Available online at www.sciencedirect.com

ScienceDirect

Journal of Sound and Vibration 302 (2007) 178–184

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

Determination of limit cycle by He's parameter-expanding method for strongly nonlinear oscillators

Lan Xu*

College of Science, Donghua University, 1882 Yan'an Xilu Road, Shanghai 200051, People's Republic of China

Received 26 September 2006; received in revised form 12 November 2006; accepted 19 November 2006
Available online 2 January 2007

Abstract

This paper applies He's parameter-expanding method (PEM) to determine the limit cycles of strongly nonlinear oscillators by which one iteration leads to an accurate solution. Comparison of the obtained results with those of the exact solution shows that the method is very effective and convenient and quite accurate to both linear and nonlinear physics and engineering problems.

1. Introduction

Many asymptotic techniques including modified Lindstedt–Poincare method [1–5], variational iteration method [6–10], homotopy perturbation method [11–21], energy balance method [22–25] were used to handle strongly nonlinear systems. He's parameter-expanding methods (PEMs) [26] including modified Lindstedt–Poincare method [2] and bookkeeping parameter method [27] were paid attention recently; it is proven that the PEMs are very effective to determine the limit cycle of strongly nonlinear oscillators with high accuracy [28].

2. Parameter-expanding method

We consider the following nonlinear oscillator [26]:

$$u'' + au + bu^3 + cu^{1/3} = 0, \quad u(0) = A, \quad u'(0) = 0. \tag{1}$$

By simple analysis [26] we know that Eq. (1) has periodic solution when $a + bA^2 + cA^{-2/3} > 0$. In case $a \leqslant 0$, traditional perturbation methods do not work even when the parameters b and c are small.

According to the PEM [26,27], the solution is expanded into a series of an artificial parameter, p, in the form:

$$u = u_0 + pu_1 + p^2u_2 + \cdots, \tag{2}$$

where p is a bookkeeping parameter, $p = 1$.

*Fax: +86 21 62378066.
E-mail address: xlanmail@21cn.com.

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2006.11.011

The coefficients a, b and c can be, respectively, expanded into a series in p in a similar way [11]

$$a = \omega^2 + p\omega_1 + p^2\omega_2 + \cdots, \tag{3}$$

$$b = pb_1 + p^2 b_2 + \cdots, \tag{4}$$

$$c = pc_1 + p^2 c_2 + \cdots. \tag{5}$$

Substituting Eqs. (2)–(5) into Eq. (1) and equating the terms with the identical powers of p, we have

$$p^0 : u_0'' + \omega^2 u_0 = 0, \tag{6}$$

$$p^1 : u_1'' + \omega^2 u_1 + \omega_1 u_0 + b_1 u_0^3 + c_1 u_0^{1/3} = 0. \tag{7}$$

Considering the initial conditions $u_0(0) = A$ and $u_0'(0) = 0$, the solution of Eq. (6) is $u_0 = A\cos\omega t$. Substituting the result into Eq. (7), we have

$$u_1'' + \omega^2 u_1 + \omega_1 A\cos\omega t + \tfrac{3}{4}b_1 A^3 \cos\omega t + \tfrac{1}{4}b_1 A^3 \cos 3\omega t + c_1 A^{1/3}(\cos\omega t)^{1/3} = 0. \tag{8}$$

We expand the term $(\cos\omega t)^{1/3}$ into a Fourier series representation as follows:

$$(\cos\omega t)^{1/3} = \sum_{n=0}^{\infty} a_{2n+1}\cos(2n+1)\omega t, \tag{9}$$

where

$$a_{2n+1} = \frac{3\Gamma\left(\frac{7}{3}\right)}{2^{4/3}\Gamma\left(n+\frac{5}{3}\right)\Gamma\left(\frac{2}{3}-n\right)} \tag{10}$$

with $a_1 = 1.15959526696$ and the interval of t in Eq. (9) is $[-\pi/\omega, \pi/\omega]$. Therefore, the first several terms are

$$(\cos\omega t)^{1/3} = a_1(\cos\omega t - \frac{\cos 3\omega t}{5} + \frac{\cos 5\omega t}{10} - \frac{7\cos 7\omega t}{110} + \cdots). \tag{11}$$

Substituting Eq. (11) into Eq. (8) we can obtain the following equation:

$$u_1'' + \omega^2 u_1 + \omega_1 A\cos\omega t + \tfrac{3}{4}b_1 A^3\cos\omega t + \tfrac{1}{4}b_1 A^3 \cos 3\omega t + c_1 A^{1/3} a_1(\cos\omega t - \frac{\cos 3\omega t}{5} + \cdots) = 0. \tag{12}$$

No secular term in u_1 requires that

$$\omega_1 A + \tfrac{3}{4}b_1 A^3 + c_1 A^{1/3} a_1 = 0. \tag{13}$$

If the first-order approximation is enough, then, setting $p = 1$ in Eqs. (3)–(5), we have

$$a = \omega^2 + \omega_1, \tag{14}$$

$$b = b_1, \tag{15}$$

$$c = c_1. \tag{16}$$

Solving Eqs. (13)–(16), we have

$$\omega = \sqrt{\frac{3}{4}bA^2 + 1.15959526696cA^{-2/3} + a}. \tag{17}$$

In order to verify the correctness of the obtained frequency, we consider some special cases.

Case 1: If $a = 0$, $b = 0$, $c = 1$, Eq. (1) reduces to

$$u'' + u^{1/3} = 0, \quad u(0) = A, \quad u'(0) = 0. \tag{18}$$

Then, from Eq. (17), we can obtain the frequency of the nonlinear oscillator as follows:

$$\omega = \sqrt{1.15959526696A^{-2/3}} = 1.07684A^{-1/3}. \tag{19}$$

The exact frequency is $\omega = 1.070451A^{-1/3}$. Therefore, it can be easily proved that the maximal relative error is less than 0.597%.

Hence, we can obtain the following zero-order approximate solution [29]:

$$u = A\cos(1.07684A^{-1/3}t) \tag{20}$$

which agrees very well with the exact solution [26] as shown in Fig. 1.

Case 2: If $a = 0,\ b = 1,\ c = 0$, Eq. (1) becomes

$$u'' + u^3 = 0, \quad u(0) = A, \quad u'(0) = 0. \tag{21}$$

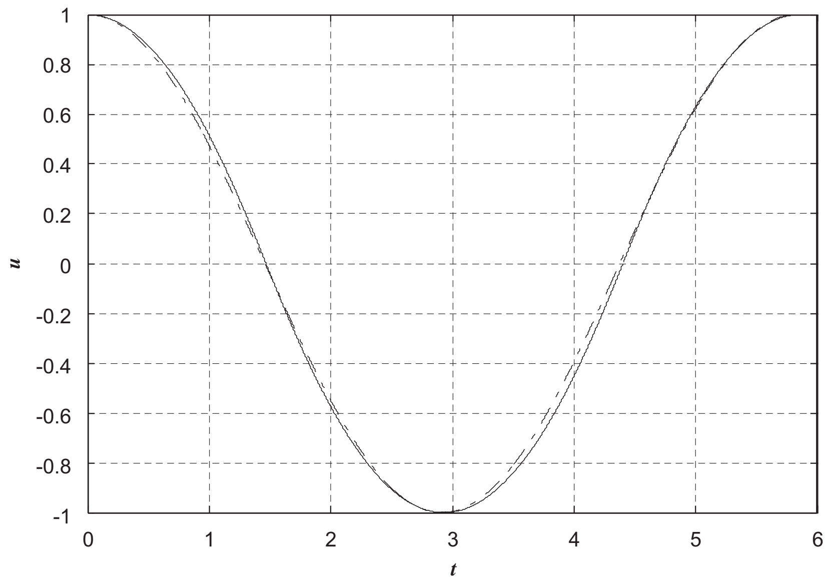

Fig. 1. Comparison of the approximate solution with the exact solution: dashed line: approximated solution and solid line: the exact solution.

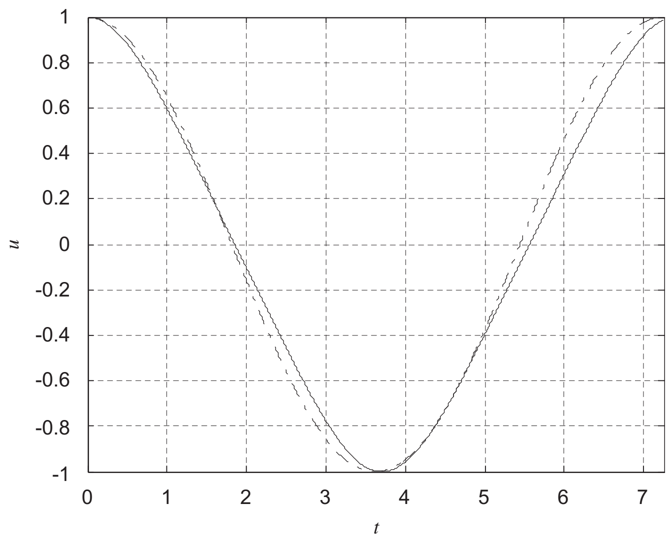

Fig. 2. Comparison of the approximate solution with the exact solution: dashed line: approximated solution and solid line: the exact solution.

Then, from Eq. (17), we can obtain the frequency of the nonlinear oscillator as follows:

$$\omega = \sqrt{\tfrac{3}{4}A^2} = 0.866A. \tag{22}$$

Hence, the approximated period is

$$T = \frac{2\pi}{0.866A} = \frac{7.2554}{A}. \tag{23}$$

The exact period [26] is $T = 7.4163A^{-1}$. Therefore, it can be easily proved that the maximal relative error is less than 2.17%.

According to Eq. (22), we can obtain the following zero-order approximate solution [29]:

$$u = A\cos(0.866At) \tag{24}$$

which agrees very well with the exact solution [26] as illustrated in Fig. 2.

Case 3: If $a = 0,\ b = 1,\ c = 1$, Eq. (1) reduces to

$$u'' + u^3 + u^{1/3} = 0, \quad u(0) = A, \quad u'(0) = 0. \tag{25}$$

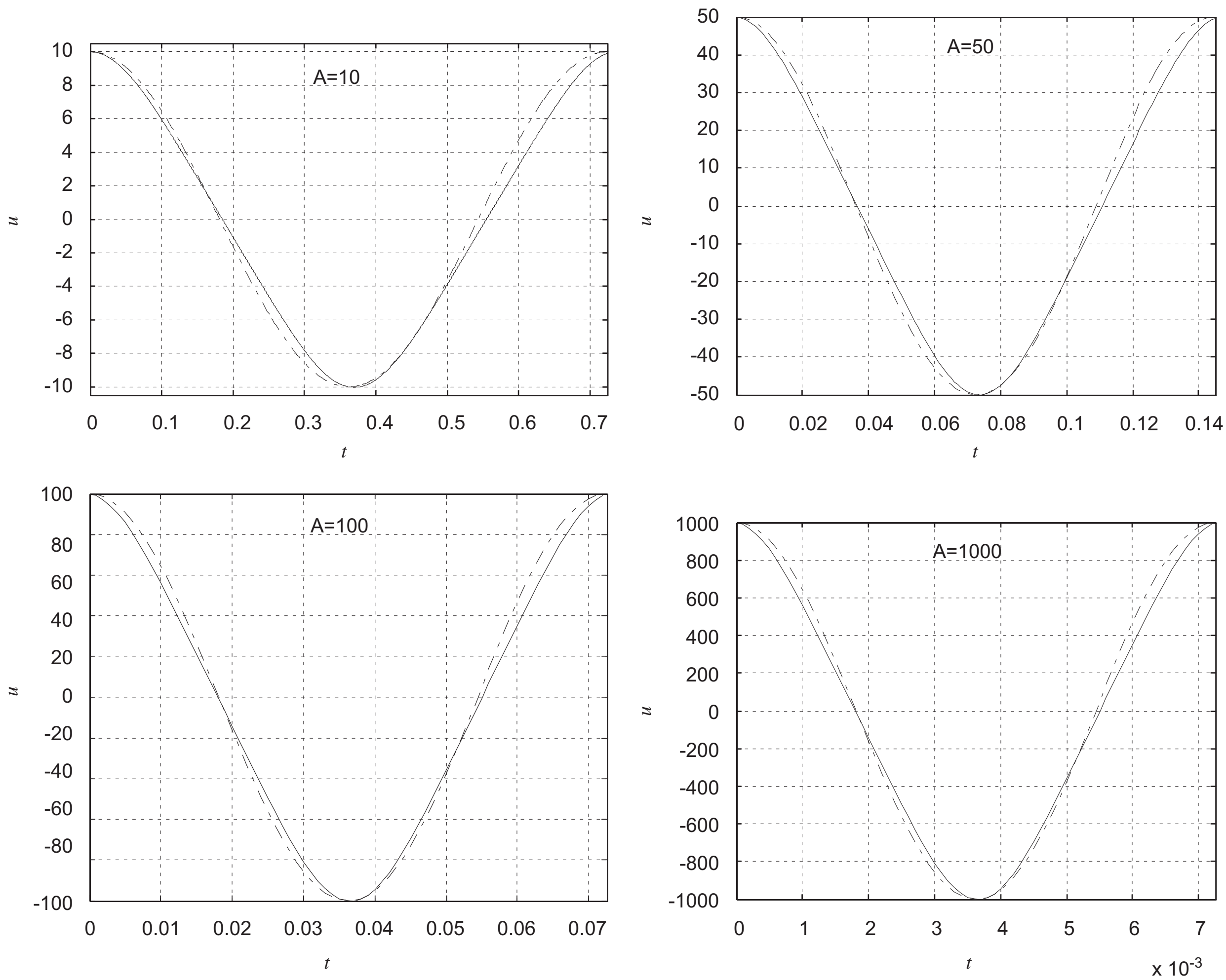

Fig. 3. Comparison of the approximate solution with the exact solution: dashed line: approximated solution and solid line: the exact solution.

Then, from Eq. (17), we can obtain the frequency of the nonlinear oscillator as follows:

$$\omega = \sqrt{\tfrac{3}{4}A^2 + 1.15959526696A^{-2/3}}. \tag{26}$$

Therefore, we can obtain the following zero-order approximate solution [29]:

$$u = A\cos[(\tfrac{3}{4}A^2 + 1.15959526696A^{-2/3})^{1/2}t] \tag{27}$$

which agrees very well with the exact solution [26] as shown in Fig. 3.

Case 4: If $a = 1,\ b = 0,\ c = 1$, according to Eq. (1), we can obtain the following nonlinear oscillator:

$$u'' + u + u^{1/3} = 0, \quad u(0) = A, \quad u'(0) = 0. \tag{28}$$

Then, from Eq. (17), we can obtain the frequency of the nonlinear oscillator as follows:

$$\omega = \sqrt{1.15959526696A^{-2/3} + 1}. \tag{29}$$

Therefore, we can obtain the following zero-order approximate solution [29]:

$$u = A\cos[(1.15959526696A^{-2/3} + 1)^{1/2}t] \tag{30}$$

which agrees very well with the exact solution as they are shown in Fig. 4.

Case 5: If $a = 1,\ b = 1,\ c = 0$, according to Eq. (1), we can obtain the following nonlinear oscillator:

$$u'' + u + u^3 = 0, \quad u(0) = A, \quad u'(0) = 0. \tag{31}$$

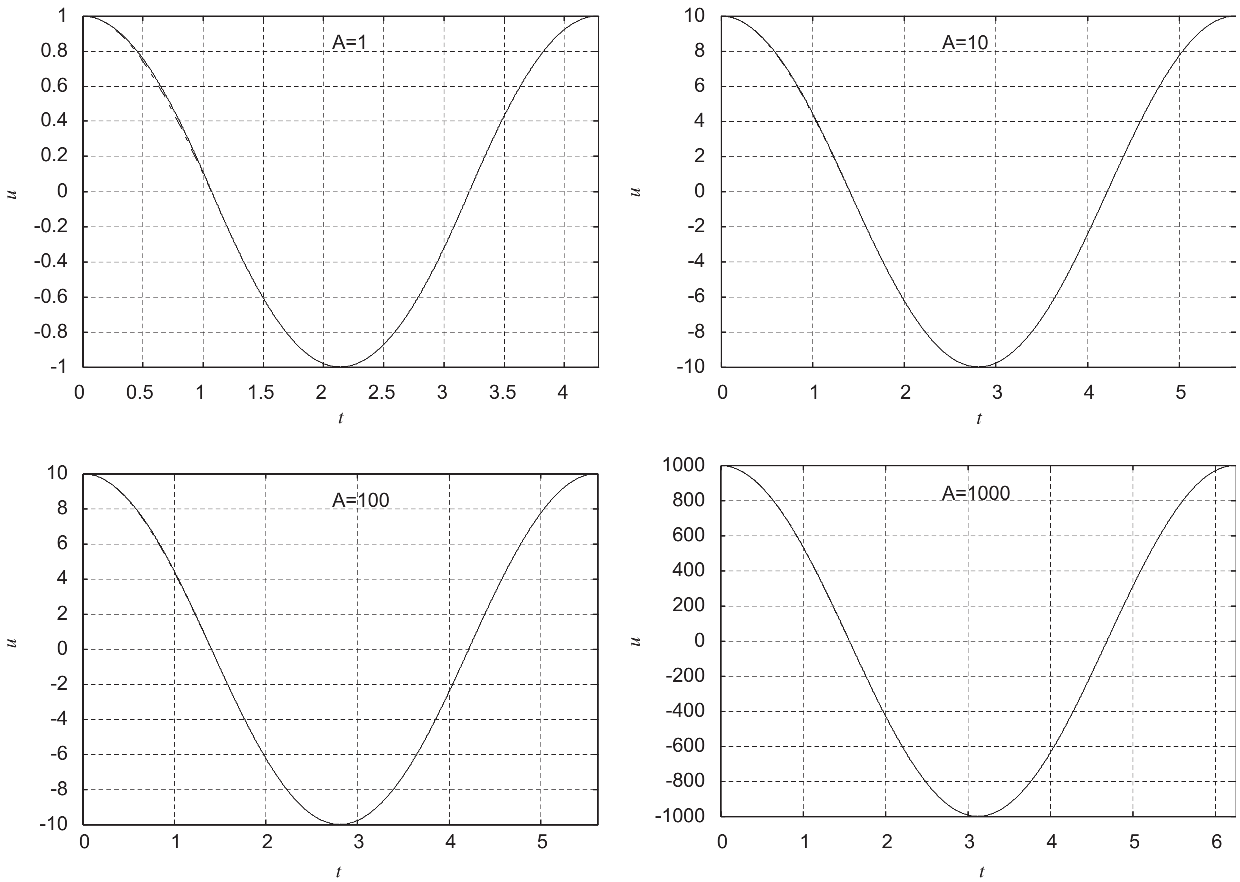

Fig. 4. Comparison of the approximate solution with the exact solution: dashed line: approximated solution and solid line: the exact solution.

Then, from Eq. (17), we can obtain the frequency of the nonlinear oscillator as follows:

$$\omega = \sqrt{\tfrac{3}{4}A^2 + 1}. \tag{32}$$

Therefore, we can obtain the following zero-order approximate solution

$$u = A\cos\left[\left(\frac{3}{4}A^2 + 1\right)^{1/2} t\right] \tag{33}$$

which agrees very well with the exact solution [26] as shown in Fig. 5.

3. Conclusion

The solution procedure of He's PEM is of deceptive simplicity and the insightful solutions obtained are of high accuracy even for the zero-order approximation [29]. The method, which is proved to be a powerful mathematical tool to the search for limit cycles of nonlinear oscillators, can be easily extended to any nonlinear equation, and the present letter can be used as paradigms for many other applications in searching for periodic solutions, limit cycles or other approximate solutions for real-life physics problems.

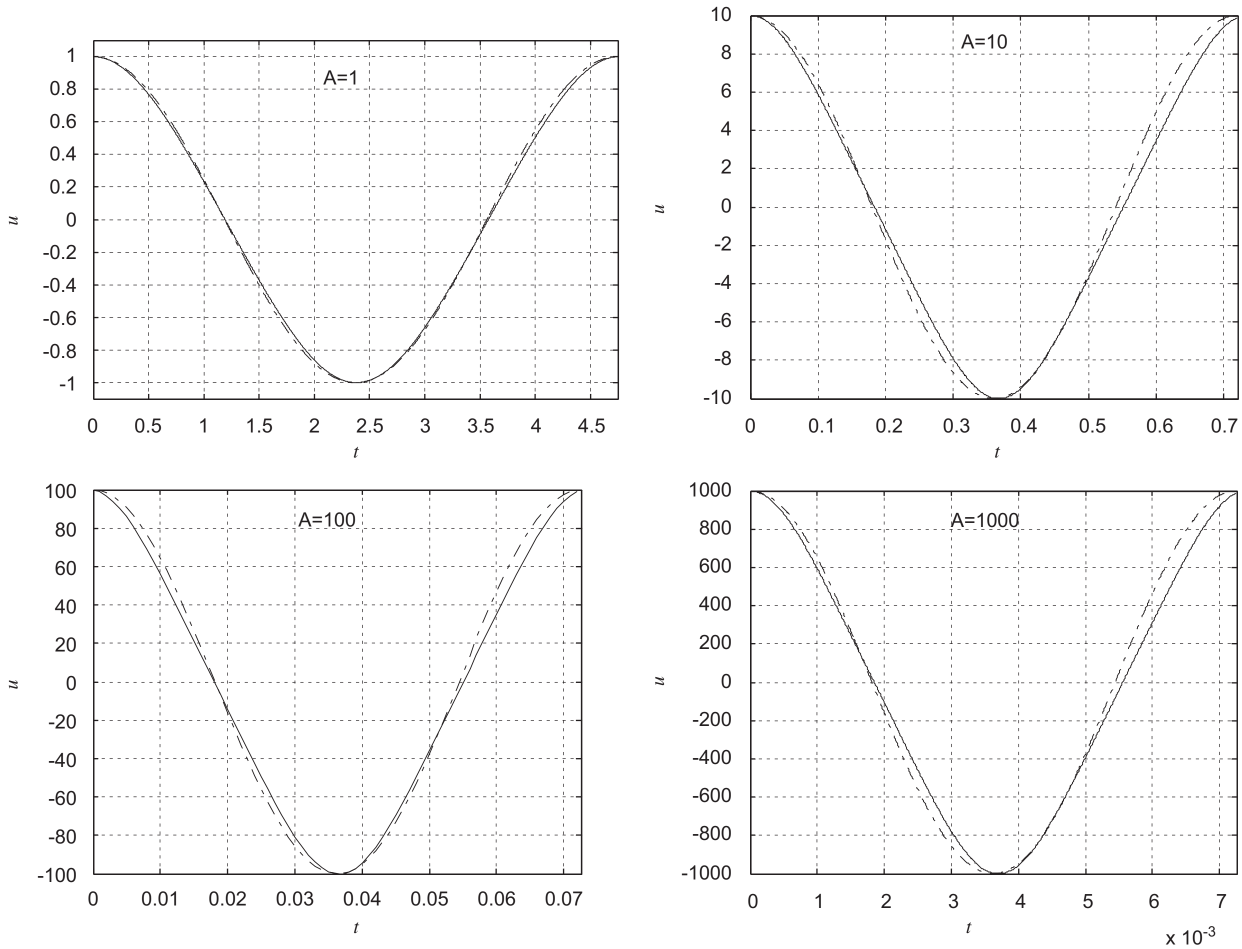

Fig. 5. Comparison of the approximate solution with the exact solution: dashed line: approximated solution and solid line: the exact solution.

References

[1] H.M. Liu, Approximate period of nonlinear oscillators with discontinuities by modified Lindstedt–Poincare method, *Chaos Solitons & Fractals* 23 (2) (2005) 577–579.

[2] J.H. He, Modified Lindstedt–Poincare methods for some strongly non-linear oscillations Part I: expansion of a constant, *International Journal of Nonlinear Mechanics* 37 (2) (2002) 309–314.

[3] J.H. He, Modified Lindstedt–Poincare methods for some strongly non-linear oscillations Part II: a new transformation, *International Journal of Nonlinear Mechanics* 37 (2) (2002) 315–320.

[4] J.H. He, Modified Lindstedt–Poincare methods for some strongly nonlinear oscillations Part III: Double series expansion, *International Journal of Nonlinear Sciences and Numerical Simulation* 2 (4) (2001) 317–320.

[5] X.C. Cai, W.Y. Wu, M.S. Li, Approximate period solution for a kind of nonlinear oscillator by He's perturbation method, *International Journal of Nonlinear Sciences and Numerical Simulation* 7 (1) (2006) 109–112.

[6] J.H. He, Variational iteration method—some recent results and new interpretations, *Journal of Computational and Applied Mathematics*, in press.

[7] J.H. He, X.H. Wu, Construction of solitary solution and compacton-like solution by variational iteration method, *Chaos Solitons & Fractals* 29 (1) (2006) 108–113.

[8] Z.M. Odibat, S. Momani, Application of variational iteration method to nonlinear differential equations of fractional order, *International Journal of Nonlinear Sciences and Numerical Simulation* 7 (1) (2006) 27–34.

[9] N. Bildik, A. Konuralp, The use of variational iteration method, differential transform method and adomian decomposition method for solving different types of nonlinear partial differential equations, *International Journal of Nonlinear Sciences and Numerical Simulation* 7 (1) (2006) 65–70.

[10] S. Momani, S. Abuasad, Application of He's variational iteration method to Helmholtz equation, *Chaos Solitons & Fractals* 27 (5) (2006) 1119–1123.

[11] J.H. He, New interpretation of homotopy perturbation method, *International Journal of Modern Physics B* 20 (18) (2006) 2561–2568.

[12] P.D. Ariel, T. Hayat, S. Asghar, Homotopy perturbation method and axisymmetric flow over a stretching sheet, *International Journal of Nonlinear Sciences and Numerical Simulation* 7 (4) (2006) 399–406.

[13] D.D. Ganji, A. Sadighi, Application of He's homotopy-perturbation method to nonlinear coupled systems of reaction–diffusion equations, *International Journal of Nonlinear Sciences and Numerical Simulation* 7 (4) (2006) 411–418.

[14] S. Abbasbandy, Application of He's homotopy perturbation method for Laplace transform, *Chaos Solitons & Fractals* 30 (5) (2006) 1206–1212.

[15] L. Cveticanin, Homotopy-perturbation method for pure nonlinear differential equation, *Chaos Solitons & Fractals* 30 (5) (2006) 1221–1230.

[16] D.D. Ganji, M. Rafei, Solitary wave solutions for a generalized Hirota–Satsuma coupled KdV equation by homotopy perturbation method, *Physics Letters A* 356 (2) (2006) 131–137.

[17] M. Rafei, D.D. Ganji, Explicit solutions of Helmholtz equation and fifth-order KdV equation using homotopy perturbation method, *International Journal of Nonlinear Sciences and Numerical Simulation* 7 (3) (2006) 321–328.

[18] A.M. Siddiqui, R. Mahmood, Q.K. Ghori, Thin film flow of a third grade fluid on a moving belt by He's homotopy perturbation method, *International Journal of Nonlinear Sciences and Numerical Simulation* 7 (1) (2006) 7–14.

[19] J.H. He, Homotopy perturbation method for solving boundary value problems, *Physics Letters A* 350 (1–2) (2006) 87–88.

[20] J.H. He, Application of homotopy perturbation method to nonlinear wave equations, *Chaos Solitons & Fractals* 26 (3) (2005) 695–700.

[21] J.H. He, Homotopy perturbation method for bifurcation of nonlinear problems, *International Journal of Nonlinear Sciences and Numerical Simulation* 6 (2) (2005) 207–208.

[22] J.H. He, Determination of limit cycles for strongly nonlinear oscillators, *Physical Review Letters* 90 (17) (2003) Art. No. 174301.

[23] J.H. He, Determination of limit cycles for strongly nonlinear oscillators (90, Art. No. 174301, 2003), *Physical Review Letters* 91 (19) (2003) Art. No. 199902.

[24] M. D'Acunto, Determination of limit cycles for a modified van der Pol oscillator, *Mechanics Research Communications* 33 (1) (2006) 93–98.

[25] M. D'Acunto, Self-excited systems: analytical determination of limit cycles, *Chaos Solitons & Fractals* 30 (3) (2006) 719–724.

[26] J.H. He, Some asymptotic methods for strongly nonlinear equations, *International Journal of Modern Physics B* 20 (10) (2006) 1141–1199.

[27] J.H. He, Bookkeeping parameter in perturbation methods, *International Journal of Nonlinear Sciences and Numerical Simulation* 2 (2001) 257–264.

[28] L. Xu, He's parameter-expanding methods for strongly nonlinear oscillators, *Journal of Computational and Applied Mathematics*, in press.

[29] J.H. He, A view to the new perturbation technique valid for large parameters—Author's reply, *Journal of Sound and Vibration* 282 (3–5) (2005) 1317–1320.