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Chaotic rotations of a liquid-filled solid

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Abstract

The disturbed Hamiltonian equations of a solid filled with a rotating ellipsoidal mass of a liquid and subjected to small-applied moments are revisited using Deprit's variables. We investigate the chaotic dynamics of the orbiting liquid-filled solid and of the liquid-filled solid sliding and rolling on a perfectly smooth plane, in either energy-conservative or energy-dissipative conditions, when appropriately perturbed. Criteria for the judgment of potential chaotic rotations of the perturbed system are formulated by means of Melnikov–Holmes–Marsden (MHM) integrals. Strategies for the solution of heteroclinic orbits of the symmetrical liquid-filled solid under torque-free conditions are outlined theoretically. Physical parameters that will probably trigger the onset of chaotic motions can be determined accordingly. Results from MHM algorithms are crosschecked with Poincare sections together with Lyapunov characteristic exponents.

1. Introduction

The chaotic dynamics of mechanical systems have attracted the attention of many scientists and engineers for several decades. Investigations on chaos help to better design and suppress the behaviors of mechanical systems as suggested in Thompson and Stewart [1], Parker and Chua [2], Chen and Leung [3] and amongst others. The investigations on rotational motions of a solid with a cavity completely liquid-filled can be traced back to a century ago. Problems of contained rotating fluids have many applications (see Refs. [4–15]). The salient features of motions of an ideal, incompressible liquid filling completely the ellipsoidal cavity of a solid are proved to be a uniform vortex and were fully investigated by Zhukovsky and Rumyantsev [4]. This interesting case is called the Zhukovsky–Rumyantsev (ZR) liquid-filled solid throughout this paper. For certain cases the ZR liquid-filled solid could be thought of as a gyrostat. Studies on the rotational motions of a gyrostat are significant to the controlling design of the artificial satellite and spinning liquid-filled projectiles (see Refs. [16–19]).

Chaotic instability of the spinning asymmetric top was discussed numerically in Ref. [20]. The chaotic dynamics of rigid bodies were studied by Ziglin [21], Holmes and Marsden [22], and Kozlov [23] using Melnikov integrals [24]. Holmes and Marsden [25] invented a Melnikov–Holmes–Marsden (MHM) integral for the disturbed multi-degrees-of-freedom Hamiltonian system. MHM integral algorithms will be the essential foundation of this article's

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contributions. The earlier studies relevant to the application of MHM integrals to chaos detection of the disturbed gyrostat system may be found in Ref. [26] using Deprit's canonical variables [27]. The instability investigations on the attitude dynamics of artificial or natural satellites without resorting to Deprit's variables may be found in Refs. [28–32]. It is noted that the fundamental Melnikov methods were extensively discussed in Refs. [33–36].

Karapetyan and Prokomina [37], Rudenko [38] and others in their lists of references investigated the stability conditions of steady motions of the liquid-filled gyrostat rolling without sliding on the plane using Lyapunov's functionals associated with first integrals. Applications on first integrals may be found in Kozlov [23] and Conte [39] amongst others.

In this paper, the chaotic rotations of ZR liquid-filled solids due to either the orbital frequency or the damped, time-periodical moments are investigated. Chaos considered in this article is in the sense of Smale's horseshoe. The authors' previous algorithms [12,13] are extended here to establish some new effective algorithms for the prediction of chaotic rotational motions of ZR liquid-filled solids. The preliminary results on chaotic motions of the ZR liquid-filled solid under gravity-gradient torques were found in Ref. [11], where the chaotic motions were regarded as the heteroclinic bifurcation due to the perturbation moment of second order $O(N^2)$ where N is the orbital frequency. Some new progresses recently made are presented in detail here. Some results of the ZR liquid-filled gyrostat under dissipative moments plus periodic torques have been derived from the disturbed Hamiltonian mechanics in terms of Eulerian angles and conjugate angular momenta in Ref. [12]. Kuang et al. [13] further investigated the liquid-filled ellipsoid based upon the noslip contact condition between the ellipsoid and the perfectly *rough* horizontal plane. The model scenarios studied in this article are different from all the previous publications of authors.

The paper is organized in the sequence of formulating the set of disturbed Hamiltonian equations, obtaining the heteroclinic orbit and applying MHM integrals for the determination of chaos as follows. In Section 1, the background of dynamics of ZR liquid-filled solids under small torques is introduced briefly. In Section 2, Eulerian equations of ZR liquid-filled solids subjected to small moments and the Helmholtz equations of a liquid in an ellipsoidal cavity together with Poisson's equations of directional cosines are described for subsequent development. In Section 3, a set of canonical variables are constructed to establish the disturbed Hamiltonian equations of the system under investigation. In Section 4, the heteroclinic orbit of rotational motions of the torque-free symmetrical ZR liquid-filled solid and the key issue in the construction of the heteroclinic orbit are given for completeness. In Section 5, the MHM integral of the ZR liquid-filled solid under periodically forced moments is derived and its application is discussed. In Section 6, criteria to determine the onset of chaos for the disturbed ZR liquid-filled solid under the action of conservative moments due to the orbital frequency are established according to MHM integrals. In Section 7, the derivation of MHM integrals for the perturbed liquid-filled ellipsoid sliding and rolling on the perfectly smooth plane is investigated. In Section 8, the 4th order Runge–Kutta algorithms are used to simulate the long-term chaotic dynamics which are crosschecked with Poincare sections and Lyapunov characteristic exponents. In Section 9, singularities involved in MHM integrals are discussed and the paper is concluded with a few remarks.

2. Governing equations

We consider a mechanical system of a ZR liquid-filled solid circularly orbiting the Earth. O is the center of mass of the solid containing an ellipsoidal cavity fully filled with an ideal liquid shown in Fig. 1. Define two Cartesian coordinate systems $OXYZ$ and $Oxyz$ having a common origin at the center of mass O of the system. $OXYZ$ is the local-vertical-local-horizontal (LVLH) coordinate system with OX along the orbit direction, OY normal to the orbit plane and OZ toward the Earth center. Effects arising from the orbital frequency will be included. $OXYZ$ will degenerate to the inertial reference system when the orbital frequency vanishes. $Oxyz$ coincides with principal axes of inertia of the ZR liquid-filled solid. Let $b_x = b_y$ and b_z be the semi-axes of the cavity and e_k be the offsets of the center of the cavity from O. The bounding surface of the cavity is $\sum_{k=x,y,z}(Y_k - e_k)^2/b_k^2 = 1$, where Y_k are coordinates measured in $Oxyz$ and the subscript index $k = x,y,z$ unless stated otherwise throughout this paper. We designate $e_x = e_y = 0$ and $e_z \neq 0$, which represent that the axis of symmetry of the cavity coincides with the body-fixed axis Oz. Let Ω_k be the components of the angular velocity of the ZR liquid-filled solid. Π_k are the components of the vorticity of the contained liquid according to the expression $\mathrm{curl}\vec{V} = (\Pi_x(t), \Pi_y(t), \Pi_z(t))$, where $\vec{V}$ is the velocity of liquid particle. I_{ck} denote the sums of

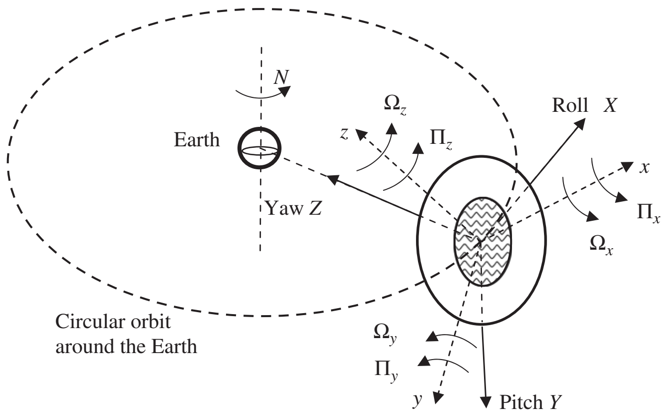

Fig. 1. Configuration of the orbiting ZR liquid-filled solid with the orbital frequency N.

moments of inertia of the solid and of the Zhukovsky solid equivalent to the consolidated liquid with respect to $Oxyz$. J_{ck} are the differences between the moments of inertia of a consolidated liquid and of the Zhukovsky solid equivalent to the contained liquid with respect to $Oxyz$. The theorem on the angular momentum about the center of gravity O with respect to the mobile axes $Oxyz$ gives

$$I_{cx}\frac{\mathrm{d}\Omega_x}{\mathrm{d}t}+J_{cx}\frac{\mathrm{d}\Pi_x}{\mathrm{d}t}+(I_{cz}\Omega_z+J_{cz}\Pi_z)\Omega_y-(I_{cy}\Omega_y+J_{cy}\Pi_y)\Omega_z=L_x, \quad (x\,y\,z), \tag{1}$$

where $I_{cx}=I_{ex}+(1/5)M_f(b_y^2-b_z^2)^2/(b_y^2+b_z^2)+M_f(e_y^2+e_z^2)$, $(x\,y\,z)$, and M_f is the mass of liquid filling the ellipsoidal cavity (see Ref. [4]). The notation $(x\,y\,z)$ abbreviates that one of the three equations is written down and the other two can be obtained by cyclic permutation of indices $(x\rightarrow y\rightarrow z)$. Designate $L_k=\varepsilon\,M_k$ $(\Omega_x,\ldots,\Pi_x,\ldots,\gamma_x,\ldots,t)$ as the applied moments to the ZR liquid-filled solid. γ_k are the direction cosines of axis OZ in relation to $Oxyz$. ε is a small parameter $(0<\varepsilon\ll 1)$. $\mathrm{d}/\mathrm{d}t$ denotes differentiation with respect to time. The Helmholtz equations of uniform vortex motion for the liquid contained in the ellipsoidal cavity in x, y and z axes, are

$$I_{gx}\frac{\mathrm{d}\Pi_x}{\mathrm{d}t}=I_{gz}\Pi_z\frac{J_{cy}(\Pi_y-\Omega_y)}{I_{gy}}-I_{gy}\Pi_y\frac{J_{cz}(\Pi_z-\Omega_z)}{I_{gz}}, \quad (x\,y\,z), \tag{2}$$

where $I_{gx}=(2/5)M_f b_y b_z$, $(x\,y\,z)$ and $J_{cx}=(4/5)M_f b_y^2 b_z^2/(b_y^2+b_z^2)$, $(x\,y\,z)$ (see Ref. [4]). It is remarked that Helmholtz equations (2) hold only for the uniform vortex motion of the ideal, incompressible liquid contained in the ellipsoidal cavity under the conditions that the center of the cavity is located on the symmetrical axis of the ZR liquid-filled solid and that the disturbing moments do not change the form of Helmholtz equations (2). Since the line spanned by γ_k remains always vertical, Poisson's equations become,

$$\frac{\mathrm{d}\gamma_x}{\mathrm{d}t}=\Omega_z\gamma_y-\Omega_y\gamma_z, \quad (x\,y\,z), \tag{3}$$

where $\gamma_x^2+\gamma_y^2+\gamma_z^2=1$.

The governing equations of rotational motions of the ZR liquid-filled solid comprise Eulerian equations (1) of the moment of momentum of the whole system, Helmholtz equations (2) of the rotating ellipsoidal mass of the liquid and Poisson's equations (3) of direction cosines. From these equations, one can derive the disturbed Hamiltonian equations of the ZR liquid-filled solid in terms of a set of Deprit's variables and construct the heteroclinic orbit of a torque-free ZR liquid-filled solid in the following sections.

3. Disturbed Hamiltonian equations

The Hamiltonian equations of rotational motions of the ZR liquid-filled solid disturbed by small moments will be derived in this section. State variables in the systems (1), (2) and (3) are $\Omega_k(t)$, $\Pi_k(t)$, and $\gamma_k(t)$. These variables are not readily suitable for applying the MHM integrals to determine the occurrence of chaotic rotational motions. In order to derive the disturbed Hamiltonian equations that are directly suitable for the applications of MHM integrals, appropriate transformations into a new set of canonical variables in $\Gamma = (L,G,H,l,g,h,L_f,l_f)$ are introduced, where the first six variables are Deprit's variables depicted in Fig. 2 of Kuang et al. [9,13] and the remaining two newly introduced variables, L_f and l_f, represent the uniform vortex motion of the contained liquid. The fundamental theory of Hamiltonian mechanics may be found in Ref. [40]. Procedures for the derivation of the disturbed Hamiltonian equations of the ZR liquid-filled solid are briefly exposed here. The kinetic energy of the whole ZR liquid-filled solid is given by

$$T = \frac{1}{2}\sum_{k=x,y,z}(I_{wk}\Omega_k^2 + I_{fk}\theta_k^2 + 2I_{gk}\Omega_k\theta_k), \tag{4}$$

where $I_{wx} = I_{cx} + J_{cx}$, $(x\,y\,z)$, with I_{wk} denoting the principal moments of inertia of the entire system with respect to $Oxyz$, I_{fk} and I_{gk} being the principal moments of inertia of the contained liquid $I_{fx} = M_f({b_y}^2 + {b_z}^2)/5$, $(x\,y\,z)$; and

$$\theta_x = (\Pi_x - \Omega_x)\frac{2b_y b_z}{b_y^2 + b_z^2}, \quad (x\,y\,z). \tag{5}$$

The transformations between variables $\Omega_k(t)$, $\Pi_k(t)$ and canonical variables in Γ take the form

$$\begin{aligned}
&\frac{\partial T}{\partial \Omega_x} = I_{wx}\Omega_x + I_{gx}\theta_x \triangleq G\sin b\sin l, &&\frac{\partial T}{\partial \Omega_y} = I_{wy}\Omega_y + I_{gy}\theta_y \triangleq G\sin b\cos l,\\
&\frac{\partial T}{\partial \Omega_z} = I_{wz}\Omega_z + I_{gz}\theta_z \triangleq G\cos b = L, &&\frac{\partial T}{\partial \theta_x} = I_{fx}\theta_x + I_{gx}\Omega_x \triangleq G_f\sin b_f\sin l_f,\\
&\frac{\partial T}{\partial \theta_y} = I_{fy}\theta_y + I_{gy}\Omega_y \triangleq G_f\sin b_f\cos l_f, &&\frac{\partial T}{\partial \theta_z} = I_{fz}\theta_z + I_{gz}\Omega_z \triangleq G_f\cos b_f = L_f,
\end{aligned} \tag{6}$$

where $\partial T/\partial\Omega_k$ and $\partial T/\partial\theta_k$ are partial derivatives computed from Eq. (4). Canonical variables in Γ are functions of time t only. So are the angular variables b and b_f. The physical meanings of the original Deprit's variables in Γ may be found in Refs. [9,13]. The definition of newly introduced canonical variables L_f and l_f may be realized from Eqs. (6) where

$$\left(\frac{\partial T}{\partial\Omega_x}\right)^2 + \left(\frac{\partial T}{\partial\Omega_y}\right)^2 + \left(\frac{\partial T}{\partial\Omega_z}\right)^2 = G^2, \quad \left(\frac{\partial T}{\partial\theta_x}\right)^2 + \left(\frac{\partial T}{\partial\theta_y}\right)^2 + \left(\frac{\partial T}{\partial\theta_z}\right)^2 = G_f^2 = \text{const.} \tag{7}$$

The second of Eqs. (7) expresses the Helmholz theorem about the constant vorticity of the liquid contained in the ellipsoidal cavity. The relations between direction cosines and Deprit's variables are

$$\begin{aligned}
\gamma_x &= (\sin I\sin g)\cos l + (\cos I\sin b + \sin I\cos b\cos g)\sin l,\\
\gamma_y &= -(\sin I\sin g)\sin l + (\cos I\sin b + \sin I\cos b\cos g)\cos l,\\
\gamma_z &= \cos I\cos b - \sin I\sin b\cos g,
\end{aligned} \tag{8}$$

where $\gamma_x = \sin\theta\sin\phi$, $\gamma_y = \sin\theta\cos\phi$, and $\gamma_z = \cos\theta$. Eulerian angles ψ, θ and ϕ are shown in Fig. 2 of Kuang et al. [9,13]. Consider the body-fixed rotational sequence $\psi\to\theta\to\phi$ for the rotational motion of the ZR liquid-filled solid in the LVLH coordinate system $OXYZ$ (see Ref. [30]). The angular velocity components can be expressed in terms of Eulerian angles and their angular rates as

$$\Omega_x = \Omega_{rx} - N\,d_{x2}, \quad \Omega_y = \Omega_{ry} - N\,d_{y2}, \quad \Omega_z = \Omega_{rz} - N\,d_{z2}, \tag{9}$$

where

$$\Omega_{rx} = \frac{d\psi}{dt}\sin\theta\sin\phi + \frac{d\theta}{dt}\cos\phi, \quad d_{x2} = \cos\phi\sin\psi + \sin\phi\cos\psi\cos\theta,$$
$$\Omega_{ry} = \frac{d\psi}{dt}\sin\theta\cos\phi - \frac{d\theta}{dt}\sin\phi, \quad d_{y2} = -\sin\phi\sin\psi + \cos\phi\cos\psi\cos\theta,$$
$$\Omega_{rz} = \frac{d\psi}{dt}\cos\theta + \frac{d\phi}{dt}, \quad d_{z2} = -\cos\psi\sin\theta,$$

and $N = \sqrt{\mu/d_e^3}$ is the orbital frequency. μ is the gravitational attraction constant of the Earth. d_e is the distance of the mass center of the ZR liquid-filled solid from the Earth. When $N = 0$, then $OXYZ$ becomes an inertial coordinate system. Angles $\psi - h$, g, and $\phi - l$ are related to angles I, θ, and b with the usual identities of spherical trigonometry for the spherical triangle PNM of Fig. 2 of Refs. [9,13], i.e.

$$\sin\theta\cos(\phi - l) = \cos I\sin b + \sin I\cos b\cos g,$$
$$\sin\theta\sin(\phi - l) = \sin I\sin g, \quad \sin\theta\sin(\psi - h) = \sin b\sin g,$$
$$\sin\theta\cos(\psi - h) = \sin I\cos b + \cos I\sin b\cos g, \tag{10}$$

where I is the angle between momenta H and G, and the angle b is a function of new momenta G and L, i.e., $\cos I = H/G$, and $\cos b = L/G$. From relations (9) and (10), one can transform Eqs. (1)–(3) into the disturbed Hamiltonian equations in terms of canonical variables in Γ as follows:

$$\frac{dL}{dt} = -\frac{\partial T}{\partial l} + \varepsilon f_L, \quad \frac{dG}{dt} = -\frac{\partial T}{\partial g} + \varepsilon f_G, \quad \frac{dH}{dt} = -\frac{\partial T}{\partial h} + \varepsilon f_H, \quad \frac{dl}{dt} = \frac{\partial T}{\partial L} + \varepsilon f_l,$$
$$\frac{dg}{dt} = \frac{\partial T}{\partial G} + \varepsilon f_g, \quad \frac{dh}{dt} = \frac{\partial T}{\partial H} + \varepsilon f_h, \quad \frac{dL_f}{dt} = \frac{\partial T}{\partial l_f} + \varepsilon f_{L_f}, \quad \frac{dl_f}{dt} = -\frac{\partial T}{\partial L_f} + \varepsilon f_{l_f}, \tag{11}$$

where the Hamiltonian of the system under investigation has been constructed as the kinetic energy, T in Eq. (4), of the ZR liquid-filled solid whose state variables Ω_k and θ_k are related to canonical variables in Γ from Eqs. (6) such as

$$\Omega_k = \frac{I_{fk}\frac{\partial T}{\partial \Omega_k} - I_{gk}\frac{\partial T}{\partial \theta_k}}{I_{wk}I_{fk} - I_{gk}^2}, \quad \theta_k = \frac{-I_{gk}\frac{\partial T}{\partial \Omega_k} + I_{wk}\frac{\partial T}{\partial \theta_k}}{I_{wk}I_{fk} - I_{gk}^2}. \tag{12}$$

Partial derivatives in Eqs. (11) are

$$\frac{\partial T}{\partial l} = (G\sin b\cos l)\Omega_x - (G\sin b\sin l)\Omega_y, \quad \frac{\partial T}{\partial g} = \frac{\partial T}{\partial h} = \frac{\partial T}{\partial H} = 0,$$
$$\frac{\partial T}{\partial L} = \frac{(-\cos b\sin l)}{\sin b}\Omega_x + \frac{(-\cos b\cos l)}{\sin b}\Omega_y + \Omega_z, \tag{13}$$

$$\frac{\partial T}{\partial G} = \frac{\sin l}{\sin b}\Omega_x + \frac{\cos l}{\sin b}\Omega_y, \quad \frac{\partial T}{\partial l_f} = (G_f\sin b_f\cos l_f)\theta_x - (G_f\sin b_f\sin l_f)\theta_y,$$
$$\frac{\partial T}{\partial L_f} = \frac{(-\cos b_f\sin l_f)}{\sin b_f}\theta_x + \frac{(-\cos b_f\cos l_f)}{\sin b_f}\theta_y + \theta_z, \tag{14}$$

$$f_l = (M_x\cos l - M_y\sin l)/(G\sin b), \quad f_{l_f} = f_{L_f} = 0, \quad f_L = M_z,$$
$$f_G = M_z\cos b + (M_x\sin l + M_y\cos l)\sin b, \tag{15}$$

$$
\begin{aligned}
f_H = {} & M_x[(\sin I \sin g)\cos l + (\cos I \sin b + \sin I \cos b \cos g)\sin l] \\
& + M_y[-(\sin I \sin g)\sin l + (\cos I \sin b + \sin I \cos b \cos g)\cos l] \\
& + M_z[\cos I \cos b - \sin I \sin b \cos g] \\
& - (\sin I \sin h)R_e G/(3N),
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
f_g = {} & M_x \frac{-(\sin I \cos b + \cos I \sin b \cos g)\cos l + (\cos I \sin b \cos b \sin g)\sin l}{G \sin I \sin b} \\
& + M_y \frac{(\sin I \cos b + \cos I \sin b \cos g)\sin l + (\cos I \sin b \cos b \sin g)\cos l}{G \sin I \sin b} \\
& + M_z \frac{(-\cos I \sin b \sin g)}{G \sin I} - \frac{R_e \cos h}{3N \sin I},
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
f_h = {} & M_x \frac{\cos g \cos l - \sin g \sin l \cos b}{G \sin I} \\
& - M_y \frac{\cos g \sin l + \sin g \cos l \cos b}{G \sin I} + M_z \frac{\sin b \sin g}{G \sin I} + \frac{R_e \cos h \cos I}{3N \sin I},
\end{aligned} \tag{18}
$$

where

$$
R_e = \frac{(\sin b \sin g)^2 + (\sin I \cos b + \cos I \sin b \cos g)^2}{(\sin I \sin g)^2 + (\cos I \sin b + \sin I \cos b \cos g)^2},
$$

Substituting transformations (8) and (12) into $M_k = M_k(\Omega_x, \ldots, \Pi_x, \ldots, \gamma_x, \ldots, t)$ in Eqs. (13)–(18), we can deduce that the disturbed Hamiltonian equations (11) governing the rotational motions of the orbiting ZR liquid-filled solid are expressed in terms of canonical variables in Γ. Terms containing N^{-1} in Eqs. (16)–(18) reflect the influence of the orbital rate on variables H, g and h. In Eqs. (11), the small parameter $\varepsilon = 3N^2$ represents a coefficient of gravity-gradient torques in the case of the orbiting ZR liquid-filled solid. When N approaches zero, the disturbed Hamiltonian equations can be modified to describe the motion of the ZR liquid-filled solid under small moments $L_k = \varepsilon M_k$ with ε being arbitrarily small and independent of N by annihilating the terms containing $1/N$ in Eqs. (16)–(18).

Note that the disturbed Hamiltonian equations (11) of the ZR liquid-filled solid are analogous to those of the gyrostat model derived by Kuang et al. [9] and of the rigid-body model derived by Tong and Taborrak [26]. The influences of the orbital rate and of the uniform vortex motion of a liquid on Deprit's variables are included in the disturbed Hamiltonian equations. The relations between Eulerian angular rates and angular velocities are given by Eqs. (9) where the orbital rate complicates the derivation of the disturbed Hamiltonian equations. The derivations become much lengthier than those of Kuang et al. [9], due to the involvement of the orbit rate and of the uniform vortex of the contained liquid, and are omitted here for brevity.

The disturbed Hamiltonian equations (11) in terms of canonical variables in Γ are ready for the application of MHM integrals to determine whether there is a transversal intersection between the stable and unstable manifolds of the liquid-filled symmetrical solid subjected to small torques. To this end, we need to explore the Hamiltonian structure of undisturbed equations of rotational motions of the ZR liquid-filled solid and the associated homoclinic/heteroclinic orbit. Two applications: (1) chaotic oscillations of the orbiting ZR liquid-filled solid, either conservative or dissipative; and (2) chaotic dynamics of the sliding liquid-filled spheroidal solid rolling on the perfectly smooth horizontal plane, either conservative or dissipative, will be investigated. We shall first obtain the homoclinic/heteroclinic orbit of the symmetrical ZR liquid-filled solid under torque-free conditions in the next section.

4. Heteroclinic orbits

To keep the idea of this article self-sustained, the acquisition of heteroclinic orbits and the construction of heteroclinic orbits of the symmetrical ZR liquid-filled solid are briefly presented in Sections 4.1 and 4.2, respectively. The detailed procedures may be found in Ref. [12].

4.1. The acquisition of heteroclinic orbits

Rotational motions of the ZR liquid-filled solid are always governed by Eulerian equations (1), Helmholtz equations (2) and Poisson equations (3). However, angular velocity components $\bar{\Omega}_k(t)$ of the ZR liquid-filled solid and uniform vortex components $\bar{\Pi}_k(t)$ of the contained liquid are uncoupled from direction cosines under torque-free motions. From Eqs. (1) and (2) when $L_k = 0$, one obtains

$$I_{cx}\frac{\mathrm{d}\bar{\Omega}_x}{\mathrm{d}t} + J_{cx}\frac{\mathrm{d}\bar{\Pi}_x}{\mathrm{d}t} + (I_{cz} - I_{cy})\bar{\Omega}_y\bar{\Omega}_z + J_{cz}\bar{\Pi}_z\bar{\Omega}_y - J_{cy}\bar{\Pi}_y\bar{\Omega}_z = 0, \quad (x\,y\,z), \tag{19}$$

$$\frac{\mathrm{d}\bar{\Pi}_x}{\mathrm{d}t} = 2b_x^2\left[\frac{\bar{\Omega}_z\bar{\Pi}_y}{b_x^2 + b_y^2} - \frac{\bar{\Omega}_y\bar{\Pi}_z}{b_x^2 + b_z^2}\right] - 2\bar{\Pi}_y\bar{\Pi}_z\frac{b_x^2(b_z^2 - b_y^2)}{(b_x^2 + b_y^2)(b_x^2 + b_z^2)}, \quad (x\,y\,z), \tag{20}$$

where the over-bar denotes the particular solution of the torque-free ZR liquid-filled solid. The symmetry of the geometrical shape of the ZR liquid-filled solid about z-axis, i.e., $b_x = b_y$ and $I_{cx} = I_{cy}$, yields $J_{cx} = J_{cy}$. The torque-free Euler–Helmholtz equations (19) and (20) have first integrals,

$$\sum_{k=x,y,z}(I_{ck}\bar{\Omega}_k^2 + J_{ck}\bar{\Pi}_k^2) = 2E = \text{const}, \quad \sum_{k=x,y,z}(I_{ck}\bar{\Omega}_k + J_{ck}\bar{\Pi}_k)^2 = G_p^2 = \text{const},$$
$$(b_yb_z)^2\bar{\Pi}_x^2 + (b_xb_z)^2\bar{\Pi}_y^2 + (b_yb_x)^2\bar{\Pi}_z^2 = \Pi^2 = \text{const}, \quad \bar{\Omega}_z = \text{const}. \tag{21}$$

The first of Eqs. (21) denotes the first integral of energy of the torque-free ZR liquid-filled solid. The second represents the first integral of angular momenta. The third implies the Helmholtz theorem about the constant vorticity of the liquid in the ellipsoidal cavity. The fourth is the first integral of Eulerian equations due to the geometrical symmetry. Using elliptic integral theory, periodic solutions of the undisturbed symmetrical ZR liquid-filled solid were presented in Ref. [12]. For further discussion, we single out the heteroclinic orbits of the torque-free symmetrical ZR liquid-filled solid from first integrals (21) associated with the torque-free Euler–Helmholtz equations (19) and (20) in terms of $\bar{\Omega}_k$ and $\bar{\Pi}_k$ as follows:

$$\bar{\Omega}_x(t) = \left((S_0 + S_1\bar{\Pi}_z + S_2\bar{\Pi}_z^2)\bar{\Pi}_x - \frac{(b_x^2 + b_z^2)}{2b_z^2}\bar{\Pi}_y\frac{\mathrm{d}\bar{\Pi}_z}{\mathrm{d}t}\right)\Big/\left(R_0 + R_2\bar{\Pi}_z^2\right),$$

$$\bar{\Omega}_y(t) = \left((S_0 + S_1\bar{\Pi}_z + S_2\bar{\Pi}_z^2)\bar{\Pi}_y + \frac{(b_x^2 + b_z^2)}{2b_z^2}\bar{\Pi}_x\frac{\mathrm{d}\bar{\Pi}_z}{\mathrm{d}t}\right)\Big/\left(R_0 + R_2\bar{\Pi}_z^2\right),$$

$$\bar{\Omega}_z = \text{constant}, \quad \bar{\Pi}_z(t) = (D_1 + D_2\tanh^2 u)/(D_3 + D_4\tanh^2 u),$$

$$\bar{\Pi}_x(t) = \exp(\bar{\Pi}_R(t))[\bar{\Pi}_x(t_a)\cos(\bar{\Pi}_I(t)) - \bar{\Pi}_y(t_a)\sin(\bar{\Pi}_I(t))],$$

$$\bar{\Pi}_y(t) = \exp(\bar{\Pi}_R(t))[\bar{\Pi}_x(t_a)\sin(\bar{\Pi}_I(t)) + \bar{\Pi}_y(t_a)\cos(\bar{\Pi}_I(t))],$$

$$\mathrm{d}\bar{\Pi}_z/\mathrm{d}t = \left(2\beta(D_2D_3 - D_1D_4)(\tanh u - \tanh^3 u)\right)/(D_3 + D_4\tanh^2 u)^2, \tag{22}$$

where $\bar{\Pi}_I(t)$ and $\bar{\Pi}_R(t)$ are defined by

$$\bar{\Pi}_R(t) = \int_{t_a}^{t} P_R(s)\mathrm{d}s, \quad \bar{\Pi}_I(t) = \int_{t_a}^{t} P_I(s)\mathrm{d}s, \quad P_R(t) = -\frac{b_x^2}{b_z^2}\bar{\Pi}_z\frac{\mathrm{d}\bar{\Pi}_z}{\mathrm{d}t}\Big/(R_0 + R_2\bar{\Pi}_z^2),$$

$$P_I(t) = \frac{2b_x^2(S_0 + S_1\bar{\Pi}_z + S_2\bar{\Pi}_z^2)\bar{\Pi}_z}{(b_x^2 + b_z^2)(R_0 + R_2\bar{\Pi}_z^2)} + \frac{(b_z^2 - b_x^2)}{(b_z^2 + b_x^2)}\bar{\Pi}_z - \bar{\Omega}_z, \tag{23}$$

in which $u = \beta(t - t_a)$ and t_a is a constant. Quantities β, S_0, S_1, S_2, R_0, R_2, and D_i for $i = 1,2,3,4$ which can be found in Refs. [12,13] are functions with arguments M_f, I_{ck}, J_{ck}, E, G_p, Π, and $\bar{\Omega}_z$. Detailed derivation procedures are omitted here for brevity. From the first integral of conservation of angular momenta and the third of Helmholtz equations (20), we find that $\bar{\Pi}_z$ satisfies the following equation:

$$(\mathrm{d}\bar{\Pi}_z/\mathrm{d}t)^2 = a_4\bar{\Pi}_z^4 + a_3\bar{\Pi}_z^3 + a_2\bar{\Pi}_z^2 + a_1\bar{\Pi}_z + a_0, \tag{24}$$

where coefficients $a_4, a_3, \ldots,$ and a_0 are functions of arguments M_f, I_{ck}, J_{ck}, E, G_p, Π, and $\bar{\Omega}_z$. Let Π_{zi} for $i = 1,2,3,4$ be real roots of the polynomial equation $\sum_{j=0}^{4} a_j \Pi_z^j = 0$, and satisfy constraints $\Pi_{z1} > \Pi_{z2} = \Pi_{z3} > \Pi_{z4}$, then particular orbits of the torque-free ZR liquid-filled solid become heteroclinic orbits in Eqs. (22). Periodic solutions of Eq. (24) can be found in Ref. [12]. From heteroclinic orbits (22), the limiting values

$$\lim_{t\to\pm\infty} P_R(t) = \lim_{t\to\pm\infty} P_I(t) = 0, \tag{25}$$

constitute the necessary conditions for the existence of heteroclinic orbits because integrals (23) will converge under conditions (25). Otherwise, integrals (23) diverge. Thus, the 7 parameters M_f, I_{ck}, J_{ck}, and 6 integral constants E, G_p, Π, $\bar{\Omega}_z$, $\Pi_x(t_a)$ and $\Pi_y(t_a)$ meeting Eqs. (25) will produce heteroclinic orbits. Obtaining at first heteroclinic orbits is important before applying MHM integrals. Complete solutions of heteroclinic orbits under torque-free conditions depend on the six initial conditions or equivalently on the six first integrals E, G_p, Π, $\bar{\Omega}_z$, $\bar{\Pi}_x(t_a)$ and $\bar{\Pi}_y(t_a)$.

4.2. The construction of heteroclinic orbits

One may perform a blanket search for heteroclinic orbits from the original 13 dimensional space spanned by four integral constants E, G_p, Π, $\bar{\Omega}_z$ and the set of nine parameters M_f, I_{ck}, J_{ck}, $\bar{\Pi}_x(t_a)$, and $\bar{\Pi}_y(t_a)$. However, it is time-consuming. When we fix the set of 9 parameters above-mentioned, by using the Newton–Raphson reiteration algorithms we may search for the numerical solution of the five unknowns Π_{z1}, Π_{z2}, Π_{z4}, E, and G_p based on the Vieta's theorem and Eqs. (25). For simulation, the semi-axis lengths of the ellipsoidal cavity are designated throughout this article as $b_x = 0.3\,\mathrm{m}$, $b_y = 0.3\,\mathrm{m}$ and $b_x = 0.1\,\mathrm{m}$. The mass of the contained liquid is assumed as $M_f = 20\,\mathrm{kg}$ and the total mass of the ZR liquid-filled solid $m = 25\,\mathrm{kg}$. The sums of moments of inertia of the rigid shell and of Zhukovsky equivalent body of the ZR liquid-filled solid with respect to $Oxyz$ are $I_{cx} = I_{cy} = 3\,\mathrm{kg\,m^2}$, and $I_{cz} = 7\,\mathrm{kg\,m^2}$. Fig. 2 is depicted to describe energy and momentum curves corresponding to heteroclinic orbits of the symmetrical torque-free ZR liquid-filled solid when given $\Pi = 6$. Fig. 2 demonstrates that many scenarios of rotational motions of the ZR liquid-filled solid are of heteroclinic structures. The acquired solutions are crosschecked by the following alternative algorithms.

When given two of the real roots Π_{z1}, Π_{z4}, and 7 parameters M_f, I_{ck}, and J_{ck}, an effective algorithm can be invented to find four integral constants E, G_p, Π, and $\bar{\Omega}_z$. Since $\lim_{t\to\pm\infty}\bar{\Pi}_z(t) = \bar{\Pi}_{z2}$, we could deduce that Eqs. (25) reduce to a polynomial equation of degree 10 for Π_{z2}

$$\sum_{j=0}^{10} d_j\, \Pi_{z2}^j = 0, \tag{26}$$

where coefficients d_j for $j = 0,1,2,\ldots,10$ are computed by symbolic manipulations and are functions of arguments b_k, I_{ck}, J_{ck}, Π_{z1} and Π_{z4}. It is noted that the Vieta theorem of a polynomial equation was adopted in the derivation of Eq. (26). When essential physical parameters are given, heteroclinic orbits in terms of $\bar{\Omega}_k(t)$ and $\bar{\Pi}_k(t)$ are computed from Eqs. (22). Heteroclinic orbits constructed from Eqs. (22) will be exploited to investigate the effect of small moments on rotational motions of the ZR liquid-filled solid in the following sections.

5. The MHM integral of the periodically forced ZR liquid-filled solid

After obtaining heteroclinic orbits, we are in a position to investigate chaotic oscillations in the damped, periodically perturbed ZR liquid-filled solid using the MHM integral developed originally by Holmes and Marsden [25]. The Melnikov-type integral is effective for determining the onset of transversal homoclinic/heteroclinic orbits in the Poincare map of the perturbed system by measuring the "distance" between the

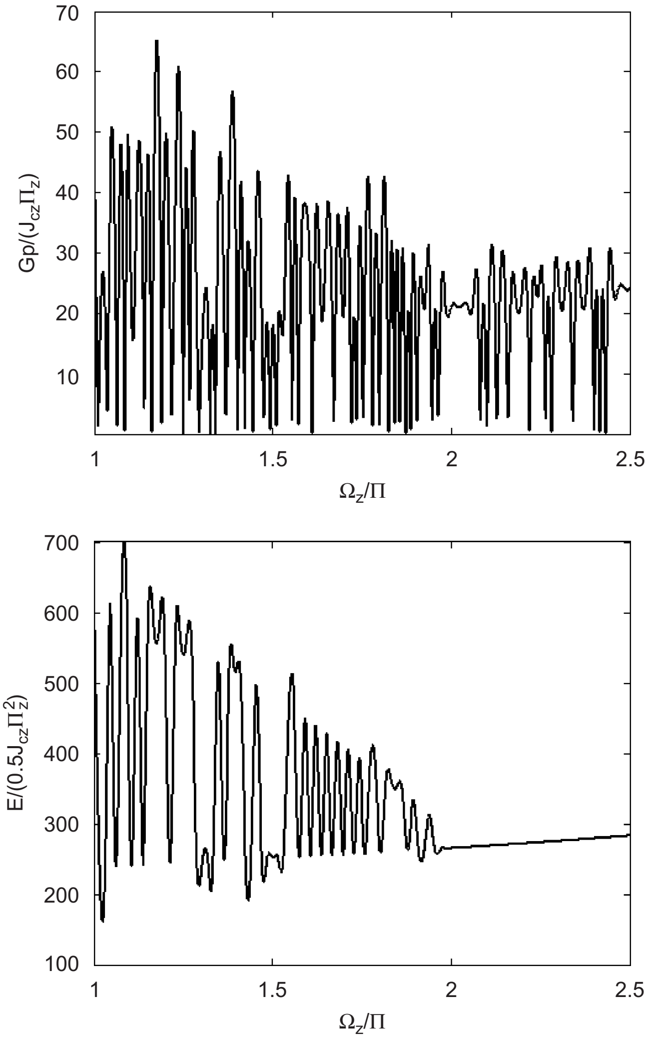

Fig. 2. Energy and momentum curves corresponding to the heteroclinic orbits of the torque-free symmetric ZR liquid-filled solid.

stable and unstable manifolds associated with the saddle point of the system such as the ZR liquid-filled solid. Putting the ZR liquid-filled solid in the external viscous media and neglecting the effect due to the orbital frequency, one can express forcing functions of the disturbed Hamiltonian equations (11) and (15)–(19) after deleting terms with a coefficient $1/N$. Assume that external moments with linear damping mechanisms take the form

$$T_k = \varepsilon M_k = \varepsilon\left[-\sigma_k \bar{\Omega}_k + \Upsilon_k + \Lambda_k \sin(\Omega_{\text{ex}} t)\right], \tag{27}$$

where $\sigma_k > 0$ representing the damping effect of the surrounding media of the ZR liquid-filled solid; $\Lambda_k \triangleq \Lambda_k (\Omega_x, \ldots, \Pi_x, \ldots, \gamma_z)$ and $\Upsilon_k \triangleq \Upsilon_k(\Omega_x, \ldots, \Pi_x, \ldots, \gamma_z)$ are known functions. Ω_{ex} is a constant representing the forcing frequency. Dissipation due to the contained liquid is not included in Eqs. (2) because Helmholtz equations cannot be obtained exactly from Navier–Stokes equations of a viscous liquid and the Hamiltonian structure of the ZR liquid-filled solid under torque-free conditions will be destroyed when considering the dissipation of the liquid. In order to develop a criterion for the judgment of onsets of chaotic rotational dynamics, we need to study the existence of real zeros of the MHM integral in association with of heteroclinic orbits. The MHM integral in the case of the periodically excited ZR liquid-filled solid takes the form

$$M(t_0) = \int_{-\infty}^{+\infty} f(\bar{L}, \bar{G}, \bar{H}, \bar{l}, \bar{g}, \bar{h}, \bar{L}_F, \bar{l}_f, \quad t + t_0)\,\mathrm{d}t, \tag{28}$$

where $\alpha_p = (\partial T/\partial G)(\Gamma_p)$, $L_F \triangleq -L_f$,

$$f = \frac{\partial T}{\partial L} f_L + \frac{\partial T}{\partial l} f_l + \left(\frac{\partial T}{\partial G} - \alpha_p\right) f_G + \frac{\partial T}{\partial g} f_g + \frac{\partial T}{\partial H} f_H + \frac{\partial T}{\partial h} f_h + \frac{\partial T}{\partial L_F} f_{L_F} + \frac{\partial T}{\partial l_f} f_{l_f}.$$

$t_0 \in [0, 2\pi/\Omega_{\mathrm{ex}}]$ is the initial time. The notation Γ_p represents the set of the hyperbolic fixed point in terms of canonical coordinates L_p, G_p, H_p, l_p, g_p, h_p, L_{Fp} and l_{fp} corresponding to canonical variables in Γ, in which the subscript "p" denotes the "hyperbolic" property as expected. Partial derivatives $\partial T/\partial L, \ldots \partial T/\partial L_F$ in the MHM integral (28) of the disturbed Hamiltonian equations (11) are presented in Eqs. (12)–(14), respectively. The integrand f of the MHM integral is a function of amplitudes of disturbances: f_L, f_G, f_H, f_l, f_g and f_h, as given in Eqs. (15)–(19) after deleting terms with a coefficient $1/N$ and the integrand f is given in Deprit's variables $\bar{l}(t)$, $\bar{L}(t)$, $\bar{l}_f(t)$, $\bar{L}_f(t)$, and $\bar{g}(t)$ which can be derived, in principle, from Eqs. (6) for heteroclinic orbits $\bar{\Omega}_k(t)$ and $\bar{\Pi}_k(t)$. Plugging Eqs. (12)–(19) after deleting terms with a coefficient $1/N$ into (28) and grouping terms according to M_k yield the following version of the MHM integral:

$$M(t_0) = \int_{-\infty}^{+\infty} \left\{ \sum_{k=x,y,z} \Delta_k(t) M_k(\bar{\Omega}_x, \ldots, \bar{\Pi}_x, \ldots, \quad \bar{\gamma}_x \ldots, \quad t + t_0) \right\} \mathrm{d}t, \tag{29}$$

where, $G_p = \sqrt{\sum_{k=x,y,z} (I_{ck}\Omega_{kp} + J_{ck}\Pi_{kp})^2}$, $\bar{\gamma}_k(t) = [I_{ck}\bar{\Omega}_k(t) + J_{ck}\bar{\Pi}_k(t)]/G_p$, $\Delta_k(t) = \bar{\Omega}_k(t) - \alpha_p \bar{\gamma}_k(t)$, and

$$\alpha_p = \frac{G_p\left[(I_{cx}\Omega_{xp} + J_{cx}\Pi_{xp})\Omega_{xp} + (I_{cy}\Omega_{yp} + J_{cy}\Pi_{yp})\Omega_{yp}\right]}{(I_{cx}\Omega_{xp} + J_{cx}\Pi_{xp})^2 + (I_{cy}\Omega_{yp} + J_{cy}\Pi_{yp})^2}.$$

The MHM integral (29) for the damped, periodically perturbed ZR liquid-filled solid shows that external moments M_k are functions with arguments $\bar{\Omega}_k$, $\bar{\Pi}_k$, and $\bar{\gamma}_k$. The MHM integral (29) may be degenerated into the criterion for dealing with chaotic dynamics of the damped, periodically disturbed gyrostat [8,9]. Tedious manipulations in deriving the MHM integral (29) are omitted for brevity. It is remarked that much in the same way, one can formulate a similar MHM integral to Eq. (29) based on the disturbed Hamiltonian equations in terms of Euler's variables ψ, θ, ϕ, p_ψ, p_θ, p_ϕ, L_f and l_f (see Ref. [12]). It is amazing that the MHM integral (29) is always convergent here as $\lim_{t\to\pm\infty}\Delta_k(t) = 0$. From the MHM integral (29), one deduces that the integrand is explicitly dependent on the hyperbolic fixed point in terms of α_p. The MHM criterion (29) has symmetric properties in amplitudes of disturbance-moments M_k and heteroclinic orbits $\bar{\Omega}_k(t)$ and $\bar{\Pi}_k(t)$. Substituting Eqs. (27) into the MHM integral (29), we obtain

$$M(t_0) = I_0 + I_c \cos(\Omega_{\mathrm{ex}} t_0) + I_s \sin(\Omega_{\mathrm{ex}} t_0), \tag{30}$$

where

$$I_0 = \int_{-\infty}^{+\infty} \sum_{k=x,y,z} \left[-\sigma_k \bar{\Omega}_k(t) + \bar{\Upsilon}_k\right] \Delta_k(t)\,\mathrm{d}t,$$

$$I_c = \int_{-\infty}^{+\infty} \sum_{k=x,y,z} \left[\bar{\Lambda}_k \Delta_k(t) \sin(\Omega_{\mathrm{ex}} t)\right] \mathrm{d}t, \quad I_s = \int_{-\infty}^{+\infty} \sum_{k=x,y,z} \left[\bar{\Lambda}_k \Delta_k(t) \cos(\Omega_{\mathrm{ex}} t)\right] \mathrm{d}t,$$

$$\bar{\Lambda}_k \triangleq \Lambda_k(\bar{\Omega}_x, \ldots, \bar{\Pi}_x, \ldots, \bar{\gamma}_x), \quad \bar{\Upsilon}_k \triangleq \Upsilon_k(\bar{\Omega}_x, \ldots, \bar{\Pi}_x, \ldots, \bar{\gamma}_x).$$

From Eq. (30), if

$$\delta = \delta(\Omega_{\mathrm{ex}}) = \left| I_0 \Big/ \sqrt{I_c^2 + I_s^2} \right| \leqslant 1, \tag{31}$$

then, the MHM integral (30) will have real zeros with respect to the argument t_0. The existence of real zeros of the MHM integral (30) means transversal intersections between the stable and unstable manifolds of the

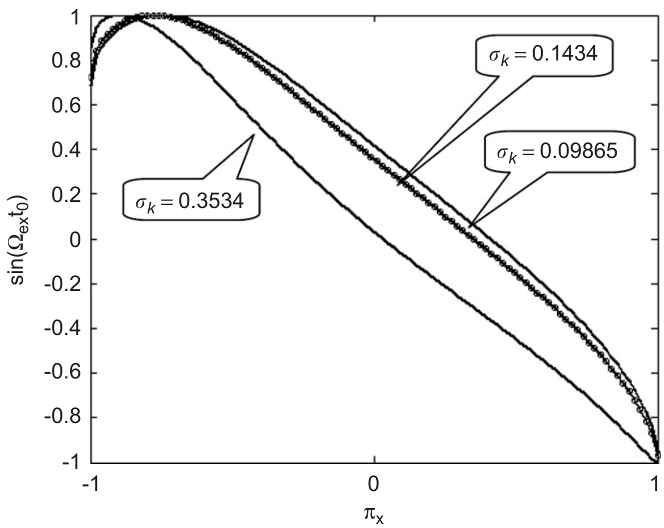

Fig. 3. The zeros of the MHM integral, $\sin(\Omega_{ex}t_0)$ versus π_x for the damped, periodically perturbed ZR liquid-filled solid.

periodically perturbed ZR liquid-filled solid. From physical parameters presented in Section 8.2 we may depict the zeros of the MHM integral, $\sin(\Omega_{\mathrm{ex}}t_0)$, against the scaled initial component of vortex $\pi_x = \Pi_x(t_a)/\sqrt{R_0 + R_2\Pi_{z2}^2}$ for the ZR liquid-filled solid with energy dissipation in Fig. 3 according to Eq. (31). Hence, one should highlight that the MHM integral (29) can be used to predict certain combinations of physical parameters which will trigger occurrences of chaotic rotations of the ZR liquid-filled solid when periodically disturbed appropriately. Applications of the MHM integral (30) will be further studied numerically in subsequent sections.

6. Chaos in the ZR liquid-filled solid in circular orbit

In this section, conditions for occurrences of chaotic motions of the symmetrical ZR liquid-filled solid in circular orbit will be established using the MHM integral developed by Holmes and Marsden [22]. When the ZR liquid-filled solid is orbiting circularly (see Fig. 1), one has $0 < \varepsilon = 3N^2 \ll 1$ since components of the gravity-gradient torque become $L_x = 3N^2(I_{wz} - I_{wy})\gamma_z\gamma_y$, $(x\,y\,z)$. It appears that the order of L_k is $O(N^2)$, but the order of the perturbation torque due to the following angular velocity (i.e., the orbital frequency N) is $O(N)$. We should consider the effect arising from the latter. The set of parameters I_{ck}, J_{ck}, E, G_p, Π, and $\bar{\Omega}_z$ aforementioned, which can possibly trigger onsets of chaotic motions of the ZR liquid-filled solid due to the effect of the orbital frequency N with appropriate initial conditions, will be determined using the MHM integral (see Ref. [22]) briefly below. The kinetic energy plus potential of the orbiting ZR liquid-filled solid takes the form

$$H_a = T_G + N\,H_1 + O(N^2), \tag{32}$$

where T_G is the kinetic energy of the orbiting ZR liquid-filled solid modified from Eq. (4) and H_1 denotes the contribution of circular motions to the potential energy of the orbiting ZR liquid-filled solid. T_G and H_1 are defined as

$$T_G = 0.5\sum_{k=x,y,z}(I_{wk}\Omega_{rk}^2 + I_{fk}\theta_k^2 + 2I_{gk}\Omega_{rk}\theta_k), \quad H_1 = -\sum_{k=x,y,z}\left[(I_{wk}\Omega_{rk} + I_{gk}\theta_k)d_{k2}\right]. \tag{33}$$

Using transformations similar to Eqs. (6), we can formulate the disturbed Hamiltonian equations of the ZR liquid-filled solid in circular orbit with the approximate Hamiltonian $T = T_G$. Lengthy derivations are here omitted for brevity. When $N\equiv 0$, the rotational motion reduces to a torque-free motion as seen from Eqs. (19)

and (20). Both the momentum $G = G_p$ and the new momentum H are constants, and so is the canonical variable h. Rotational motions of the orbiting ZR liquid-filled solid under torque-free conditions are expressed by a Hamiltonian system of the form

$$\frac{d\bar{L}}{dt} = -\frac{\partial T_G}{\partial \bar{l}}, \quad \frac{d\bar{l}}{dt} = \frac{\partial T_G}{\partial \bar{L}}, \quad \frac{d\bar{L}_f}{dt} = \frac{\partial T_G}{\partial \bar{l}_f}, \quad \frac{d\bar{l}_f}{dt} = -\frac{\partial T_G}{\partial \bar{L}_f}, \tag{34}$$

where the over-bar denotes heteroclinic orbits of the torque-free symmetrical ZR liquid-filled solid. It is highlighted that along heteroclinic orbits (22) there are two first integrals H and h which are derivable from Eqs. (11). Hamiltonian equations (34) in terms of generalized Deprit's variables are equivalent to the torque-free Euler–Helmholtz equations (19) and (20). Heteroclinic orbits in terms of $\bar{L}$, $\bar{l}$, $\bar{L}_f$, and $\bar{l}_f$ are related to those in terms of $\bar{\Omega}_{rk}(t) = \bar{\Omega}_k(t)$ and $\bar{\Pi}_k(t)$ in Eqs. (22). Chaotic dynamics of the disturbed Hamiltonian system in the case of the ZR liquid-filled solid in circular orbit may be determined by the MHM integral developed by Holmes and Marsden [22]. From heteroclinic orbits (22), the torque-free Euler–Helmholtz equations (19) and (20) have equilibrium points of the form

$$\lim_{x\to\pm\infty} \bar{\Omega}_k(t) = \Omega_{kp}, \quad \lim_{x\to\pm\infty} \bar{\Pi}_k(t) = \Pi_{kp}, \tag{35}$$

where Ω_{kp} and Π_{kp} can be computed from the limiting processes applying to heteroclinic orbits (22). It is noted that Π_{zp} should be equal to the selected root Π_{z2} of the polynomial equation of degree 10 in Eq. (26). The resolution of Π_{zp} is quite onerous because one must at first compute coefficients d_j for $j = 0,1,\ldots,10$ of Eq. (26). In order to determine the hyperbolicity of fixed points in Eqs. (35), one should compute linearized eigenvalues about fixed points of Jacobian matrix of the vector field (34). If eigenvalues have nonzero real parts then fixed points are hyperbolic. The computation of the above-mentioned Jacobian matrix of the vector field in terms of canonical variables $\bar{l}$, $\bar{L}$, $\bar{l}_f$, and $\bar{L}_F$ where $\bar{L}_F = -\bar{L}_f$ is omitted here for brevity. Following Holmes and Marsden [22], one can determine if the stable and unstable manifolds of periodic orbits intersect transversally by calculating the MHM integral

$$M(g_0) = \int_{-\infty}^{+\infty} \{T_G, H_1/\Omega_g\}[\bar{l}(t), \bar{L}(t), \bar{l}_f(t), \bar{L}_F(t), \bar{g}(t) + g_0]\,dt, \tag{36}$$

where $g_0 \in [0, 2\pi)$ is the initial value of the canonical variable g, and { } is the Poisson bracket in canonical variables l, L, l_f and L_f, i.e.,

$$\left\{T_G, \frac{H_1}{\Omega_g}\right\} = \frac{1}{\Omega_g}\{T_G, H_1\} - \{T_G, \Omega_g\}\frac{H_1}{\Omega_g^2}, \quad \Omega_g = \left.\frac{dg}{dt}\right|_{N=0} = \frac{\partial T_G}{\partial G},$$

$$\{T_G, H_1\} = \left(\frac{\partial T_G}{\partial l}\frac{\partial H_1}{\partial L} - \frac{\partial T_G}{\partial L}\frac{\partial H_1}{\partial l}\right) + \left(\frac{\partial T_G}{\partial l_f}\frac{\partial H_1}{\partial L_F} - \frac{\partial T_G}{\partial L_F}\frac{\partial H_1}{\partial l_f}\right),$$

$$\{T_G, \Omega_g\} = \left(\frac{\partial T_G}{\partial l}\frac{\partial \Omega_g}{\partial L} - \frac{\partial T_G}{\partial L}\frac{\partial \Omega_g}{\partial l}\right) + \left(\frac{\partial T_G}{\partial l_f}\frac{\partial \Omega_g}{\partial L_F} - \frac{\partial T_G}{\partial L_F}\frac{\partial \Omega_g}{\partial l_f}\right) \equiv -\frac{d\Omega_g}{dt}, \tag{37}$$

where the second of Eqs. (37) is derived from the fifth of Eqs. (11). Partial derivatives $\partial T_G/\partial L,\ldots,\partial\Omega_g/\partial l_f$ in Eqs. (37) are associated with heteroclinic orbits $\bar{\Omega}_k(t)$ and $\bar{\Pi}_k(t)$ of the torque-free ZR liquid-filled solid in Eq. (22) with six integral constants E, G_p, Π, $\bar{\Omega}_z$, H and h. Similarly, heteroclinic orbits in terms of $\bar{l}(t)$, $\bar{L}(t)$, $\bar{l}_f(t)$, $\bar{L}_f(t)$, and $\bar{g}(t)$ correspond to heteroclinic orbits in terms of $\bar{\Omega}_k(t)$ and $\bar{\Pi}_k(t)$ through transformations (6). Using the latter, one can simplify the computation of the MHM integrals. An explicit evaluation of the MHM integral is quite restricted by the algebraic complexity of heteroclinic orbits. In order to evaluate the MHM integral (36), one may set

$$g(t) = \int_0^t \Omega_g(t)\,dt + g_0 \triangleq \bar{g}(t) + g_0, \quad \Omega_g(t) = \frac{(L_2\bar{\Pi}_z^2 + L_1\bar{\Pi}_z + L_0)G_p}{N_2\bar{\Pi}_z^2 + N_1\bar{\Pi}_z + N_0}, \tag{38}$$

where $N_2 = r_2 I_{cx}^2 + R_2 J_{cx}^2 + 2I_{cx}J_{cx}S_2$, $N_0 = r_0 I_{cx}^2 + R_0 J_{cx}^2 + 2I_{cx}J_{cx}S_0$, $L_2 = r_2 I_{cx} + S_2 J_{cx}$, $L_1 = S_1 J_{cx}$, $L_0 = r_0 I_{cx} + S_0 J_{cx}$, and $N_1 = 2I_{cx}J_{cx}S_1$. Then, one can transform the MHM integral (36) into the

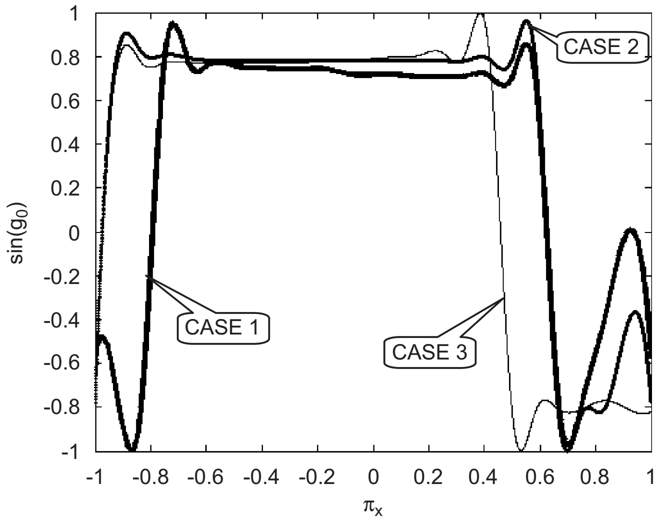

Fig. 4. The real roots of the MHM integral, $\sin(g_0)$versus π_x for the orbiting ZR liquid-filled solid with energy conservation.

following form,

$$M(g_0) = \int_{-\infty}^{+\infty} \left\{ \sum_{j=0}^{12} Q_j \tan^j(0.5g_0) \bigg/ \left[\sum_{j=0}^{6} H_{d\,j} \tan^j(0.5g_0) \right]^2 \right\} \mathrm{d}t, \tag{39}$$

where Q_j for $j = 0,1,\ldots,12$ and H_{dj} for $j = 0,1,\ldots,6$ are functions with arguments $\bar{g}(t)$, $\bar{\Omega}_k(t)$ and $\bar{\Pi}_k(t)$ and their lengthy expressions are omitted for brevity. It can be proved that along heteroclinic orbits of the torque-free symmetrical ZR liquid-filled solid, following limiting processes are always true

$$\left.\frac{\partial T_G}{\partial l}\right|_{t\to\pm\infty} = \left.\frac{\partial T_G}{\partial L}\right|_{t\to\pm\infty} = \left.\frac{\partial T_G}{\partial l_f}\right|_{t\to\pm\infty} = \left.\frac{\partial T_G}{\partial L_f}\right|_{t\to\pm\infty} = -\left.\frac{\partial T_G}{\partial L_F}\right|_{t\to\pm\infty} = 0. \tag{40}$$

Thus, the MHM integral (39) derivable from Eq. (36) associated with Eqs. (37) converges rapidly when $\Omega_g(t)\neq 0$ and $1/\Omega_g(t)\neq 0$ for $t\in(-\infty,+\infty)$ where $\Omega_g(t)$ is given in Eqs. (38). The existence of a real angle $g_0\in[0,2,\pi)$ making the MHM integral (39) have simple zero, i. e., $M(g_0) = 0$, implies transversal intersections between the stable and unstable manifolds in rotational motions of the ZR liquid-filled solid disturbed due to the effect of the orbital frequency N. Coefficients Q_i for $i = 0$, 1, 2, 3, 4 in Eq. (39) are functions of arguments E, G_p, Π, $\bar{\Omega}_z$, H, h, M_f, I_{ck}, and J_{ck}. The real zero g_0 of the MHM integral (39) contains much information for determining physical parameters which will trigger the possible onset of chaotic rotational motions of the ZR liquid-filled solid disturbed by small moments due to the orbital frequency. The MHM integral (39) is associated with heteroclinic orbits (22) of the torque-free symmetrical ZR liquid-filled solid established in Section 4. It is remarked that when there is no essential singularity in the integrand of the MHM integral, the MHM technique can effectively be used to measure the "distance" between the stable and unstable manifolds of the Hamiltonian system subjected to small perturbations. Otherwise, the MHM integral will fail. From the analysis in Section 8.1, Fig. 4 is drawn from the MHM integral (39) using the bisection method to compute real zeros of the MHM integral, $\sin(g_0)$, against the scaled initial component of vortex $\pi_x = \Pi_x(t_a)/\sqrt{R_0 + R_2\Pi_{z2}^2}$. Note that the perturbation H_1 for the disturbed system under investigation contains the argument h because of relations in Eqs. (9) and (10). The argument h may be designated for the determination of initial conditions for the time evolution of state variables d_{x2}, d_{y2} and d_{z2}.

Chaotic oscillations in the perturbed ZR liquid-filled solid sliding with rolling on the horizontal plane will be presented in next section.

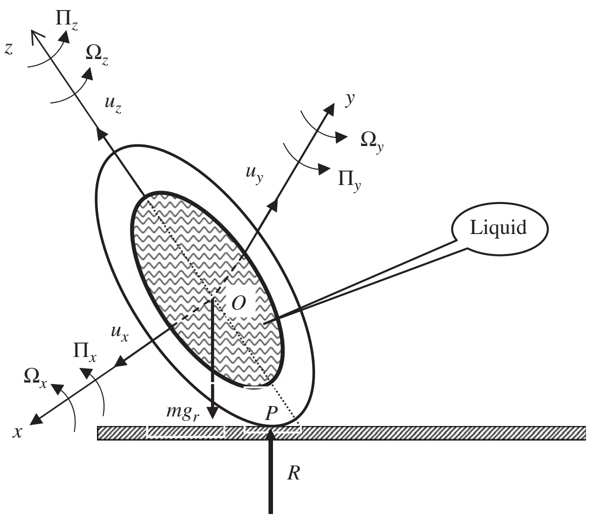

Fig. 5. Configuration of the sliding liquid-filled ellipsoid rolling on the perfectly smooth horizontal plane.

7. Chaos in a liquid-filled ellipsoid sliding and rolling on a perfectly smooth plane

Kuang et al. [13] investigated chaotic motions of the ZR liquid-filled solid rolling without sliding on the perfectly rough plane. Now we consider an ellipsoid with an axisymmetrical ellipsoidal cavity sliding and rolling on a perfectly smooth horizontal plane. The system is shown in Fig. 5. The cavity is completely filled with an ideal, incompressible fluid in uniform vortex motion that is fully determined by three variables as discussed in Section 2. The entire system has a total of six rotational and three translational degrees of freedom and is called the liquid-filled ellipsoid hereafter. Chaotic dynamics of the liquid-filled ellipsoid either dissipative or conservative will be discussed as below.

7.1. A liquid-filled ellipsoid with energy dissipation plus periodic perturbations

Let the principal axes at the origin O of the mass center of the entire system be the reference frame $Oxyz$; the mass of the entire system be m and the acceleration of gravity be g_r; and the inertial reference frame be $OXYZ$. Other notations were given in Section 2: $M_g = mg_r$ denotes the weight of the system due to gravity; ξ_k the coordinates of the point of contact P along $Oxyz$; R_k the resolved parts along $Oxyz$ of the normal reaction and two friction components at the point of contact P; u_k the velocity components of the center of gravity along $Oxyz$; and γ_k the direction-cosines of the outward direction of the normal to the surface at the point of contact P with respect to $Oxyz$. The motion of the sliding liquid-filled ellipsoid rolling on the perfectly smooth horizontal plane is governed fully by Eulerian equations (1) in x, y and z axes, together with

$$L_x = \xi_y R_z - \xi_z R_y, \quad (x, y, z). \tag{41}$$

The equations of motion of the center of gravity (see Ref. [41]) are

$$m\frac{\mathrm{d}u_x}{\mathrm{d}t} = mg_r\gamma_x + R_x + m(\Omega_z u_y - \Omega_y u_z), \quad (x, y, z). \tag{42}$$

Helmholtz equations of uniform vortex motion for an ideal, incompressible liquid contained in the ellipsoidal cavity, in axes x, y and z, are described by Eqs. (2). Since the line spanned by γ_k remains always vertical, Poisson's equations remain as Eqs. (3). Let V_{px}, V_{py}, and V_{pz} be the resolved parts of the velocity at the point of contact P in the positive directions of $Oxyz$. The component velocities may be determined from the body-fixed angular rates as

$$V_{px} = u_x - (\xi_y \Omega_z - \xi_z \Omega_y), \quad (x, y, z). \tag{43}$$

Since the resultant of R_x, R_y, and R_z is a reaction R normal to the fixed horizontal plane, we have

$$R_x = -R\gamma_x, \quad R_y = -R\gamma_y, \quad R_z = -R\gamma_z, \tag{44}$$

where the negative sign is due to the fact that γ_x, γ_y and γ_z are the direction-cosines of the outward direction of the normal. It is required that the velocity of the point of contact resolved normal to the perfectly smooth horizontal plane should be zero:

$$\gamma_x V_x + \gamma_y V_y + \gamma_z V_z = 0. \tag{45}$$

Eq. (45) is understood as the constraint condition for the liquid-filled ellipsoid rolling and slipping on a perfectly smooth horizontal plane. Let the equation of the bounding surface be $F(\xi_x, \xi_y, \xi_z) \triangleq (\xi_x + \mu_x)^2/a_x^2 + (\xi_y + \mu_y)^2 a_y^2 + (\xi_z + \mu_z)^2/a_z^2 = 1$, where a_k are semi-diameters of the ellipsoid and μ_k are the offsets of the center of the gravity O from the geometric center G_e. The point of contact is where the normal to the liquid-filled ellipsoid is parallel to the unit vector along the vertical axis. Taking advantage of the fact that the normal to the surface of the liquid-filled ellipsoid is parallel to grad(F), the gradient of $F(\xi_x,\xi_y,\xi_z)$, one can write (see Ref. [42]),

$$\xi_x = \gamma_x a_x^2 \Big/ \sqrt{(\gamma_x a_x)^2 + (\gamma_y a_y)^2 + (\gamma_z a_z)^2} - \mu_x, \quad (x, y, z), \tag{46}$$

which are the desired relationships governing the coordinates of the point of contact P between the liquid-filled ellipsoid and the horizontal plane. Assume that the offset of the center of the gravity O from G_e is a small amplitude ε and the shape of the liquid-filled ellipsoid is slightly deviated from the sphere measured by a small amplitude ε. These assumptions may be expressed as

$$a_k = a + \varepsilon A_k, \quad \mu_k = \varepsilon B_k, \tag{47}$$

where A_k, B_k for $k = x,y,z$ are known constants. Expanding the coordinates of the contact point in terms of ε and substituting Eqs. (47) into Eqs. (46), one obtains when $0 < \varepsilon \ll 1$,

$$\xi_k = a\gamma_k + \varepsilon f_{ek} + O(\varepsilon^2), \tag{48}$$

where $f_{ek} = 2\gamma_k A_k - (A_x\gamma_x^2 + A_y\gamma_y^2 + A_z\gamma_z^2)\gamma_k - B_k$. Eqs. (48) are used to replace Eqs. (46) for the desired approximation of Eulerian equations in terms of ω_k and γ_k. Further assume that I_{ck} are then perturbed to $I_{ck} + \varepsilon I_{ck1}$, where I_{ck1} are constants because of Eqs. (47), when ignoring the induced inertial products. Eliminating the unknown variables R_x, R_y, and R_z, in Eqs. (41) associated with Eqs. (44), and in Eqs. (42) associated with Eqs. (3), (45)–(47), one obtains the approximate Eulerian equations of the liquid-filled ellipsoid to the first order of ε as follows:

$$I_{cx}\frac{d\Omega_x}{dt} + J_{cx}\frac{d\Pi_x}{dt} + (I_{cz}\Omega_z + J_{cz}\Pi_z)\Omega_y - (I_{cy}\Omega_y + J_{cy}\Pi_y)\Omega_z = \varepsilon M_x + O(\varepsilon^2), \quad (x, y, z), \tag{49}$$

where after ignoring the terms with $d\Pi_k/dt$ in the functions M_k, respectively, we have

$$\begin{aligned} M_x = {} & (J_{cz}\Pi_z\Omega_y - J_{cy}\Pi_y\Omega_z)I_{cx1}/I_{cx} + [-I_{cx1}(I_{cy} - I_{cz}) + I_{cx}(I_{cy1} - I_{cz1})]\Omega_y\Omega_z/I_{cx} \\ & + mg_r[2\gamma_z\gamma_y(A_z - A_y) + B_y\gamma_z - B_z\gamma_y], \quad (x, y, z). \end{aligned} \tag{50}$$

One observes that functions M_k for $k = x,y,z$ are linear in terms of A_k and B_k. As aforementioned, there are five first integrals when $\varepsilon = 0$: firstly, the energy integral; secondly, the generalized Jellett's integral in the case of the symmetry $I_{cx} = I_{cy}$, $I_{cx} = J_{cy}$ and $a_k = a$; thirdly, the integral of constant vorticity; and fourthly, the integral $\Omega_z = \bar{\Omega}_z =$ constant due to the assumption of symmetry. One may deduce that the generalized Jellett's integral holds for the symmetrical liquid-filled ball, no matter whether there is a sliding friction between the spherical base and the horizontal plane. The generalized Jellett's integral may be found in Ref. [37]. When $e_z = 0$, the generalized Jellett's integral states that the scalar product of the angular momentum vector of the liquid-filled ellipsoid for the mass center O and the radius vector of the contact point is a constant. The first integrals of the liquid-filled ellipsoid rolling without sliding when $\varepsilon = 0$ play a key role in the formulation of periodic solutions and homoclinic/heteroclinic orbits of the undisturbed system. It can be inferred that equations of motions of the liquid-filled ball rolling on the horizontal plane possess the same structures as

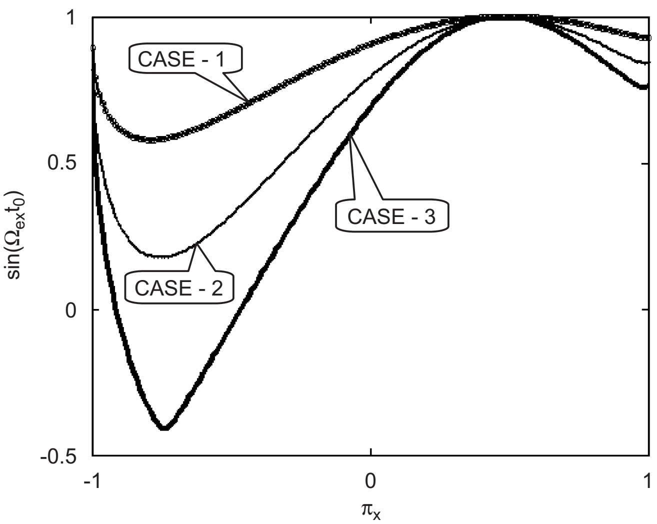

Fig. 6. The zeros of MHM integral, $\sin(\Omega_{ex}t_0)$ versus π_x for the periodically perturbed liquid-filled ellipsoid sliding and rolling with energy dissipation.

those of the ZR liquid-filled solid rotating about a fixed point, such as Eqs. (1)–(3). Therefore, investigations performed in Sections 2–5 are valid for the case of the liquid-filled ellipsoid rolling and sliding. It is remarked that equations of rotational and translational motions of the slipping liquid-filled ball are uncoupled under the first order approximation in ε. Therefore, the derived dynamical equations consisting of the first order perturbed Eulerian equations (49), Helmholtz equations (2) and Poisson's equations (3), can be transformed into the disturbed Hamiltonian equations (11) together with Eqs. (12)–(18). Strategies to establish relationships among its physical parameters by utilizing the MHM integrals to investigate chaotic motions have been highlighted Section 5 in the case of energy dissipation. Specifically, conjecture that the ZR liquid-filled solid is at all time subjected to the action of a dissipative couple $T_{fk} = -\varepsilon\sigma_k\Omega_k$, where σ_k are positive constants representing air resistance (see Ref. [42]). Besides, we consider that the acceleration of gravity is varying periodically in time such as, $g_r = g_e + C_e \sin(\Omega_{\rm ex}t)$, where g_e represents the average level of the acceleration of gravity; C_e is the amplitude of periodical part of g_r; and Ω_{ex} is the external forcing frequency. The periodically varying acceleration of gravity may arise due to a vertical motion/shaking of the horizontal plane. Heteroclinic orbits of the undisturbed liquid-filled ellipsoid rolling and sliding on the perfectly smooth horizontal plane are identical to Eqs. (22). Thus, the MHM integral for the disturbed liquid-filled ellipsoid with energy dissipation plus periodic excitations is the same in form as Eq. (30) in which

$$\begin{aligned}\bar{\Lambda}_x &= mC_e[2\bar{\gamma}_z\bar{\gamma}_y(A_z - A_y) + B_y\bar{\gamma}_z - B_z\bar{\gamma}_y],\\ \bar{\Upsilon}_x &= (J_{cz}\bar{\Pi}_z\bar{\Omega}_y - J_{cy}\bar{\Pi}_y\bar{\Omega}_z)I_{cx1}/I_{cx}\\ &\quad + [-I_{cx1}(I_{cy} - I_{cz}) + I_{cx}(I_{cy1} - I_{cz1})]\bar{\Omega}_y\bar{\Omega}_z/I_{cx} + \bar{\Lambda}_x g_e/C_e, \quad (x, y, z).\end{aligned} \tag{51}$$

The corresponding zeros of the MHM integral characterizing the possible transversal intersections between the stable and unstable manifolds of the disturbed liquid-filled ellipsoid with energy dissipation plus periodic excitations are determined by Eq. (31) associated with Eqs. (51). Given appropriate physical parameters in Section 8.2.2, we may depict the real zeros of the MHM integral against the scaled initial component of vortex $\pi_x = \Pi_x(t_a)/\sqrt{R_0 + R_2\Pi_{z2}^2}$ of the disturbed liquid-filled ellipsoid as shown in Fig. 6.

7.2. A liquid-filled ellipsoid with energy conservation

We shall investigate the possible onset of chaotic dynamics of the perturbed liquid-filled ellipsoid with energy conservation. Multiplying Eqs. (2) by Π_x, Π_y, Π_z respectively, Eqs. (3) by $mg_r\xi_x$, $mg_r\xi_y$, $mg_r\xi_z$

respectively, Eqs. (49) by Ω_x, Ω_y, Ω_z respectively, and summing the results, one obtains the approximated Hamiltonian for the perturbed slipping liquid-filled ellipsoid in rotation as $H_a = 0.5\sum_{k=x,y,z}\left[I_{ck}\Omega_k^2 + J_{ck}\Pi_k^2 + mg_r\gamma_k\xi_k\right]$. We again assume that I_{ck} are perturbed to $I_{ck} + \varepsilon I_{ck1}$, for constants I_{ck1} when ignoring the induced products of moments of inertia. By invoking Eqs. (48), the Hamiltonian corresponding to Eqs. (2), (3) and (49) may be approximately transformed into the form,

$$H_a = T_G + \varepsilon H_1 + O(\varepsilon^2),$$

$$T_G = \frac{1}{2}\sum_{k=x,y,z}(I_{wk}\Omega_k^2 + I_{fk}\theta_k^2 + 2I_{gk}\Omega_k\theta_k) + mg_r a,$$

$$H_1 = \sum_{k=x,y,z}\left(\frac{1}{2}I_{ck1}\Omega_k^2 + mg_r(A_k\gamma_k^2 - B_k\gamma_k)\right), \tag{52}$$

where Ω_k, θ_k and γ_k are functions of canonical variables in Γ, defined in Eqs. (12) and (8), respectively. The principal term T_G corresponds to the torque-free rotational motion of the slipping liquid-filled ellipsoid. The perturbation amplitude term H_1 reflects the contribution of gravitational forces mg_r and the slight shape changes of the disturbed slipping liquid-filled ellipsoid rolling on the perfectly smooth horizontal plane. The equations of rotational motions uncoupled from the ones of translational motions of the slipping liquid-filled ellipsoid may be transformed into the disturbed Hamiltonian equations using appropriate canonical variables in Γ as discussed in Section 3. The Hamiltonian T_G describes a Hamiltonian system in Eqs. (34) whose heteroclinic orbits are the torque-free rotation of the ZR liquid-filled solid rotating about the center of mass "fixed" in space as written in Eqs. (22). The perturbed Hamiltonian is independent of the variable h as described in Eqs. (52). Therefore the conjugate momentum H is a constant of motion when ignoring the influence of terms higher than $O(\varepsilon^2)$. The approximate Hamiltonian (52) can be used to establish the MHM integral to probe the potential occurrences of chaotic motion. For the torque-free rotation of the slipping liquid-filled ellipsoid rolling on the perfectly smooth plane, there are two hyperbolic points which are connected by heteroclinic orbits (22). Under appropriate perturbations in the translation and in the shape of the bounding volume, the highly degenerate structure is expected to break and perhaps to yield transversality of heteroclinic orbits. The existence of transverse homoclinic/heteroclinic orbits implies the existence of "horseshoes" and occurrences of chaos. Following Holmes and Marsden [22], we want to prove that the MHM integral (36) together with Eqs. (37), (38) and (40) has simple zeros implying Smale's horseshoes. In order to find the perturbation H_1 and its derivatives with respect to l, L, l_f and L_f in terms of g_0, we substitute Eqs. (8) and Eqs. (12) into Eqs. (52) to get

$$H_1 = \sum_{j=0}^{4}\frac{K_j\tan^j(0.5g_0)}{\left[1+\tan^2(0.5g_0)\right]^2}, \tag{53}$$

where $K_j = K_j(\bar{\Omega}_x, \ldots, \bar{\Pi}_z, \bar{g})$ are functions of $\bar{g}(t)$, $\bar{\Omega}_k(t)$ and $\bar{\Pi}_k(t)$and omitted here for brevity due to their complexity. From Eq. (53) one obtains

$$\frac{\partial H_1}{\partial l} = \sum_{j=0}^{4}\frac{\partial K_j}{\partial l}\frac{\tan^j(0.5g_0)}{\left[1+\tan^2(0.5g_0)\right]^2}, \quad \frac{\partial H_1}{\partial L} = \sum_{j=0}^{4}\frac{\partial K_j}{\partial L}\frac{\tan^j(0.5g_0)}{\left[1+\tan^2(0.5g_0)\right]^2}. \tag{54}$$

The derivatives $\partial K_j/\partial l$ and $\partial K_j/\partial L$ for $j = 0,1,2,3,4$ are also complicated functions of $\bar{g}(t)$, $\bar{\Omega}_k(t)$ and $\bar{\Pi}_k(t)$ and omitted here for brevity. Substituting Eqs. (53) and (54) into Eq. (36), the MHM integral takes the final form,

$$M(g_0) = \sum_{j=0}^{4}P_j\tan^j(0.5g_0)/[1+\tan^2(0.5g_0)]^2, \tag{55}$$

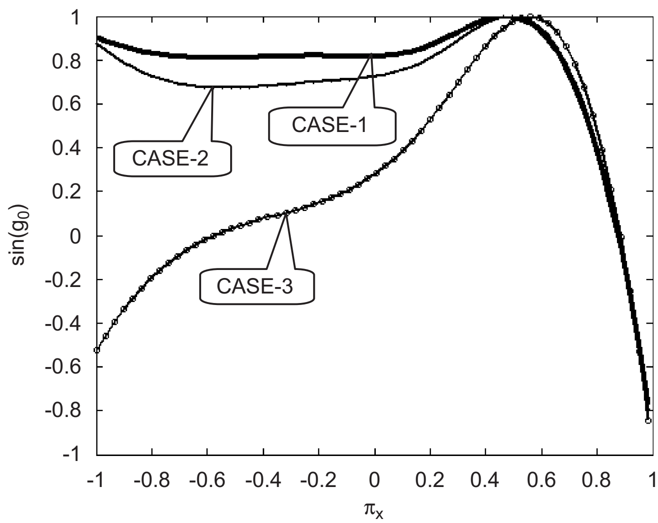

Fig. 7. The real zeros of the MHM integral, $\sin(g_0)$ versus π_x for the perturbed liquid-filled ellipsoid sliding with energy conservation.

where

$$P_j = \int_{-\infty}^{+\infty} \left\{ T_G, \frac{K_j}{\Omega} \right\} (\bar{l}(t), \bar{L}(t), \bar{g}(t)) \mathrm{d}t$$

for $j = 0,1,2,3,4$. Remarkably, the limiting properties given in Eqs. (40) ensure that coefficients P_j in Eq. (55) converge quickly towards normal limits. The real zeros of the MHM integral (55) will provide necessary conditions for possible chaotic motion. Given I_{ck}, I_{ck1}, J_{ck}, A_k, B_k, a and constants G_p, E, Π, $\bar{\Omega}_z$, and H, the real roots of the 4th order polynomial $M(g_0) = 0$ with respect to $\sin(g_0)$ or $\tan(g_0)$ can be obtained. G_p, E, Π, $\bar{\Omega}_z$, and H are evaluated from the first integrals of the undisturbed liquid-filled ellipsoid slipping and rolling on the perfectly smooth plane. According to the parameters given in Section 8.2.1, the real zeros of the MHM integral (55) in terms of $\sin(g_0)$, against the scaled initial component of vortex $\pi_x = \Pi_x(t_a)/\sqrt{R_0 + R_2\Pi_{z2}^2}$ are computed and depicted in Fig. 7 where the first integral $H = \sum_{k=x,y,z}[(I_{ck}\Omega_{kp} + J_{ck}\Pi_k)\gamma_{kp}]$. It is remarked that the MHM integral (55) is a local criterion for the existence of transversal intersections between the stable and unstable manifolds and hence for the possible chaotic motion of the slipping liquid-filled ellipsoid rolling on the perfectly smooth plane when energy is conserved.

8. Numerical simulations

The above theoretical analyses using MHM integrals demonstrated that rotational motions of the ZR liquid-filled solids, either conservative or dissipative, are chaotically sensitive to initial conditions. Numerical examples are presented below.

8.1. Chaotic rotations of the ZR liquid-filled solid in circular orbit

8.1.1. The conservative case

When numerically integrating the MMH integral (36), one will encounter potential singularities in the denominator and numerator of the function $\Omega_g(t)$ in Eqs. (38). The vanishing of the denominator of $\Omega_g(t)$ leads to $I_{wk}\bar{\Omega}_k + I_{gk}\bar{\theta}_k = 0$ for $k = x,y$ whose physical meaning is that the vector of the instantaneous angular momentum coincides with the body-fixed z-axis at certain time $t_f \in (-\infty, +\infty)$. Fortunately, no such singularities are detected for the selected examples.

Heteroclinic orbits of the symmetrical ZR liquid-filled solid have been established in Section 4.2. Consider the following three Cases when the torque-free symmetrical ZR liquid-filled solid has heteroclinic orbits,

Case 1:

$$E = 52.0992, \quad G_p = 7.9427, \quad \Pi = 0.7842, \quad \bar{\Omega}_z = 0.4254,$$
$$\Pi_{z1} = 5.0, \quad \Pi_{z2} = -0.5963, \quad \Pi_{z3} = -0.5963, \quad \Pi_{z4} = -7.0;$$

Case 2:

$$E = 63.1278, \quad G_p = 9.0726, \quad \Pi = 0.8595, \quad \bar{\Omega}_z = 0.6381,$$
$$\Pi_{z1} = 5.0, \quad \Pi_{z2} = -0.8945, \quad \Pi_{z3} = -0.8945, \quad \Pi_{z4} = -8.0;$$

Case 3:

$$E = 90.044, \quad G_p = 11.602, \quad \Pi = 1.0173, \quad \bar{\Omega}_z = 1.0634,$$
$$\Pi_{z1} = 5, \quad \Pi_{z2} = -1.4908, \quad \Pi_{z3} = -1.4908, \quad \Pi_{z4} = -10. \tag{56}$$

Take the Case 3 as an example to explain algorithms established in Section 4. Given the two real roots of the polynomial equation

$$a_4\Pi_z^4 + a_3\Pi_z^3 + a_2\Pi_z^2 + a_1\Pi_z + a_0 = 0, \tag{57}$$

to be $\Pi_{z1} = 5$ and $\Pi_{z4} = -10$, one obtains all 10 roots of the 10th degree polynomial Eq. (26) to be 0(double), 2.5(double), $2.5 \pm i13.601$ (double), 10.3122, and -1.4908, where $i^2 = 1$. Choose -4.4908 as the double root $\Pi_{z2} = \Pi_{z3} = -1.4908$ of Eq. (57), satisfying constraints $\Pi_{z1} > \Pi_{z2} = \Pi_{z3} > \Pi_{z4}$ in generating heteroclinic orbits of the torque-free symmetrical ZR liquid-filled solid. Then, from algorithms of Kuang et al. [12], one gets $\bar{\Omega}_z = 1.0634\,\mathrm{rad/s}$, $G_p = 11.6017\,\mathrm{N\,m\,s}$, $\Pi = 1.0173\,\mathrm{rad/s}$, and $E = 90.0435\,\mathrm{N\,m}$. Hence, we can compute heteroclinic orbits of the torque-free ZR liquid-filled solid using Eqs. (22) for the Case 3 when two integral constants $\bar{\Pi}_y(t_a)$ and $\bar{\Pi}_x(t_a)$ are appropriately designated. It can be shown that $\lim_{t\to\pm\infty}\bar{\Omega}_k(t) = \Omega_{kp}$ and $\lim_{t\to\pm\infty}\bar{\Pi}_k(t) = \Pi_{kp}$ as expected. The integral constant $H = \sum_{k=x,y,z}[(I_{ck}\bar{\Omega}_{kp} + J_{ck}\bar{\Pi}_k)\bar{\gamma}_{kp}]$ along heteroclinic orbits is used for the computation of the MHM integral. For the ZR liquid-filled solid with b_x, b_y, b_z, and I_{ck} given above together with other integral constants, the real roots of the MHM integral (39) in terms of $\sin g_0$ against the scaled initial component of vortex $\pi_x = \Pi_x(t_a)/\sqrt{R_0 + R_2\Pi_{z2}^2}$ are depicted in Fig. 4 using the bisection method. As there are real roots in the MHM integral, there exist transversal intersections between the stable and unstable manifolds of rotational motions of the ZR liquid-filled solid disturbed by small moments arisen from the orbital frequency. Hence, rotational motions for the appropriate selection of N together with initial conditions are chaotic in the sense of Smale's horseshoes. Except for heteroclinic orbits, one may acquire periodic orbits of the torque-free ZR liquid-filled solid using the Jacobian elliptic integral (see Ref. [12]). When given six integral constants, we may numerically solve periodic solutions of the torque-free ZR liquid-filled solid with ease but heteroclinic orbits with difficulty. That is why we re-highlighted the algorithms for the solution of heteroclinic orbits in Section 4. Although the computing algorithms in Section 4 are useful, we fail in zoning precisely the parametric regions for periodic solutions from those for heteroclinic orbits. Except for the three cases studied, other discrete sets of four integral constants producing heteroclinic orbits of the torque-free symmetrical ZR liquid-filled solid may be determined similarly in Fig. 2.

In order to study random-like behaviors (see Ref. [43]) exhibited by Eqs. (1)–(3) for the orbiting ZR liquid-filled solid, one can at first select some physical parameters as given below. The distance of the mass center of the ZR liquid-filled solid from the Earth center is $d_e = 7178.137\,\mathrm{km}$. The gravitational attraction constant of the Earth is $\mu = 3.986 \times 10^{14}\,\mathrm{N\,m^2/kg}$. The theoretical results are numerically crosschecked using Poincare sections in Fig. 8 by the 4th Runge–Kutta algorithms. Initial conditions are selected from one of two hyperbolic fixed points computed. Take the Case 3 as an example, we designate

$$\Omega_{xp} = 0.3838, \quad \Omega_{yp} = -1.5724, \quad \Omega_{zp} = 1.0634,$$
$$\Pi_{xp} = 7.9704, \quad \Pi_{yp} = -32.6559, \quad \Pi_{zp} = -1.4908, \tag{58}$$

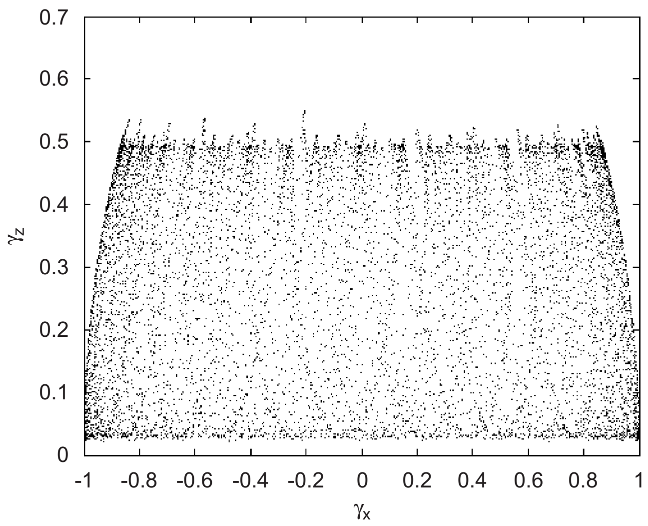

Fig. 8. Poincare sections γ_x versus γ_z at the hyperplane $\phi = 0.5\pi(\text{mod}2\pi)$ of the orbiting ZR liquid-filled solid with energy conservation; 9350 points.

together with directional cosines being initiated from $\gamma_{kp} = (I_{ck}\Omega_{kp} + J_{ck}\Pi_{kp})/G_p$. In addition, initial conditions of d_{x2}, d_{y2} and d_{z2} can be associated with the above-mentioned initial conditions and the given value $h = 0.15\pi$. Because Poincare maps of periodic orbits of the dynamical system are closed curves, one deduces from Fig. 8 that the orbits of long-term behaviors of the disturbed ZR liquid-filled solid arising from the orbital frequency are bounded and non-periodic. Other combinations for directional cosines can be drawn similarly. In order to show the long-term chaotic characteristic properties, twelve Lyapunov exponents for the disturbed ZR liquid-filled solid orbitor are evaluated from standard algorithms based on Gram-Schmidt reorthonormalization procedures (see Ref. [44]) as follows: 1.070438, 0.037519, 0.003842, −0.006816, −0.006058, −0.007581, −0.001091, −0.000337, −0.00883, −0.005395, −0.026089 and −1.057547. One may check that the sum of Lyapunov exponents of the conservative system (1)–(3) together with Eqs. (9) is approximately vanishing and the largest Lyapunov exponent is positive, i.e., 1.070438. The calculated Lyapunov exponents show that numerical results conform with the prediction of the MHM integral (39) in the case of the disturbed ZR liquid-filled solid due to the orbital frequency.

8.1.2. The dissipative case

When the ZR liquid-filled solid is subjected to damped and periodically perturbing torques in time described in Eqs. (27), the corresponding MHM integral (29) or (30) has been discussed in Section 5. As mentioned in the last section, the physical parameters and integral constants in association with heteroclinic orbits must be identified beforehand according to the established algorithms in Section 4.2. Further, designate damping coefficients as

$$\begin{aligned} &\sigma_k = 0.3534 \ \text{(the first situ)}, \quad \sigma_k = 0.09865 \ \text{(the second situ)}, \\ &\sigma_k = 0.1434 \ \text{(the third situ)}. \quad \text{for } k = x, y, z, \end{aligned} \tag{59}$$

and amplitudes of the external forcing terms in Eqs. (27) as

$$\Lambda_x = 3.5638, \quad \Lambda_y = 5.7854, \quad \Lambda_z = 4.7896. \tag{60}$$

The real zeros of the MHM integral, $\sin(\Omega_{ex}t_0)$, against the scaled initial component of vortex $\pi_x = \Pi_x(t_a)/\sqrt{R_0 + R_2\Pi_{z2}^2}$ are depicted from Eqs. (30) and (31) in Fig. 3 for three different situations in Eqs. (59) together with Eqs. (60) and when $\Omega_{\text{ex}} = 4$. The existence of real zeros of the MHM integral implies the possible existence of chaotic motions of the ZR liquid-filled solid disturbed by damped and time-periodical excitations, when the dimensionless "small" parameter ε is appropriately selected. In practical computations, ε may be a

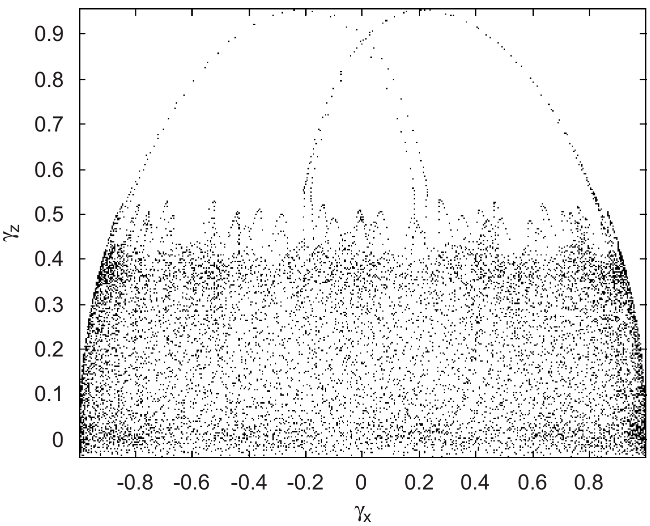

Fig. 9. Poincare sections γ_x versus γ_z at the hyperplane $\phi = 0.5\pi(\text{mod}2\pi)$ of the damped, periodically perturbed ZR liquid-filled solid orbitor; 11014 points.

big value. The MHM integral does not depend on the smallness of ε but chaotic dynamics of the original system (i.e., the ZR liquid-filled solid under the action of damped and time-periodically perturbed moments) do. Taking initial conditions (58), chaotic dynamics of damped and time-periodically perturbed ZR liquid-filled solid are depicted using Poincare sections in Fig. 9 where $\Omega_{\text{ex}} = 4$ and $\varepsilon = 0.01$. Nine Lyapunov exponents corresponding to Fig. 9 may be calculated as 0.103485, 0.016834, 0.005638, −0.004618, −0.005917, −0.006571, −0.007956, −0.008702, and −0.092307. In order to use the standard Gram–Schmidt reorthonormalization procedures, one should transform the non-autonomous system into an appropriate autonomous form. The largest Lyapunov exponent is positive, i.e., 0.103485 implying that simulated dynamics is chaotic. The sum of nine Lyapunov exponents is negative, i.e., -1.14322×10^{-4}, which shows that the dynamical system of the damped and time-periodically perturbed ZR liquid-filled solid is dissipative.

8.2. Chaotic oscillations of the liquid-filled ellipsoid

8.2.1. Energy conserving case

Consider the chaotic motion of the sliding liquid-filled ellipsoid rolling on the perfectly smooth plane. Designate physical parameters of the liquid-filled ellipsoid as in Section 4.2. In addition, perturbation parameters are given as: $I_{cx1} = 0.5$, $I_{cy1} = 0.9$, $I_{cz1} = 0.75$, $A_x = 45$, $A_y = 80$, $A_z = 76$, $B_x = 50$, $B_y = 39$, and $B_z = 125.6$. We focus on the real zeros of the MHM integral (55) for three Cases in Eqs. (56). The corresponding real zeros of the MHM integral (55) in terms of $\sin(g_0)$ against the scaled initial component of vortex $\pi_x = \Pi_x(t_a)/\sqrt{R_0 + R_2\Pi_{z2}^2}$ are graphed in Fig. 7. For simulated cases, $\sin(g_0)$ does not vary with respect to the argument h, an integral constant of the torque-free liquid-filled ellipsoid. The existence of real zeros of the MHM integral (55) ensures that the disturbed motions with energy conservation can be chaotic depending on the value of ε and initial conditions. Taking $\varepsilon = 0.001$ and initial conditions (58), long-term dynamics may be simulated and depicted using Poincare sections in Fig. 10. Lyapunov exponents conforming with Fig. 10 are evaluated below: 17.255575, 4.536381, 0.394175, 0.065544, 0.013406, −0.006513, −0.0019738, −5.053901 and −17.184933. The sum of all Lyapunov exponents is approximately zero as expected. As the largest Lyapunov exponent is positive, the motion is chaotic in the sense of Smale's horseshoes.

8.2.2. Energy dissipating case

Designate the damping coefficient $\sigma = 0.01$, the amplitude of the gravity $C_e = 8$, the external excitation frequency $\Omega_{\text{ex}} = 4$, the average level $g_e = 9.8$ and the small parameter $\varepsilon = 0.001$. Using the MHM integral (29)

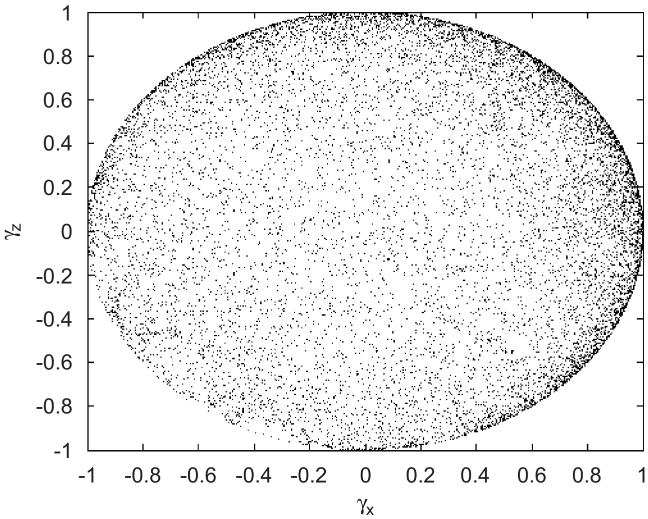

Fig. 10. Poincare sections γ_x versus γ_z at the hyperplane $\phi = 0.5\pi(\text{mod}2\pi)$ for the perturbed liquid-filled ellipsoid sliding and rolling on the plane with energy conservation; 11655 points.

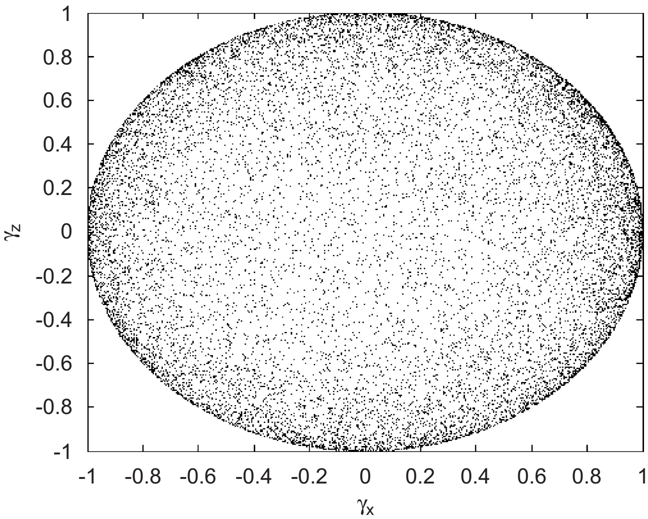

Fig. 11. Poincare sections γ_x versus γ_z at the hyperplane $\phi = 0.5\pi(\text{mod}2\pi)$ for the liquid-filled ellipsoid sliding and rolling on the plane with energy dissipation; 14610 points.

associated with Eqs. (51), we may draw the real zeros of the MHM integral (29), $\sin(\Omega_{ex}t_0)$, against the scaled initial component of vortex $\pi_x = \Pi_x(t_a)/\sqrt{R_0 + R_2\Pi_{z2}^2}$ in Fig. 6 for Case 1–3 in Eqs. (56). The existence of the real zeros of the MHM integral means that there are many sets of physical parameters which may lead to chaotic motions under appropriate dissipative and periodic moments. Taking $\varepsilon = 0.001$ and initial conditions (58), we may depict chaotic motions using Poincare sections in Fig. 11. Other combinations can be drawn similarly. Lyapunov exponents conforming with Fig. 11 are computed as 7.204723, 0.761418, 0.015400, −0.001417, −0.002344, −0.004755, −0.004719, −0.777833 and −7.198459. Since the largest Lyapunov exponent is positive, conforming to the characteristic of chaotic dynamics, we may deduce that MHM integrals help us a lot in determining the set of physical parameters which will potentially lead to chaos in the sense of Smale's horseshoes.

9. Conclusions

In the computation of MHM integrals we encounter three types of potential singularities in the integrands of MHM integrals for the determination of the transversal intersections between the stable and unstable manifolds of the disturbed ZR liquid-filled solids either conservative or dissipative as discussed in Ref. [11].

The key contribution of this paper is the formulation of the MHM criteria that can be used for judging the potential occurrences of chaotic rotational motions of the ZR liquid-filled solids in the case of the orbiting ZR liquid-filled solid and the sliding liquid-filled ellipsoid rolling on the perfectly smooth horizontal plane either conservative or dissipative. The author's previous investigations were carried out for the ZR liquid-filled solid in circular orbit under the action of gravity-gradient torques of order $O(N^2)$. Present disturbances due to the orbital frequency are of order $O(N)$. The degenerate structures of heteroclinic orbits of the torque-free ZR liquid-filled solid will break perhaps to yield transverse heteroclinic orbits due to disturbance toques of order $O(N)$. The existence of real zeros in MHM integrals due to Holmes and Marsden [22,25] generically means that there is no first integral for rotational motions of the ZR liquid-filled solid under certain disturbing torques. The significance of the investigation in the paper is that, based upon 1) the disturbed Hamiltonian equations in terms of generalized Deprit's canonical variables, 2) homoclinic/heteroclinic orbits and 3) MHM integrals, effective algorithms for predicting physical parameters corresponding to the potential chaotic rotations of the ZR liquid-filled solid under the action of small moments are formulated. Theoretical conclusions predicted from MHM integrals agree well with the extracted information from Poincare sections and Lyapunov exponents of the disturbed ZR liquid-filled solids, either conservative or dissipative. The dynamics of the disturbed liquid-filled ellipsoid rolling and sliding on the perfectly smooth plane are found to be chaotic theoretically and their complexities are detected theoretically.

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