1$	$h_0=0.1$	$h_0=0.2$	$h_0=0.1$	$h_0=0.1$	$h_0=0.2$	$h_0=0.1$	$h_0=0.1$	$h_0=0.2$
$\eta=-0.5$											
I	−0.5	0	0.1669	0.1648	0.3177	0.1945	0.1921	0.3711	0.2647	0.2617	0.5064
		0.5	0.2594	0.2533	0.4748	0.3040	0.2971	0.5577	0.4178	0.4083	0.7673
		−0.5	0.1806	0.1779	0.3407	0.2101	0.2071	0.3977	0.2857	0.2819	0.5427
	0	0	0.2743	0.2669	0.4962	0.3218	0.3134	0.5837	0.4440	0.4325	0.8064
		0.5	0.3663	0.3516	0.6333	0.4308	0.4138	0.7465	0.5947	0.5712	1.0306
		−0.5	0.2883	0.2796	0.5154	0.3384	0.3284	0.6069	0.4681	0.4545	0.8411
	0.5	0	0.3817	0.3647	0.6504	0.4496	0.4299	0.7681	0.6236	0.5961	1.0655
		0.5	0.4743	0.4459	0.7692	0.5589	0.5258	0.9089	0.7732	0.7271	1.2567
II	−0.5	0	0.7771	0.7436	1.3320	0.8911	0.8530	1.5298	1.1678	1.1175	2.0032
		0.5	1.0396	0.9653	1.6318	1.1904	1.1071	1.8774	1.5544	1.4471	2.4604
		−0.5	0.8534	0.8088	1.4211	0.9794	0.9287	1.6339	1.2852	1.2183	2.1428
	0	0	1.1234	1.0304	1.7066	1.2868	1.1825	1.9653	1.6809	1.5466	2.5781
		0.5	1.3559	1.2036	1.8971	1.5509	1.3813	2.1884	2.0199	1.8047	2.8758
		−0.5	1.2066	1.0927	1.7738	1.3828	1.2549	2.0449	1.8075	1.6427	2.6859
	0.5	0	1.4446	1.2637	1.9539	1.6528	1.4512	2.2559	2.1531	1.8973	2.9673
		0.5	1.6619	1.4049	2.0852	1.8991	1.6138	2.4107	2.4676	2.1092	3.1761
III	−0.5	0	1.8084	1.6538	2.7321	2.0678	1.8908	3.1234	2.6873	2.4510	4.0347
		0.5	2.3398	2.0303	3.1173	2.6643	2.3170	3.5678	3.4344	2.9897	4.6123
		−0.5	2.0026	1.7985	2.8920	2.2894	2.0556	3.3050	2.9732	2.6616	4.2640
	0	0	2.5453	2.1615	3.2388	2.8979	2.4669	3.7062	3.7336	3.1815	4.7879
		0.5	2.9990	2.4170	3.4422	3.4055	2.7587	3.9440	4.3650	3.5549	5.1057
		−0.5	2.7483	2.2834	3.3456	3.1289	2.6062	3.8275	4.0294	3.3598	4.9413
	0.5	0	3.2105	2.5271	3.5302	3.6455	2.8849	4.0435	4.6710	3.7171	5.2310
		0.5	3.6245	2.7122	3.6585	4.1074	3.0982	4.1930	5.2430	3.9942	5.4316

Table 1 (*continued*)

Mode	α	β	$\mu=-0.5$			$\mu=0.0$			$\mu=1.0$		
			Ω_c	Ω_s		Ω_c	Ω_s		Ω_c	Ω_s	
			$h_0=0.1$	$h_0=0.1$	$h_0=0.2$	$h_0=0.1$	$h_0=0.1$	$h_0=0.2$	$h_0=0.1$	$h_0=0.1$	$h_0=0.2$
$\eta=0.0$											
I	−0.5	0	0.1521	0.1501	0.2891	0.1775	0.1753	0.3384	0.2427	0.2398	0.4639
		0.5	0.2377	0.2319	0.4339	0.2792	0.2727	0.5111	0.3856	0.3767	0.7069
		−0.5	0.1642	0.1617	0.3094	0.1914	0.1886	0.3619	0.2613	0.2578	0.4959
	0	0	0.2508	0.2438	0.4523	0.2949	0.2870	0.5335	0.4088	0.3980	0.7411
		0.5	0.3362	0.3222	0.5782	0.3964	0.3802	0.6838	0.5500	0.5277	0.9500
		−0.5	0.2632	0.2550	0.4689	0.3096	0.3002	0.5535	0.4301	0.4174	0.7712
	0.5	0	0.3497	0.3335	0.5926	0.4129	0.3941	0.7019	0.5755	0.5496	0.9798
		0.5	0.4358	0.4085	0.7011	0.5148	0.4831	0.8313	0.7161	0.6721	1.1578
II	−0.5	0	0.6852	0.6553	1.1725	0.7881	0.7543	1.3520	1.0391	0.9947	1.7846
		0.5	0.9247	0.8568	1.4424	1.0619	0.9860	1.6669	1.3948	1.2978	2.2038
		−0.5	0.7527	0.7129	1.2508	0.8665	0.8214	1.4441	1.1442	1.0850	1.9099
	0	0	0.9993	0.9143	1.5077	1.1481	1.0530	1.7442	1.5088	1.3873	2.3093
		0.5	1.2125	1.0715	1.6775	1.3909	1.2343	1.9446	1.8221	1.6247	2.5806
		−0.5	1.0733	0.9692	1.5660	1.2338	1.1172	1.8139	1.6227	1.4736	2.4056
	0.5	0	1.2916	1.1243	1.7264	1.4823	1.2962	2.0034	1.9425	1.7078	2.6622
		0.5	1.4916	1.2519	1.8426	1.7095	1.4440	2.1415	2.2340	1.9024	2.8512
III	−0.5	0	1.5875	1.4514	2.3963	1.8204	1.6651	2.7518	2.3796	2.1739	3.5872
		0.5	2.0705	1.7908	2.7387	2.3642	2.0515	3.1498	3.0646	2.6671	4.1123
		−0.5	1.7589	1.5790	2.5376	2.0168	1.8114	2.9136	2.6348	2.3630	3.7948
	0	0	2.2527	1.9063	2.8457	2.5722	2.1842	3.2726	3.3329	2.8393	4.2706
		0.5	2.6673	2.1351	3.0250	3.0372	2.4472	3.4832	3.9146	3.1799	4.5561
		−0.5	2.4328	2.0136	2.9398	2.7778	2.3077	3.3803	3.5983	2.9994	4.4090
	0.5	0	2.8554	2.2316	3.1027	3.2515	2.5586	3.5718	4.1898	3.3252	4.6691
		0.5	3.2348	2.3959	3.2164	3.6761	2.7494	3.7045	4.7183	3.5765	4.8486
$\eta=1.0$											
I	−0.5	0	0.1245	0.1228	0.2361	0.1460	0.1441	0.2777	0.2013	0.1989	0.3842
		0.5	0.1971	0.1919	0.3573	0.2327	0.2268	0.4233	0.3245	0.3166	0.5923
		−0.5	0.1340	0.1317	0.2514	0.1568	0.1544	0.2955	0.2159	0.2128	0.4088
	0	0	0.2071	0.2009	0.3705	0.2446	0.2376	0.4395	0.3424	0.3328	0.6175
		0.5	0.2798	0.2670	0.4749	0.3315	0.3168	0.5652	0.4650	0.4448	0.7960
		−0.5	0.2166	0.2092	0.3824	0.2558	0.2475	0.4539	0.3588	0.3475	0.6395
	0.5	0	0.2899	0.2753	0.4843	0.3440	0.3270	0.5774	0.4844	0.4611	0.8168
		0.5	0.3633	0.3383	0.5730	0.4315	0.4025	0.6843	0.6070	0.5670	0.9678
II	−0.5	0	0.5287	0.5047	0.8993	0.6117	0.5847	1.0449	0.8162	0.7813	1.4012
		0.5	0.7261	0.6688	1.1136	0.8387	0.7749	1.2982	1.1144	1.0341	1.7468
		−0.5	0.5811	0.5490	0.9585	0.6729	0.6368	1.1155	0.8994	0.8528	1.4999
	0	0	0.7846	0.7130	1.1620	0.9068	0.8271	1.3565	1.2060	1.1054	1.8295
		0.5	0.9621	0.8403	1.2938	1.1101	0.9753	1.5145	1.4710	1.3033	2.0497
		−0.5	0.8423	0.7548	1.2046	0.9742	0.8766	1.4084	1.2971	1.1737	1.9043
	0.5	0	1.0244	0.8802	1.3288	1.1826	1.0229	1.5575	1.5683	1.3691	2.1125
		0.5	1.1921	0.9824	1.4178	1.3742	1.1427	1.6649	1.8166	1.5303	2.2639
III	−0.5	0	1.2141	1.1074	1.8211	1.4002	1.2793	2.1095	1.8514	1.6936	2.7997
		0.5	1.6092	1.3787	2.0856	1.8477	1.5913	2.4212	2.4216	2.1003	3.2232
		−0.5	1.3464	1.2053	1.9291	1.5528	1.3927	2.2351	2.0528	1.8437	2.9659
	0	0	1.7512	1.4666	2.1668	2.0109	1.6937	2.5158	2.6354	2.2366	3.3489
		0.5	2.0938	1.6463	2.3040	2.3975	1.9029	2.6782	3.1243	2.5145	3.5740
		−0.5	1.8915	1.5482	2.2385	2.1723	1.7888	2.5990	2.8468	2.3633	3.4591
	0.5	0	2.2412	1.7190	2.3635	2.5667	1.9880	2.7469	3.3449	2.6289	3.6640
		0.5	2.5570	1.8459	2.4525	2.9220	2.1372	2.8508	3.7920	2.8310	3.8055

Table 2
Value of frequency parameter Ω for simply supported plate

Mode	α	β	$\mu = -0.5$			$\mu = 0.0$			$\mu = 1.0$		
			Ω_c	Ω_s		Ω_c	Ω_s		Ω_c	Ω_s	
			$h_0 = 0.1$	$h_0 = 0.1$	$h_0 = 0.2$	$h_0 = 0.1$	$h_0 = 0.1$	$h_0 = 0.2$	$h_0 = 0.1$	$h_0 = 0.1$	$h_0 = 0.2$
$\eta = -0.5$											
I	−0.5	0	0.1002	0.0997	0.1965	0.1138	0.1132	0.2234	0.1459	0.1453	0.2873
		0.5	0.1262	0.1253	0.2454	0.1434	0.1425	0.2795	0.1863	0.1852	0.3641
		−0.5	0.1143	0.1135	0.2227	0.1295	0.1288	0.2531	0.1656	0.1648	0.3248
	0	0	0.1396	0.1384	0.2699	0.1583	0.1571	0.3069	0.2046	0.2031	0.3980
		0.5	0.1646	0.1626	0.3144	0.1880	0.1859	0.3602	0.2482	0.2456	0.4770
		−0.5	0.1531	0.1514	0.2939	0.1733	0.1717	0.3339	0.2231	0.2212	0.4319
	0.5	0	0.1774	0.1750	0.3367	0.2022	0.1996	0.3850	0.2652	0.2621	0.5072
		0.5	0.2029	0.1993	0.3793	0.2331	0.2292	0.4372	0.3120	0.3070	0.5873
II	−0.5	0	0.6110	0.5942	1.1045	0.6973	0.6781	1.2604	0.9047	0.8789	1.6296
		0.5	0.7678	0.7357	1.3242	0.8722	0.8359	1.5052	1.1189	1.0715	1.9258
		−0.5	0.6855	0.6620	1.2112	0.7839	0.7570	1.3847	1.0213	0.9849	1.7967
	0	0	0.8490	0.8066	1.4275	0.9662	0.9182	1.6257	1.2440	1.1810	2.0862
		0.5	0.9891	0.9246	1.5884	1.1216	1.0491	1.8039	1.4323	1.3388	2.2982
		−0.5	0.9289	0.8746	1.5213	1.0591	0.9973	1.7358	1.3684	1.2869	2.2340
	0.5	0	1.0737	0.9934	1.6776	1.2195	1.1291	1.9087	1.5624	1.4452	2.4381
		0.5	1.2063	1.0969	1.8033	1.3660	1.2436	2.0471	1.7388	1.5822	2.6002
III	−0.5	0	1.5395	1.4404	2.4772	1.7573	1.6425	2.8198	2.2748	2.1182	3.6133
		0.5	1.9357	1.7519	2.8586	2.1963	1.9877	3.2431	2.8080	2.5349	4.1205
		−0.5	1.7256	1.5901	2.6659	1.9705	1.8136	3.0348	2.5524	2.3386	3.8866
	0	0	2.1290	1.8932	3.0149	2.4172	2.1492	3.4223	3.0938	2.7428	4.3505
		0.5	2.4684	2.1237	3.2487	2.7916	2.4045	3.6823	3.5446	3.0494	4.6613
		−0.5	2.3206	2.0262	3.1528	2.6361	2.3013	3.5805	3.3770	2.9383	4.5536
	0.5	0	2.6657	2.2488	3.3671	3.0166	2.5477	3.8189	3.8343	3.2337	4.8383
		0.5	2.9761	2.4306	3.5255	3.3580	2.7491	3.9947	4.2429	3.4752	5.0468
$\eta = 0.0$											
I	−0.5	0	0.0901	0.0897	0.1767	0.1025	0.1020	0.2012	0.1318	0.1312	0.2595
		0.5	0.1134	0.1126	0.2203	0.1291	0.1282	0.2513	0.1681	0.1671	0.3282
		−0.5	0.1027	0.1020	0.2000	0.1166	0.1159	0.2277	0.1495	0.1487	0.2931
	0	0	0.1254	0.1243	0.2422	0.1425	0.1413	0.2758	0.1845	0.1832	0.3586
		0.5	0.1477	0.1459	0.2815	0.1690	0.1670	0.3229	0.2235	0.2211	0.4288
		−0.5	0.1375	0.1360	0.2636	0.1559	0.1543	0.3000	0.2011	0.1994	0.3890
	0.5	0	0.1592	0.1569	0.3013	0.1817	0.1792	0.3450	0.2388	0.2359	0.4558
		0.5	0.1820	0.1784	0.3386	0.2093	0.2054	0.3908	0.2806	0.2758	0.5263
II	−0.5	0	0.5356	0.5210	0.9690	0.6131	0.5965	1.1097	0.8003	0.7781	1.4456
		0.5	0.6798	0.6511	1.1709	0.7746	0.7423	1.3362	0.9998	0.9578	1.7231
		−0.5	0.6011	0.5807	1.0630	0.6896	0.6662	1.2199	0.9041	0.8727	1.5954
	0	0	0.7515	0.7137	1.2619	0.8579	0.8152	1.4431	1.1118	1.0559	1.8674
		0.5	0.8814	0.8227	1.4103	1.0025	0.9369	1.6087	1.2884	1.2042	2.0671
		−0.5	0.8220	0.7735	1.3443	0.9402	0.8852	1.5404	1.2229	1.1507	2.0001
	0.5	0	0.9561	0.8833	1.4884	1.0895	1.0078	1.7012	1.4051	1.2996	2.1928
		0.5	1.0796	0.9792	1.6045	1.2265	1.1144	1.8300	1.5713	1.4287	2.3458
III	−0.5	0	1.3476	1.2619	2.1734	1.5427	1.4437	2.4844	2.0087	1.8746	3.2104
		0.5	1.7094	1.5462	2.5211	1.9451	1.7605	2.8730	2.5012	2.2611	3.6833
		−0.5	1.5111	1.3938	2.3403	1.7307	1.5952	2.6758	2.2552	2.0717	3.4574
	0	0	1.8800	1.6707	2.6585	2.1407	1.9036	3.0319	2.7561	2.4473	3.8904
		0.5	2.1917	1.8812	2.8708	2.4858	2.1381	3.2698	3.1747	2.7324	4.1795
		−0.5	2.0491	1.7879	2.7798	2.3346	2.0383	3.1722	3.0088	2.6226	4.0735
	0.5	0	2.3663	1.9911	2.9743	2.6857	2.2648	3.3904	3.4339	2.8975	4.3384
		0.5	2.6524	2.1569	3.1173	3.0016	2.4497	3.5505	3.8147	3.1222	4.5313

Table 2 (*continued*)

Mode	α	β	$\mu=-0.5$			$\mu=0.0$			$\mu=1.0$		
			Ω_c	Ω_s		Ω_c	Ω_s		Ω_c	Ω_s	
			$h_0=0.1$	$h_0=0.1$	$h_0=0.2$	$h_0=0.1$	$h_0=0.1$	$h_0=0.2$	$h_0=0.1$	$h_0=0.1$	$h_0=0.2$
$\eta=1.0$											
I	−0.5	0	0.0718	0.0714	0.1406	0.0819	0.0815	0.1606	0.1059	0.1055	0.2084
		0.5	0.0902	0.0894	0.1746	0.1029	0.1021	0.1998	0.1347	0.1338	0.2624
		−0.5	0.0817	0.0811	0.1590	0.0931	0.0925	0.1815	0.1200	0.1194	0.2351
	0	0	0.0997	0.0987	0.1918	0.1135	0.1125	0.2191	0.1478	0.1466	0.2865
		0.5	0.1172	0.1154	0.2217	0.1343	0.1325	0.2550	0.1785	0.1763	0.3404
		−0.5	0.1092	0.1079	0.2087	0.1242	0.1228	0.2381	0.1610	0.1595	0.3105
	0.5	0	0.1263	0.1242	0.2372	0.1444	0.1422	0.2724	0.1907	0.1880	0.3617
		0.5	0.1440	0.1407	0.2648	0.1659	0.1624	0.3064	0.2235	0.2190	0.4149
II	−0.5	0	0.4090	0.3979	0.7398	0.4711	0.4584	0.8533	0.6224	0.6057	1.1279
		0.5	0.5301	0.5068	0.9083	0.6077	0.5816	1.0447	0.7942	0.7608	1.3686
		−0.5	0.4592	0.4435	0.8116	0.5299	0.5121	0.9383	0.7035	0.6799	1.2461
	0	0	0.5853	0.5547	0.9772	0.6724	0.6380	1.1267	0.8826	0.8383	1.4828
		0.5	0.6959	0.6469	1.1016	0.7967	0.7420	1.2677	1.0372	0.9679	1.6578
		−0.5	0.6395	0.6003	1.0391	0.7361	0.6920	1.2009	0.9701	0.9129	1.5872
	0.5	0	0.7536	0.6929	1.1596	0.8644	0.7966	1.3376	1.1297	1.0432	1.7565
		0.5	0.8597	0.7742	1.2572	0.9832	0.8882	1.4477	1.2765	1.1567	1.8912
III	−0.5	0	1.0257	0.9607	1.6556	1.1808	1.1063	1.9078	1.5553	1.4559	2.5067
		0.5	1.3245	1.1941	1.9386	1.5156	1.3691	2.2289	1.9714	1.7835	2.9092
		−0.5	1.1506	1.0615	1.7833	1.3255	1.2233	2.0565	1.7479	1.6113	2.7042
	0	0	1.4560	1.2892	2.0425	1.6676	1.4797	2.3509	2.1723	1.9306	3.0735
		0.5	1.7163	1.4618	2.2134	1.9580	1.6744	2.5457	2.5299	2.1732	3.3182
		−0.5	1.5864	1.3786	2.1340	1.8182	1.5837	2.4586	2.3715	2.0692	3.2188
	0.5	0	1.8518	1.5454	2.2905	2.1142	1.7719	2.6372	2.7355	2.3036	3.4432
		0.5	2.0926	1.6807	2.4043	2.3821	1.9250	2.7666	3.0633	2.4946	3.6044

3. Method of solution

By taking a new independent variable

$$x \equiv 2R - 1, \tag{9}$$

the range $0 \leqslant R \leqslant 1$ is transformed to $-1 \leqslant x \leqslant 1$, the applicability range of the Chebyshev collocation technique and Eqs. (7) and (8) now reduce to

$$U_1 \frac{\mathrm{d}W}{\mathrm{d}x} + U_2 \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + U_3 \frac{\mathrm{d}\psi}{\mathrm{d}x} + (U_4 + \Omega^2 U_5)\psi = 0, \tag{10}$$

$$V_1 \frac{\mathrm{d}^2 W}{\mathrm{d}x^2} + V_2 \frac{\mathrm{d}W}{\mathrm{d}x} + V_3\Omega^2 W + V_4 \frac{\mathrm{d}\psi}{\mathrm{d}x} + V_5\psi = 0, \tag{11}$$

where $U_1 = 2A_1,\ U_2 = 4A_2,\ U_3 = 2A_3,\ U_4 = A_4,\ U_5 = A_5$ and $V_1 = 4B_1,\ V_2 = 2B_2,\ V_3 = B_3,\ V_4 = 2B_4,\ V_5 = B_5$.

According to Chebyshev collocation technique, we assume

$$\frac{\mathrm{d}^2 W}{\mathrm{d}x^2} = \sum_{k=0}^{m-3} a_{k+3} T_k \tag{12}$$

Table 3
Value of frequency parameter Ω for free plate

Mode	α	β	$\mu=-0.5$			$\mu=0.0$			$\mu=1.0$		
			Ω_c	Ω_s		Ω_c	Ω_s		Ω_c	Ω_s	
			$h_0=0.1$	$h_0=0.1$	$h_0=0.2$	$h_0=0.1$	$h_0=0.1$	$h_0=0.2$	$h_0=0.1$	$h_0=0.1$	$h_0=0.2$
$\eta=-0.5$											
I	−0.5	0	0.2209	0.2189	0.4271	0.2461	0.2441	0.4765	0.3050	0.3027	0.5918
		0.5	0.2378	0.2348	0.4530	0.2669	0.2637	0.5095	0.3388	0.3348	0.6480
		−0.5	0.2553	0.2524	0.4889	0.2838	0.2807	0.5444	0.3503	0.3467	0.6737
	0	0	0.2681	0.2639	0.5058	0.3001	0.2957	0.5675	0.3786	0.3733	0.7178
		0.5	0.2935	0.2873	0.5423	0.3318	0.3250	0.6145	0.4293	0.4208	0.7969
		−0.5	0.2987	0.2932	0.5577	0.3337	0.3278	0.6246	0.4192	0.4122	0.7871
	0.5	0	0.3215	0.3136	0.5877	0.3624	0.3539	0.6643	0.4660	0.4554	0.8566
		0.5	0.3515	0.3402	0.6260	0.4001	0.3875	0.7143	0.5258	0.5096	0.9410
II	−0.5	0	0.8360	0.8066	1.4734	0.9498	0.9160	1.6716	1.2180	1.1726	2.1310
		0.5	0.9929	0.9404	1.6538	1.1210	1.0618	1.8674	1.4200	1.3436	2.3572
		−0.5	0.9604	0.9182	1.6461	1.0922	1.0436	1.8683	1.4026	1.3369	2.3808
	0	0	1.1139	1.0442	1.8030	1.2589	1.1801	2.0369	1.5971	1.4948	2.5711
		0.5	1.2570	1.1544	1.9239	1.4154	1.3006	2.1686	1.7828	1.6372	2.7249
		−0.5	1.2359	1.1460	1.9425	1.3982	1.2961	2.1956	1.7764	1.6430	2.7709
	0.5	0	1.3788	1.2513	2.0491	1.5539	1.4109	2.3108	1.9598	1.7774	2.9030
		0.5	1.5149	1.3449	2.1312	1.7027	1.5134	2.4002	2.1365	1.8990	3.0081
III	−0.5	0	1.8686	1.7291	2.9203	2.1287	1.9670	3.3142	2.7421	2.5218	4.2163
		0.5	2.2862	2.0373	3.2502	2.5855	2.3036	3.6731	3.2825	2.9159	4.6299
		−0.5	2.1198	1.9274	3.1685	2.4161	2.1930	3.5955	3.1142	2.8104	4.5690
	0	0	2.5350	2.2153	3.4480	2.8688	2.5059	3.8978	3.6456	3.1729	4.9119
		0.5	2.8939	2.4357	3.6191	3.2605	2.7470	4.0859	4.1072	3.4555	5.1323
		−0.5	2.7838	2.3840	3.6253	3.1521	2.6977	4.0989	4.0087	3.4161	5.1638
	0.5	0	3.1438	2.5902	3.7671	3.5441	2.9226	4.2547	4.4684	3.6779	5.3447
		0.5	3.4718	2.7565	3.8406	3.9014	3.1047	4.3405	4.8879	3.8920	5.4532
$\eta=0.0$											
I	−0.5	0	0.1892	0.1876	0.3660	0.2113	0.2096	0.4094	0.2631	0.2611	0.5109
		0.5	0.2056	0.2029	0.3912	0.2312	0.2283	0.4409	0.2946	0.2911	0.5632
		−0.5	0.2190	0.2166	0.4197	0.2440	0.2414	0.4685	0.3025	0.2996	0.5825
	0	0	0.2317	0.2281	0.4367	0.2599	0.2560	0.4910	0.3291	0.3245	0.6238
		0.5	0.2549	0.2492	0.4694	0.2887	0.2825	0.5330	0.3749	0.3672	0.6943
		−0.5	0.2581	0.2533	0.4815	0.2889	0.2838	0.5405	0.3644	0.3582	0.6840
	0.5	0	0.2790	0.2719	0.5084	0.3151	0.3074	0.5759	0.4067	0.3971	0.7459
		0.5	0.3060	0.2956	0.5415	0.3489	0.3373	0.6192	0.4601	0.4453	0.8195
II	−0.5	0	0.7269	0.7018	1.2836	0.8283	0.7995	1.4617	1.0688	1.0303	1.8779
		0.5	0.8729	0.8264	1.4523	0.9884	0.9362	1.6462	1.2594	1.1923	2.0943
		−0.5	0.8350	0.7990	1.4346	0.9526	0.9112	1.6350	1.2314	1.1757	2.1012
	0	0	0.9789	0.9174	1.5832	1.1098	1.0404	1.7962	1.4167	1.3271	2.2864
		0.5	1.1125	1.0200	1.6956	1.2563	1.1532	1.9195	1.5918	1.4617	2.4319
		−0.5	1.0857	1.0064	1.7055	1.2322	1.1425	1.9364	1.5758	1.4592	2.4660
	0.5	0	1.2195	1.1049	1.8055	1.3787	1.2505	2.0452	1.7498	1.5871	2.5923
		0.5	1.3468	1.1920	1.8813	1.5184	1.3464	2.1283	1.9166	1.7016	2.6906
III	−0.5	0	1.6302	1.5103	2.5558	1.8624	1.7239	2.9127	2.4129	2.2254	3.7377
		0.5	2.0135	1.7933	2.8584	2.2836	2.0349	3.2451	2.9156	2.5942	4.1276
		−0.5	1.8493	1.6838	2.7740	2.1139	1.9226	3.1619	2.7410	2.4820	4.0548
	0	0	2.2319	1.9494	3.0318	2.5331	2.2134	3.4436	3.2379	2.8236	4.3810
		0.5	2.5627	2.1515	3.1845	2.8957	2.4358	3.6138	3.6686	3.0879	4.5848
		−0.5	2.4503	2.0974	3.1871	2.7826	2.3826	3.6214	3.5601	3.0408	4.6075
	0.5	0	2.7829	2.2869	3.3129	3.1465	2.5908	3.7618	3.9905	3.2866	4.7749
		0.5	3.0862	2.4386	3.3694	3.4782	2.7583	3.8316	4.3829	3.4862	4.8692

Table 3 (*continued*)

Mode	α	β	$\mu=-0.5$			$\mu=0.0$			$\mu=1.0$		
			Ω_c	Ω_s		Ω_c	Ω_s		Ω_c	Ω_s	
			$h_0=0.1$	$h_0=0.1$	$h_0=0.2$	$h_0=0.1$	$h_0=0.1$	$h_0=0.2$	$h_0=0.1$	$h_0=0.1$	$h_0=0.2$
$\eta=1.0$											
I	−0.5	0	0.1388	0.1376	0.2684	0.1558	0.1545	0.3018	0.1956	0.1942	0.3801
		0.5	0.1534	0.1512	0.2907	0.1733	0.1709	0.3291	0.2224	0.2196	0.4239
		−0.5	0.1610	0.1592	0.3084	0.1802	0.1783	0.3461	0.2253	0.2232	0.4343
	0	0	0.1727	0.1698	0.3243	0.1945	0.1914	0.3662	0.2482	0.2445	0.4691
		0.5	0.1917	0.1868	0.3493	0.2179	0.2126	0.3983	0.2849	0.2783	0.5232
		−0.5	0.1923	0.1885	0.3573	0.2161	0.2121	0.4029	0.2745	0.2697	0.5141
	0.5	0	0.2095	0.2036	0.3779	0.2375	0.2310	0.4299	0.3085	0.3006	0.5614
		0.5	0.2309	0.2218	0.4015	0.2642	0.2541	0.4610	0.3507	0.3379	0.6154
II	−0.5	0	0.5473	0.5285	0.9675	0.6273	0.6060	1.1098	0.8193	0.7912	1.4478
		0.5	0.6722	0.6351	1.1120	0.7657	0.7242	1.2705	0.9873	0.9347	1.6418
		−0.5	0.6283	0.6015	1.0813	0.7212	0.6906	1.2421	0.9444	0.9038	1.6230
	0	0	0.7529	0.7040	1.2110	0.8588	0.8041	1.3855	1.1104	1.0404	1.7939
		0.5	0.8680	0.7918	1.3064	0.9864	0.9017	1.4918	1.2651	1.1593	1.9227
		−0.5	0.8338	0.7713	1.3032	0.9525	0.8821	1.4928	1.2345	1.1439	1.9359
	0.5	0	0.9499	0.8561	1.3889	1.0809	0.9763	1.5879	1.3897	1.2581	2.0497
		0.5	1.0604	0.9304	1.4524	1.2031	1.0591	1.6586	1.5377	1.3591	2.1353
III	−0.5	0	1.2334	1.1435	1.9375	1.4169	1.3138	2.2264	1.8569	1.7193	2.9061
		0.5	1.5527	1.3781	2.1855	1.7710	1.5749	2.5038	2.2870	2.0368	3.2437
		−0.5	1.3986	1.2747	2.1031	1.6081	1.4657	2.4182	2.1101	1.9195	3.1576
	0	0	1.7197	1.4965	2.3157	1.9631	1.7119	2.6555	2.5389	2.2169	3.4437
		0.5	1.9979	1.6631	2.4314	2.2706	1.8978	2.7887	2.9102	2.4438	3.6123
		−0.5	1.8864	1.6086	2.4321	2.1551	1.8416	2.7911	2.7907	2.3876	3.6228
	0.5	0	2.1674	1.7654	2.5244	2.4651	2.0164	2.8987	3.1637	2.6000	3.7602
		0.5	2.4242	1.8891	2.5442	2.7482	2.1551	2.9338	3.5040	2.7702	3.8237

and

$$\frac{d^2\psi}{dx^2}=\sum_{k=0}^{m-3} b_{k+3}T_k, \tag{13}$$

where a_j and b_j ($j=3,4,\ldots,m$) are the unknown constants and T_j ($j=0,1,2,\ldots,m-3$) are the Chebyshev polynomials.

Successive integration of Eqs. (12) and (13) leads to

$$W=a_1+a_2T_1+\sum_{k=0}^{m-3} a_{k+3}T_k^2 \tag{14}$$

and

$$\psi=b_1+b_2T_1+\sum_{k=0}^{m-3} b_{k+3}T_k^2, \tag{15}$$

where a_1, a_2, b_1 and b_2 are the constants of integration and T_k^j represents the jth integral of T_k.

Substitution of W, ψ and their derivatives in Eqs. (10) and (11) gives simultaneous equations in terms of the T's, a's and b's. The satisfaction of this resultant set of equations at (m−2) collocation points given by

$$x_i=\cos\left(\frac{(2i-1)}{(m-2)}\frac{\pi}{2}\right), \quad i=1,2,\ldots,m-2, \tag{16}$$

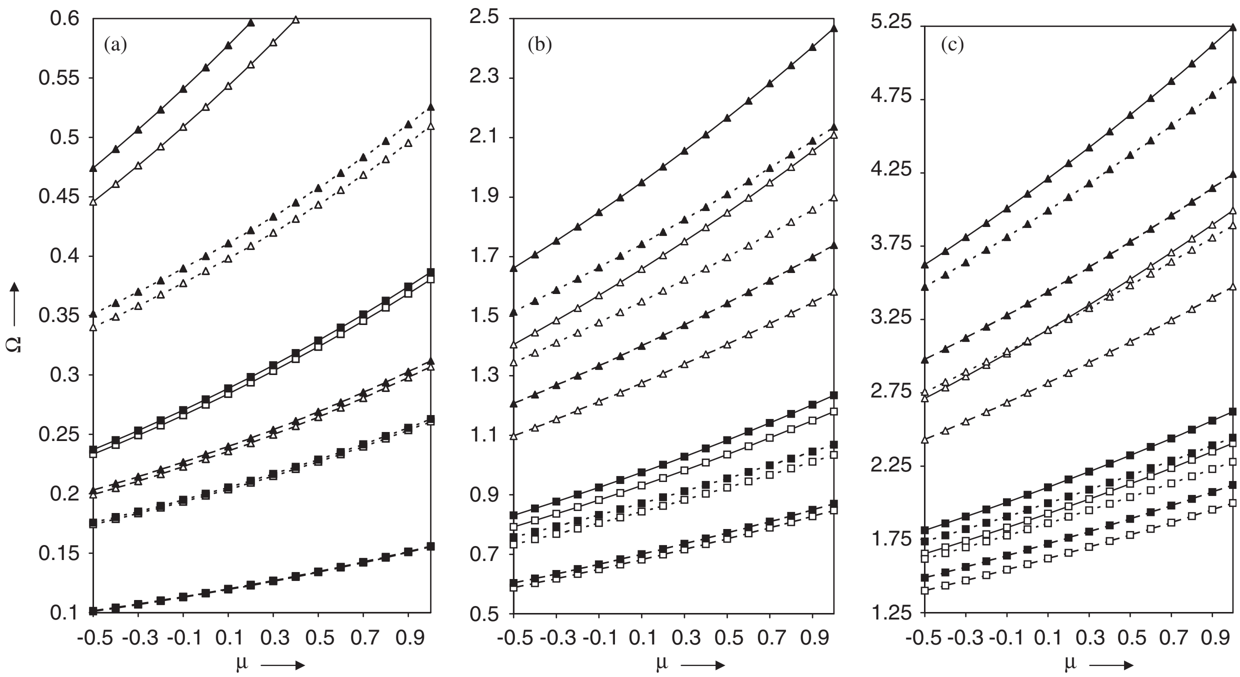

Fig. 2. Frequency parameter for the three plates for $\eta = -0.5$, $\alpha = 0.5$, $\beta = 0.5$ for (a) fundamental (b) second and (c) third mode. (——) clamped plate; (– – –) simply supported plate; (⋯⋯), free plate. (□) $h_0 = 0.05$; (△) $h_0 = 0.1$. (□, △) Mindlin plate theory; (■, ▲) Classical plate theory.

provides a set of $2(m-2)$ equations in unknowns a_j and b_j ($j = 1, 2, \ldots, m$), which can be written in matrix form as

$$[A][C] = [0], \tag{17}$$

where A and C are matrices of order $(2m-4) \times 2m$ and $2m \times 1$, respectively.

4. Edge conditions and frequency equations

By satisfying the relations for:

(1) clamped edge: $W = \psi = 0$,
(2) simply supported edge: $W = \frac{\partial \psi}{\partial R} + \frac{\upsilon}{R}\psi = 0$,
(3) free edge: $\psi + \frac{\partial W}{\partial R} = \frac{\partial \psi}{\partial R} + \frac{\upsilon}{R}\psi = 0$

at the boundary of the plate, along with the relation for
(4) regularity condition: $\psi = Q_r = 0$,

at the center of the plate, a set of four homogeneous equations is obtained. These equations together with the field equation (17) give a complete set of $2m$ equations in $2m$ unknowns.

For a clamped plate, the frequency equation can be written as

$$\begin{bmatrix} A \\ A^C \end{bmatrix} [C^*] = [0], \tag{18}$$

where A^C is a matrix of order $4 \times 2m$.

For a non-trivial solution of Eq. (18), the frequency determinant must vanish and hence,

$$\begin{vmatrix} A \\ A^C \end{vmatrix} = 0. \tag{19}$$

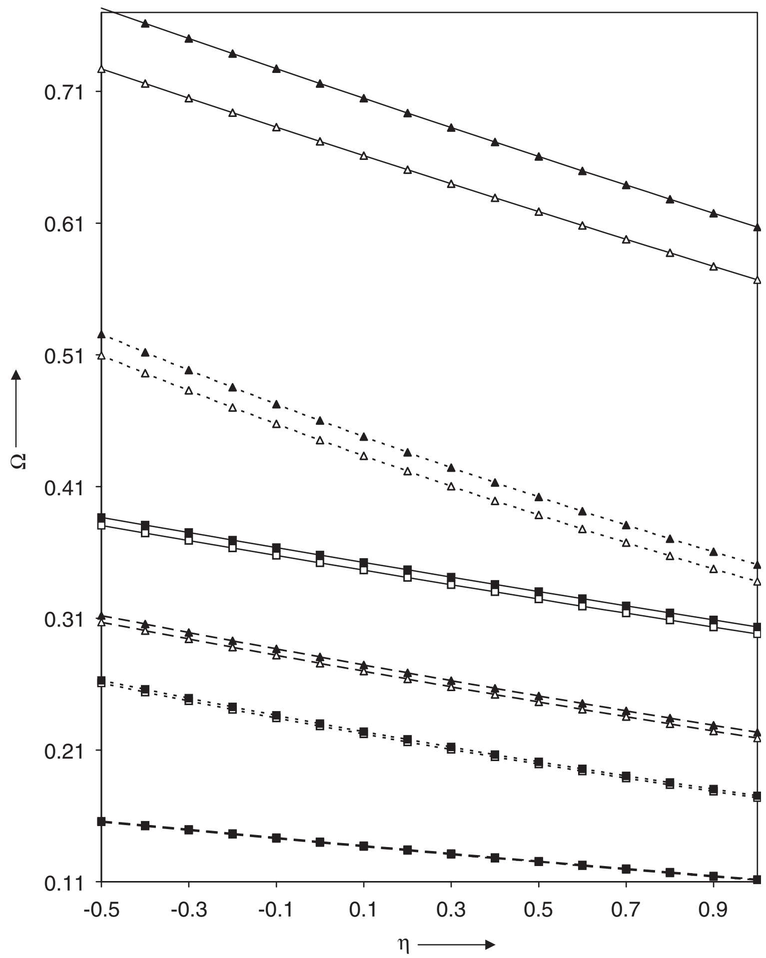

Fig. 3. Frequency parameter for clamped, simply supported and free plates for the fundamental mode of vibration for $\mu = 1.0$, $\alpha = 0.5$ and $\beta = 0.5$. (——) clamped; (– – –) simply supported; (······) free. (□) $h_0 = 0.05$; (△) $h_0 = 0.1$. (□, △) Mindlin plate theory; (■, ▲) Classical plate theory.

Similarly, for simply supported and free plates frequency determinants can be written as

$$\begin{vmatrix} A \\ A^S \end{vmatrix} = 0, \quad \begin{vmatrix} A \\ A^F \end{vmatrix} = 0, \tag{20,21}$$

respectively.

5. Numerical results and discussion

The frequency equations (19)–(21) have been solved to obtain the values of the frequency parameter Ω for various values of plate parameters. In the present work, numerical results have been computed for the first three modes of vibration to investigate the effect of non-homogeneity parameter $\mu = -0.5(0.1)1.0$, density parameter $\eta = -0.5(0.1)1.0$, thickness parameter $h_0 = 0.05(0.05)0.2$, taper parameters $\alpha = -0.5(0.1)0.5$ and $\beta = -0.5(0.1)0.5$ (such that $\alpha + \beta > -1$) on the natural frequencies by shear plate theory of Mindlin (SPT) and CPT for $v = 0.3$, where in the convention $i(j)k$, i represents the initial value, j the increment and k the final value of the parameter. For determining the results on the basis of CPT, the governing equation of motion is obtained by eliminating Q_r from Eqs. (1) and (2) after neglecting the rotatory inertia term in Eq. (1) and then substituting $\psi_r = -\partial w/\partial r$ in the resulting equation. The averaging shear constant is taken to be $\pi^2/12$.

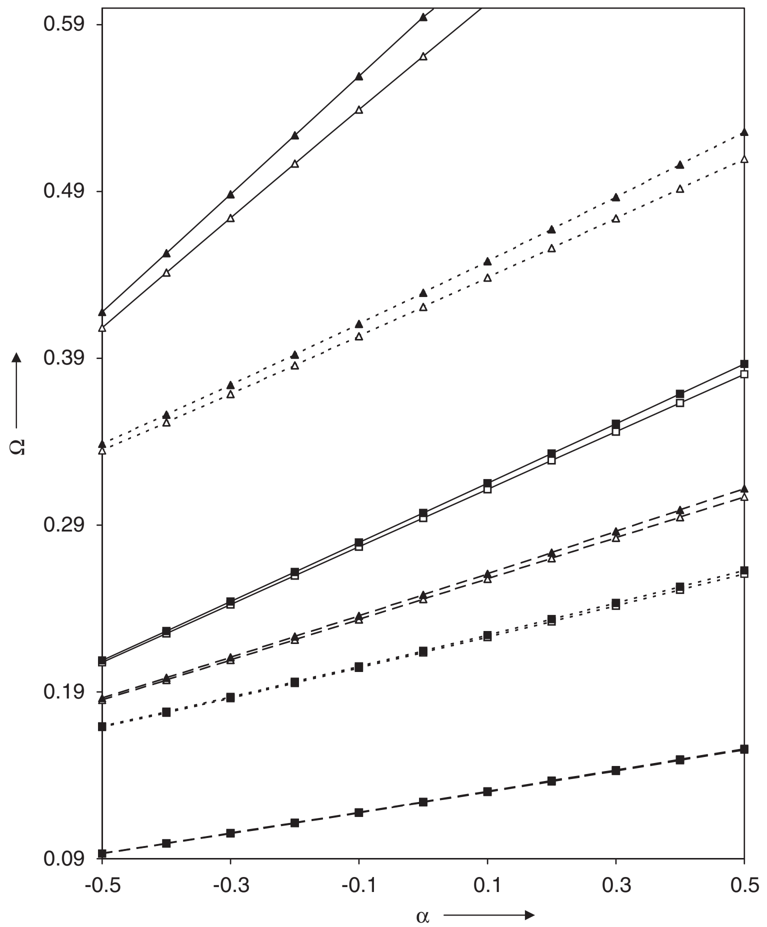

Fig. 4. Frequency parameter for clamped, simply supported and free plates for the fundamental mode of vibration for $\mu = 1.0$, $\eta = -0.5$ and $\beta = 0.5$. (——) clamped; (– – –) simply supported; (······) free. (□) $h_0 = 0.05$; (△) $h_0 = 0.1$. (□, △) Mindlin plate theory; (■, ▲) Classical plate theory.

To choose the appropriate value of the number of collocation points m, a computer program was developed and run for $m = 8(1)20$ for different sets of plate parameters for all the three edge conditions. Figs. 1B(a–c) show the convergence of the frequency parameter Ω with the number of collocation points for the first three modes of vibration for all the three edge conditions. In all the computations we have fixed $m = 15$, since further increase in value of m does not improve the results even in fourth place of decimal. The results are given in Tables 1–3 and Figs. 2–8.

Tables 1–3 present the frequency parameter obtained by CPT (Ω_c) for $h_0 = 0.1$ and SPT (Ω_s) for $h_0 = 0.1$, 0.2 taking various values of $\mu = -0.5, 0.0, 1.0$, $\eta = -0.5, 0.0, 1.0$, $\alpha = -0.5, 0.0, 0.5$ and $\beta = -0.5, 0.0, 0.5$ for clamped, simply supported and free plates, respectively. In case of classical theory, h_0 does not appear explicitly in the governing differential equation except in the final expression of Ω. Therefore, the frequencies have been computed for general value of h_0 and then transformed for the required value of $h_0(=0.1)$ by multiplying with $h_0/\sqrt{12}$. From the results, it is found that for $\alpha > 0$, $\beta > 0$, the frequency parameter for free plate is smaller than that for clamped plate and higher than that for simply supported plate. The frequency parameter increases with the increase in non-homogeneity parameter μ, taper parameters α and β, thickness parameter h_0, while it decreases with the increase in density parameter η. From the results for linear thickness variation (LTV), i.e. $\beta = 0.0$ and parabolic thickness variation (PTV), i.e. $\alpha = 0.0$, it is noticed that when the plate becomes thicker towards the edge (i.e. $\alpha > 0$, $\beta > 0$), $\Omega_{\text{LTV}} > \Omega_{\text{PTV}}$, while it is just reverse when the plate becomes thicker towards the center (i.e. $\alpha < 0$, $\beta < 0$; $\alpha + \beta > -1$).

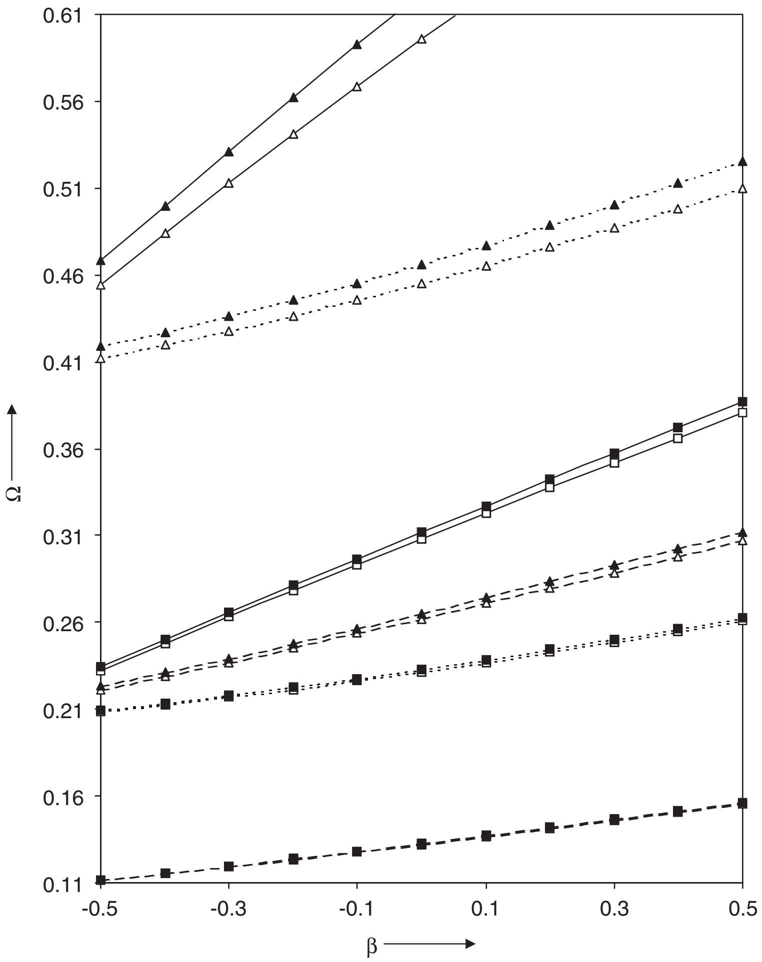

Fig. 5. Frequency parameter for clamped, simply supported and free plates for the fundamental mode of vibration for $\mu = 1.0$, $\eta = -0.5$ and $\alpha = 0.5$. (——) clamped; (– – –) simply supported; (······) free. (□) $h_0 = 0.05$; (△) $h_0 = 0.1$. (□, △) Mindlin plate theory; (■, ▲) Classical plate theory.

Figs. 2(a–c) show the effect of non-homogeneity parameter μ on the frequency parameter Ω for all the three edge conditions for the first three modes of vibration for $\eta = -0.5$, $\alpha = 0.5$, $\beta = 0.5$ and $h_0 = 0.05$, 0.1 by CPT and SPT. It is observed that the frequency parameter Ω increases with increasing values of non-homogeneity parameter μ for all the three plates. The rate of increase of Ω with non-homogeneity parameter μ for clamped plate is higher as compared to simply supported and free plates. Further, it also increases with the increase in thickness h_0 as well as with increasing number of modes. The effect of transverse shear and rotatory inertia increases with increasing value of μ. This effect also increases with increase in number of modes.

Fig. 3 shows the plots of frequency parameter Ω versus density parameter η for $\mu = 1.0$, $\alpha = 0.5$, $\beta = 0.5$ and two values of $h_0 = 0.05$, 0.1 for clamped, simply supported and free plates vibrating in fundamental mode applying CPT and SPT. It is seen that frequency parameter Ω decreases with the increasing values of density parameter η for all the three plates. The rate of decrease in Ω for simply supported plate is lower than that for clamped and free plates. The effect of transverse shear and rotatory inertia decreases with the increasing values of η. The difference between Ω_c and Ω_s is not appreciable for $h_0 = 0.05$ for simply supported and free edge conditions. However, when h_0 increases, this difference increases. The discrepancy in Ω_c and Ω_s is larger for clamped plate as compared to free and simply supported plates. A similar inference was observed when the plate vibrates in second and third modes.

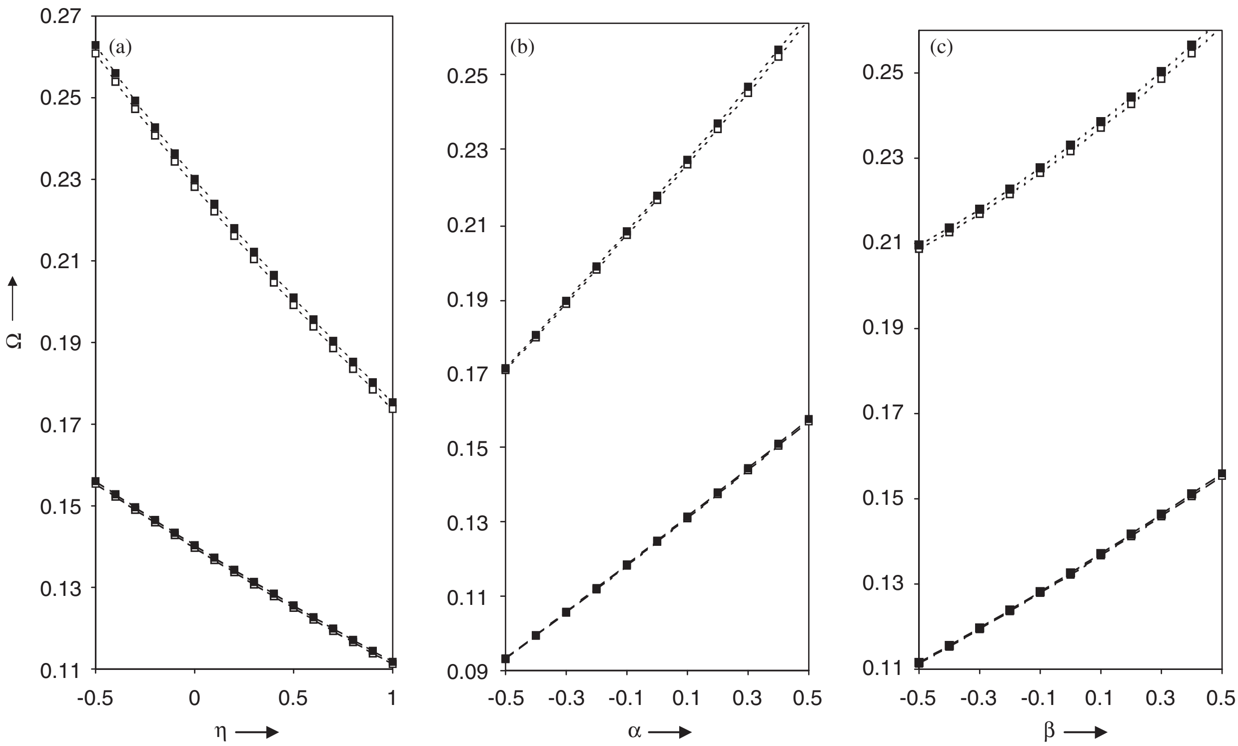

Fig. 6. Frequency parameter for simply supported and free plates for $h_0 = 0.05$, $\mu = 1.0$ and (a) $\alpha = 0.5$, $\beta = 0.5$, (b) $\eta = -0.5$, $\beta = 0.5$, (c) $\eta = -0.5$, $\alpha = 0.5$. (– – –) simply supported; (······) free. (□) Mindlin plate theory; (■) Classical plate theory.

Fig. 4 depicts the variation of Ω with taper parameter α for $\mu = 1.0$, $\eta = -0.5$, $h_0 = 0.05$, 0.1 and $\beta = 0.5$ for all the three plates vibrating in fundamental mode. It is observed that frequency parameter increases with increasing values of taper parameter α. The increase is more pronounced in case of clamped plates. The effect of transverse shear and rotatory inertia is found to be more pronounced when α and β are both positive. Fig. 5 shows the effect of taper parameter β on frequency parameter Ω for $\mu = 1.0$, $\eta = -0.5$, $h_0 = 0.05$, 0.1 and $\alpha = 0.5$ for clamped, simply supported and free plates vibrating in fundamental mode. It is clear that frequency parameter increases with the increasing value of taper parameter β. The rate of increase is higher for clamped plate as compared to simply supported and free plates. A similar behavior of frequency parameter Ω with taper parameters α and β is observed in case of second and third modes except that rate of increase of Ω with α and β is much higher as compared to the fundamental mode (graphs not given). To provide clarity between Ω_c and Ω_s for $h_0 = 0.05$, the magnified graphs are given in Figs. 6(a–c) for fundamental mode of vibration in case of simply supported and free edge conditions.

Figs. 7(a–c) show the behavior of frequency parameters Ω with thickness parameter h_0 for $\mu = 1.0$, $\eta = -0.5$, $\alpha = 0.5$ and $\beta = 0.5$ for the first three modes of vibration for clamped, simply supported and free plate, respectively. It is seen that the effect of transverse shear and rotatory inertia increases with the increase in the values of h_0 as well as the number of modes. This effect increases in the order of edge conditions: simply supported, free and clamped.

Figs. 8(a–c) show the plots for normalized transverse displacements for $\mu = 1.0$, $\eta = -0.5$, $h_0 = 0.1$, $\alpha = 0.0$, $\beta = 0.0$; $\alpha = 0.5$, $\beta = 0.0$; $\alpha = 0.5$, $\beta = 0.5$ for first three modes of vibration for clamped, simply supported and free plate, respectively. The radii of nodal circles decrease as the outer edge becomes thicker and thicker for all three edge conditions except for the fundamental mode in case of free plate. In this case the behavior is just the reverse.

A comparison of results for homogeneous ($\mu = 0.0$, $\eta = 0.0$) Mindlin circular plates of uniform thickness ($\alpha = 0.0$, $\beta = 0.0$) with those of exact results obtained by Irie et al. [30] and DQM results obtained by

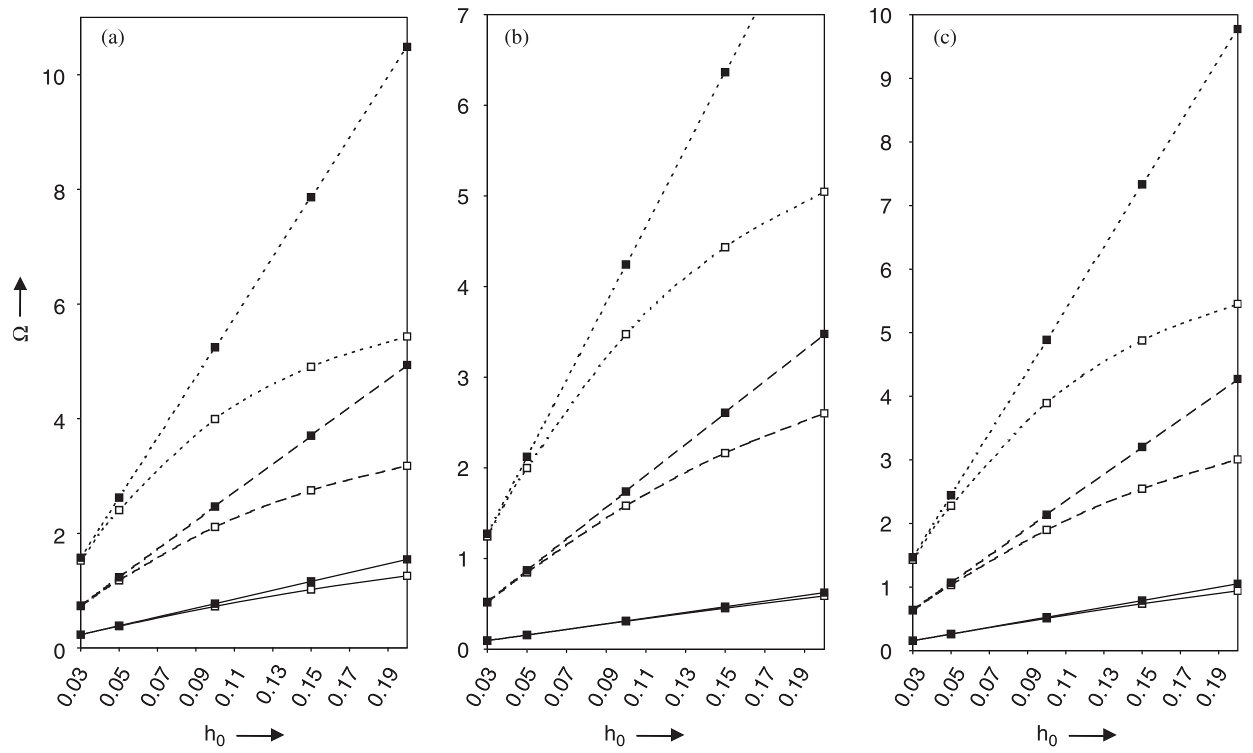

Fig. 7. Frequency parameter for first three modes of vibration for (a) clamped (b) simply supported and (c) free plate for $\mu = 1.0$, $\eta = -0.5$, $\alpha = 0.5$ and $\beta = 0.5$. (——) fundamental mode; (– – –) second mode; (······) third mode. (□) Mindlin plate theory; (■) Classical plate theory.

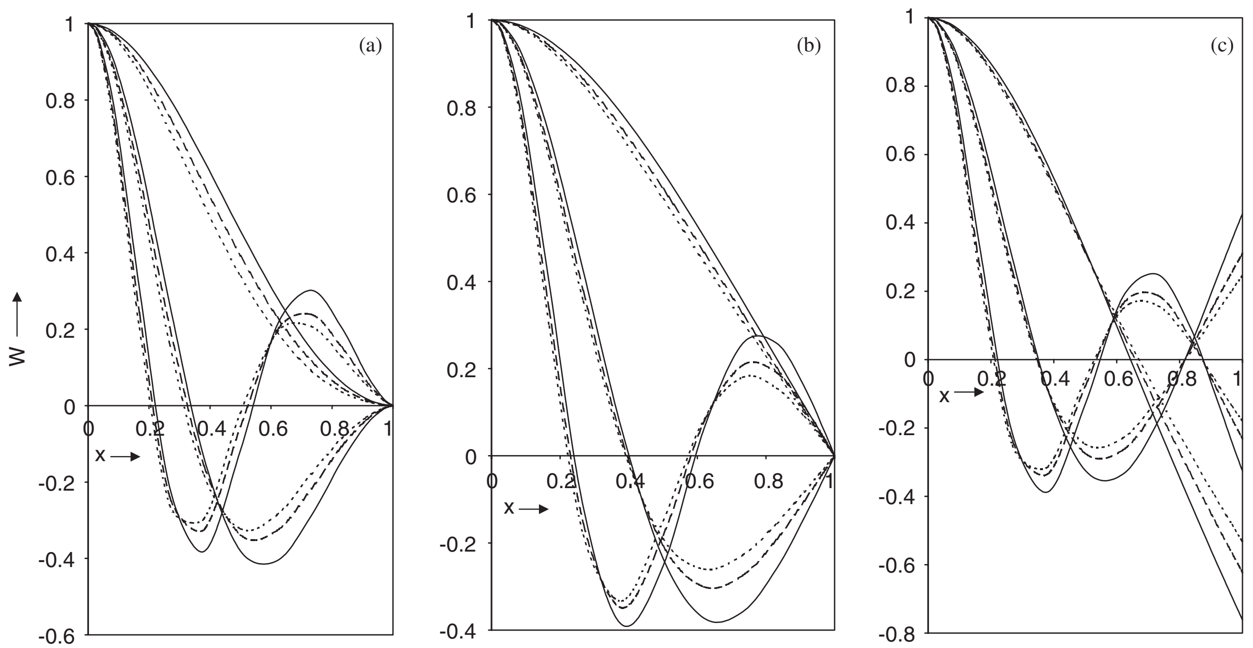

Fig. 8. Normalized displacements for the first three modes of vibration for (a) clamped (b) simply supported and (c) free plate for $\mu = 1.0$, $\eta = -0.5$ and $h_0 = 0.1$. (——) $\alpha = 0$, $\beta = 0$; (– – –) $\alpha = 0.5$, $\beta = 0$; (······) $\alpha = 0.5$, $\beta = 0.5$.

Table 4
Comparison of frequency parameter Ω for homogeneous ($\mu = 0.0$, $\eta = 0.0$) uniform thickness ($\alpha = 0.0$, $\beta = 0.0$) circular plates

h/a	Method	Boundary								
		Clamped			S–S			Free		
		Mode								
		I	II	III	I	II	III	I	II	III
0.001	Present	10.2158	39.7708	89.1024	4.9351	29.7198	74.1550	9.0031	38.4429	87.7489
	DQM	10.216[a]	39.771[a]	89.102[a]	4.9351[a]	29.720[a]	74.155[a]	9.0031[a]	38.443[a]	87.749[a]
	Exact	10.216[b]	39.771[b]	89.104[b]	4.935[b]	29.720[b]	74.156[b]	9.003[b]	38.443[b]	87.750[b]
0.05	Present	10.1447	38.8554	84.9950	4.9247	29.3233	71.7563	8.9686	37.7874	84.4430
	DQM	10.145[a]	38.855[a]	84.995[a]	4.9247[a]	29.323[a]	71.756[a]	8.9686[a]	37.787[a]	84.443[a]
0.1	Present	9.9408	36.4787	75.6643	4.8938	28.2400	65.9424	8.8679	36.0407	76.6756
	DQM	9.9408[a]	36.479[a]	75.664[a]	4.8938[a]	28.240[a]	65.942[a]	8.8679[a]	36.401[a]	76.676[a]
0.15	Present	9.6286	33.3934	65.5507	4.8440	26.7148	59.0621	8.7095	33.6744	67.8274
	DQM	9.6286[a]	33.393[a]	65.551[a]	4.8440[a]	26.715[a]	59.062[a]	8.7095[a]	33.674[a]	67.827[a]
0.2	Present	9.2400	30.2107	56.6823	4.7773	24.9945	52.5139	8.5051	31.1106	59.6450
	DQM	9.2400[a]	30.211[a]	56.682[a]	4.7773[a]	24.994[a]	52.514[a]	8.5051[a]	31.111[a]	59.645[a]
0.25	Present	8.8068	27.2529	49.4204	4.6963	23.2541	46.7745	8.2674	28.6055	52.5842
	DQM	8.8068[a]	27.253[a]	49.420[a]	4.6963[a]	23.254[a]	46.775[a]	8.2674[a]	28.605[a]	52.584[a]
	Exact	8.807[b]	27.253[b]	49.420[b]	4.696[b]	23.254[b]	46.775[b]	8.267[b]	28.605[b]	52.584[b]

[a]Values obtained by DQM [31].
[b]Exact values [30].

Liew et al. [31] for first three natural frequencies has been presented in Table 4. A close agreement of the results shows the versatility of Chebyshev collocation technique.

6. Conclusions

The effect of transverse shear and rotatory inertia together with non-homogeneity of the plate material on the natural frequencies of circular plates of quadratically varying thickness has been analyzed on the basis of Mindlin's theory. It is observed that

(i) the frequency parameter Ω increases with the increase in non-homogeneity parameter μ, taper parameters α and β as well as thickness parameter h_0 but it decreases with the increasing values of density parameter η,
(ii) the values of frequency parameter Ω_c are always greater than those of Ω_s for all the three edge conditions,
(iii) the difference between these two frequency parameters, i.e. ($\Omega_c - \Omega_s$) is found to increase with the increasing values of thickness parameter h_0 and with increase in number of modes.

Thus the effects of rotatory inertia and transverse shear cannot be neglected while predicting the vibrational behavior of moderately thick plates.

Acknowledgement

The authors express grateful thanks to the learned referees for their valuable comments.

References

[1] A.W. Leissa, *Vibration of Plates*, NASA SP-169, Washington, 1969.

[2] A.W. Leissa, Recent research in plate vibrations: classical theory, *Shock and Vibration Digest* 9 (10) (1977) 13–24.

[3] A.W. Leissa, Recent research in plate vibrations: complicating effects, *Shock and Vibration Digest* 9 (12) (1977) 21–35.

[4] A.W. Leissa, Plate vibration research, 1976–1980: classical theory, *Shock and Vibration Digest* 13 (9) (1981) 11–22.

[5] A.W. Leissa, Plate vibration research, 1976–1980: complicating effects, *Shock and Vibration Digest* 13 (10) (1981) 19–36.

[6] A.W. Leissa, Recent research in plate vibrations, 1981–1985. Part I. Classical theory, *Shock and Vibration Digest* 19 (2) (1987) 11–18.

[7] A.W. Leissa, Recent research in plate vibrations, 1981–1985. Part II. Complicating effects, *Shock and Vibration Digest* 19 (3) (1987) 10–24.

[8] U.S. Gupta, R. Lal, S.K. Jain, Effect of elastic foundation on axisymmetric vibration of polar orthotropic circular plates of variable thickness, *Journal of Sound and Vibration* 139 (1990) 503–513.

[9] P.A.A. Laura, R.H. Gutierrez, Analysis of vibrating circular plates of non-uniform thickness by the method of differential quadrature, *Ocean Engineering* 22 (1995) 97–100.

[10] R.H. Gutierrez, E. Romanelli, P.A.A. Laura, Vibration and elastic stability of thin circular plates with variable profile, *Journal of Sound and Vibration* 195 (3) (1996) 391–399.

[11] J. Wang, Generalized power series solutions of the vibration of classical circular plates with variable thickness, *Journal of Sound and Vibration* 202 (1997) 593–599.

[12] B. Singh, S.M. Hassan, Transverse vibration of a circular plate with arbitrary thickness variation, *International Journal of Mechanical Sciences* 40 (1998) 1089–1104.

[13] T.Y. Wu, G.R. Liu, Free vibration analysis of circular plates with variable thickness by the generalized differential quadrature rule, *International Journal of Solids and Structures* 38 (44–45) (2001) 7967–7980.

[14] U.S. Gupta, A.H. Ansari, Transverse vibration of polar orthotropic parabolically tapered circular plates, *Indian Journal of Pure and Applied Mathematics* 34 (6) (2003) 819–830.

[15] Y. Xiang, L. Zhang, Free vibration analysis of stepped circular Mindlin plates, *Journal of Sound and Vibration* 280 (2005) 633–655.

[16] N. Bhardwaj, A.P. Gupta, Axisymmetric vibration of polar orthotropic circular plates of quadratically varying thickness resting on elastic foundation, *International Journal of Structural Stability and Dynamics* 5 (3) (2005) 387–408.

[17] Le Thi Thu Hang, C.M. Wang, T.Y. Wu, Exact vibration results for stepped circular plates with free edges, *International Journal of Mechanical Sciences* 47 (2005) 1224–1248.

[18] R.D. Mindlin, H. Deresiewicz, Thickness—shear and flexural vibrations of a circular disk, *Journal of Applied Physics* 25 (1954) 1329–1332.

[19] M. Levinson, An accurate simple theory of the statics and dynamics of elastic plates, *Mechanics Research Communications* 7 (1980) 343–350.

[20] J.N. Reddy, A simply higher-order theory for laminated composite plate, *Transactions of ASME Journal of Applied Mechanics* 51 (1984) 745–752.

[21] A.Y.T. Leung, An unconstrained third order plate theory, *Computers and Structures* 40 (4) (1991) 871–875.

[22] K.M. Liew, Y. Xiang, S. Kitipornchai, Research on thick plate vibration: a literature survey, *Journal of Sound and Vibration* 180 (1) (1995) 163–176.

[23] R.K. Bose, Note on forced vibration of a thin non-homogeneous circular plate with central hole, *Indian Journal of Physics* 41 (1967) 886–890.

[24] S.K. Biswas, Note on the torsional vibration of a finite circular cylinderical non-homogeneous material by a particular type of twist on one of the plane surface, *Indian Journal of Physics* 43 (1969) 320–323.

[25] J.S. Tomar, D.C. Gupta, N.C. Jain, Vibrations of non-homogeneous plates of variable thickness, *Journal of Acoustical Society of America* 72 (1982) 851–855.

[26] J.S. Tomar, D.C. Gupta, N.C. Jain, Axisymmetric vibrations of an isotropic elastic non-homogeneous circular plate of linearly varying thickness, *Journal of Sound and Vibration* 85 (1982) 365–370.

[27] J.S. Tomar, D.C. Gupta, N.C. Jain, Free vibrations of an isotropic non-homogeneous infinite plate of linearly varying thickness, *Meccanica* 18 (1983) 30–33.

[28] J.S. Tomar, D.C. Gupta, N.C. Jain, Free vibrations of an isotropic non-homogeneous infinite plate of parabolically varying thickness, *Indian Journal of Pure and Applied Mathematics* 15 (1984) 211–220.

[29] G.V. Rao, B.P. Rao, I.S. Raju, Vibrations of inhomogeneous thin plates using a high-precision triangular element, *Journal of Sound and Vibration* 34 (3) (1974) 444–445.

[30] T. Irie, G. Yamada, S. Aomura, Natural frequencies of Mindlin circular plates, *Transactions of ASME Journal of Applied Mechanics* 47 (1980) 652–655.

[31] K.M. Liew, J.B. Han, Z.M. Xiao, Vibration analysis of circular Mindlin plates using the differential quadrature method, *Journal of Sound and Vibration* 205 (5) (1997) 617–630.