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Short Communication

Calculation of modes in azimuthally non-uniform lined ducts with uniform flow

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Abstract

In a recent article, a method was proposed to calculate the mode scattering by an azimuthally non-uniform impedance liner section inserted in an infinite duct. The method allowed the problem to be formulated as a two-dimensional Helmholtz eigenvalue problem, which could be solved with general purpose software rather than custom written codes, but appeared to be limited to ducts without flow. In this short communication, the relevant system of equations is reformulated so that problems with flow can also be treated. The resulting eigenvalue calculation shows good agreement with a well-tested one-dimensional solver when applied to a circular duct section with constant impedance.

1. Introduction

In a recent paper [1] one of us proposed a method to calculate sound transmission through a liner section in an infinite duct, the liner being axially uniform but azimuthally non-uniform. As discussed in that article many other approaches to the problem have been proposed but all numerical ones require custom codes to be written. The approach suggested in Ref. [1] differed by being possible to implement on commercially available finite element Helmholtz solvers. In this way a certain amount of computation speed per run, and frequency resolution can be sacrificed in order to greatly reduce the total implementation time.

The procedure proposed was, however, limited to ducts without flow, a significant limitation given that a typical application likely to require the solution of such a problem is an aircraft engine inlet with a spliced liner. This limitation is due to the presence of the eigenvalue to be found in the boundary condition of the problem. It was speculated that iteration about the solution with no flow could be used to obtain the flow solution. In this short communication a simpler way of formulating the problem as a standard Helmholtz eigenvalue problem is given, which will allow the method to be applied to problems with flow.

2. Problem

Consider a duct of constant cross-section containing a uniform mean flow. The duct is aligned with the x-axis, and the mean flow $\mathbf{U}_0 = (U, 0, 0)$ where U is constant, and the Mach number $M = U/c_0 < 1$, i.e. the

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flow is subsonic. Assume that a harmonic noise source (with exp $\mathrm{j}\omega t$ time dependence) is introduced into the duct, resulting in a harmonic pressure field that satisfies the convected Helmholtz equation

$$\left(\mathrm{j}\omega + U\frac{\partial}{\partial x}\right)^2 p = c_0^2\nabla^2 p. \tag{1}$$

Modal solutions are sought in the form

$$p_n(\mathbf{x}) = \psi_n(y,z)\mathrm{e}^{-\mathrm{j}k_n x}, \tag{2}$$

where $\psi_n(y,z)$ is the nth transverse mode shape, k_n is the axial wavenumber and the time dependence has been suppressed.

Substituting Eq. (2) into Eq. (1) gives the field equation,

$$\nabla_\perp^2\psi_n + [(k - Mk_n)^2 - k_n^2]\psi_n = 0, \tag{3}$$

where $\nabla_\perp^2$ denotes the two-dimensional Laplacian and $k = \omega/c_0$ is the wavenumber.

The specific acoustic impedance $Z_{\mathrm{spec.}}$ (non-dimensional) at the duct wall is given by the ratio of the acoustic pressure to normal acoustic particle velocity, i.e.

$$Z_{\mathrm{spec.}} = \frac{1}{\rho_0 c_0}\frac{p}{(\mathbf{u}\cdot\mathbf{n})}, \tag{4}$$

where ρ_0 is the mean density, $\mathbf{u}(\mathbf{x},t)$ is the time-harmonic acoustic particle velocity and $\mathbf{n}$ is the unit outward normal on the wall.

The boundary condition in the presence of uniform flow is given by Ingard [2]

$$\mathrm{j}\omega(\mathbf{u}\cdot\mathbf{n}) = (\mathrm{j}\omega + \mathbf{U}_0\cdot\boldsymbol{\nabla})\frac{p}{\rho_0 c_0 Z_{\mathrm{spec.}}}. \tag{5}$$

Combining Eq. (5) with Eq. (2) and the acoustic momentum equation,

$$(\mathrm{j}\omega + \mathbf{U}_0\cdot\boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho_0}\boldsymbol{\nabla}p, \tag{6}$$

leads to

$$\mathbf{n}\cdot\boldsymbol{\nabla}_\perp\psi_n + \frac{\mathrm{j}k}{Z_{\mathrm{spec.}}}\left(1 - M\frac{k_n}{k}\right)^2\psi_n = 0, \tag{7}$$

where $\boldsymbol{\nabla}$ has been replaced by $\boldsymbol{\nabla}_\perp$.

There are two difficulties with solving the resulting eigenvalue problem using a general purpose solver. The first is that the field equation (3) is quadratic in the eigenvalue. The second is that the boundary condition (7) is formulated in terms of the eigenvalue.

The solution to the first difficulty is well-known: auxiliary field variables that are multiplied by the original eigenvalue the required number of times are introduced and the resulting larger system is solved. This procedure was used, for example, by Unruh and Eversman [3] on a problem that was cubic in the eigenvalue. When applied twice to the present problem, as shown below, it removes the second difficulty at the same time. In order to allow non-specialist users to solve such transmission problems the resulting reformulation is spelled out explicitly below.

3. Matrix formulation

Introducing the variables

$$\zeta_n = k_n\psi_n, \quad \xi_n = k_n\zeta_n = k_n^2\psi_n, \tag{8}$$

means that the field equation, Eq. (3), can be expressed as

$$\nabla_\perp^2\psi_n + k^2\psi_n - 2kM\zeta_n + (M^2-1)\xi_n = 0, \tag{9}$$

which can be written

$$\begin{pmatrix} \nabla_\perp^2 \psi_n \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} k^2\psi_n - 2kM\zeta_n + (M^2-1)\xi_n \\ \zeta_n \\ \xi_n \end{pmatrix} = k_n \begin{pmatrix} 0 \\ \psi_n \\ \zeta_n \end{pmatrix} \tag{10}$$

or (with a slight abuse of notation)

$$-\boldsymbol{\nabla}_\perp \cdot \left[\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\nabla}_\perp \psi_n \\ \boldsymbol{\nabla}_\perp \zeta_n \\ \boldsymbol{\nabla}_\perp \xi_n \end{pmatrix} \right] + \begin{pmatrix} k^2 & -2kM & M^2-1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \psi_n \\ \zeta_n \\ \xi_n \end{pmatrix} = k_n \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_n \\ \zeta_n \\ \xi_n \end{pmatrix}. \tag{11}$$

Similarly for the boundary condition, expanding Eq. (7) leads to

$$\mathbf{n} \cdot \boldsymbol{\nabla}_\perp \psi_n + \frac{\mathrm{j}k}{Z_{\text{spec.}}} \left(1 - 2M\frac{k_n}{k} + M^2 \frac{k_n^2}{k^2} \right) \psi_n = 0 \tag{12}$$

or

$$\mathbf{n} \cdot \boldsymbol{\nabla}_\perp \psi_n + \frac{\mathrm{j}k}{Z_{\text{spec.}}} \psi_n - \frac{2\mathrm{j}M}{Z_{\text{spec.}}} \zeta_n + \frac{\mathrm{j}M^2}{kZ_{\text{spec.}}} \xi_n = 0, \tag{13}$$

which can be written

$$\mathbf{n} \cdot \left[\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\nabla}_\perp \psi_n \\ \boldsymbol{\nabla}_\perp \zeta_n \\ \boldsymbol{\nabla}_\perp \xi_n \end{pmatrix} \right] + \begin{pmatrix} -\dfrac{\mathrm{j}k}{Z_{\text{spec.}}} & \dfrac{2\mathrm{j}M}{Z_{\text{spec.}}} & -\dfrac{\mathrm{j}M^2}{kZ_{\text{spec.}}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_n \\ \zeta_n \\ \xi_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{14}$$

Eqs. (11) and (14) are in a form that can be solved with a standard finite element eigenvalue solver, such as that included in COMSOL Multiphysics (formerly FEMLAB). For some solvers it may be preferable to solve for an eigenvalue $\lambda_n = -\mathrm{j}k_n$ in which case the auxiliary variables ζ_n and ξ_n would be given by $\zeta_n = \lambda_n \psi_n$ and $\xi_n = \lambda_n^2 \psi_n$, etc.

4. Example

Although the method can be applied to arbitrarily shaped ducts with arbitrary impedance distribution (as long as it is axially uniform) it is hard to find reliable solutions of this nature for validation purposes. Therefore, it was applied to a circular duct with azimuthally uniform impedance, so that the results could be compared with eigenvalues found by a one-dimensional solver which uses a tracking method proposed by Eversman [4] and has been extensively tested and used in previous work [5]. The values of the eigenvalues are tracked in the complex plane as the impedance is varied, starting at the known rigid wall solution. The tracking is performed using an initial value problem formulation, and the eigenvalues are refined using the Newton–Raphson method. There is no unique tracking path in the impedance plane; the path used here was proposed by Rienstra [6], to ensure that surface wave modes are detected using this type of tracking method.

For the example chosen, the duct radius a is 0.5 m, the frequency is 500 Hz, $M = 0.5$ and $c_0 = 340\,\mathrm{m\,s^{-1}}$. The specific acoustic impedance of the wall is $Z_{\text{spec.}} = 2 - \mathrm{j}$.

The axial wavenumbers, calculated by solving Eqs. (11) and (14) using COMSOL Multiphysics with 10 quadratic elements per wavelength are plotted in Fig. 1a. A comparison with the results of the eigenvalue tracking method is shown in Fig. 1b. The two methods produce near identical results. A similar agreement is observed at higher wavenumbers.

The mesh size was chosen to be a tenth of the free-space wavelength c_0/f in the body of the duct. Close to the wall of the duct much finer elements were chosen so as to resolve the modes corresponding to surface waves which are strongly localised to the wall.

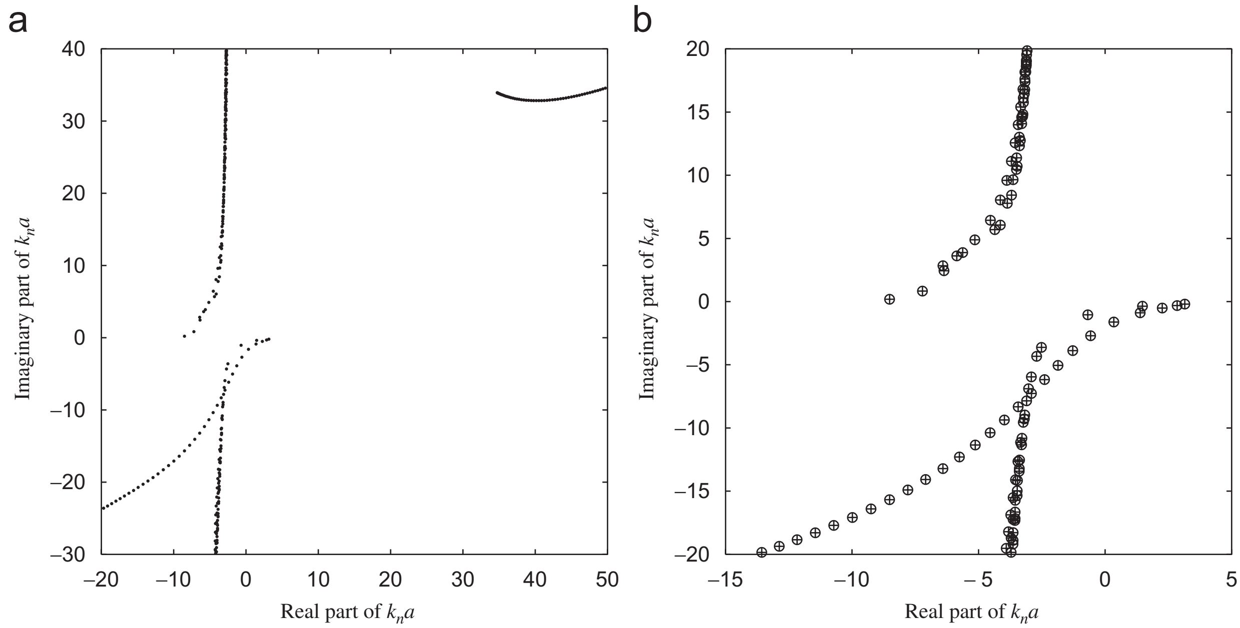

Fig. 1. (a) Spectrum of $k_n a$ (non-dimensional) found by solving Eqs. (11) and (14) with COMSOL Multiphysics. The branches at the bottom left and top right correspond to surface waves [7]. (b) Comparison of these results (+) with those obtained from eigenvalue tracking (○) for a smaller region of the complex plane. Similar agreement is found for the other solutions.

By solving the above problem over a range of values of ka it was possible to confirm that problems typical of real aerospace applications with $ka \approx 40$ could feasibly be solved on a desktop PC in a matter of hours. It is not possible to be more precise about the execution time for the following reason. Typical commercial eigenvalue solvers search for a given number of eigenvalues in the vicinity of a given point in the complex plane. When the spectrum lies on a number of disjoint branches it may be necessary to search around several points to be sure an adequate spectrum has been found. For the uniform duct problem used to illustrate the method Rienstra's asymptotic results [6] can help to locate the branches, but for more general, azimuthally non-uniform problems such as the liner-splice scattering treated in Ref. [1] no such results are available and it may be prudent to do multiple searches throughout the plane. Unfortunately, there is no way of knowing when all modes have been found.

To solve scattering problems such as the liner-splice problem the modes can be calculated as shown and then sorted into positive and negative travelling modes according to the sign of the imaginary part of k_n. Where surface waves are present this assumes that they are all stable, although in fact they may be unstable, backward-travelling waves, see Rienstra [6] for further details. This assumption will not affect the mode matching calculation which proceeds as before. The alternative mode matching approach in three-dimensional ducts with flow outlined in Astley et al. [8], whereby continuity of mass flux and momentum flux is explicitly imposed, rather than continuity of pressure and axial velocity may be expected to give improved results.

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