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Short Communication

Approximate study of the free vibrations of a cantilever anisotropic plate carrying a concentrated mass

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Abstract

This study is concerned with the vibration analysis of a cantilevered rectangular anisotropic plate when a concentrated mass is rigidly attached to its center point. Based on the classical theory of anisotropic plates, the Ritz method is employed to perform the analysis. The deflection of the plate is approximated by a set of beam functions in each principal coordinate direction. The influence of the mass magnitude on the natural frequencies and modal shapes of vibration is studied for a boron-epoxy plate and also in the case of a generic anisotropic material. The classical Ritz method with beam functions as the spatial approximation proved to be a suitable procedure to solve a problem of this analytical complexity.

1. Introduction

Within the realm of the classical theory of plates, the case of a free edge offers considerable difficulty. This is the reason why, when dealing with vibrations problems, it is quite common to make use of the Ritz variational method, commonly referred to as the Rayleigh–Ritz method in the engineering literature [1,2] where one considers the free edge as the free end of a beam and performs all the algorithmic procedures accordingly. Readers with an interest in the history of this variational method are directed to Ref. [1]. In the present study, a cantilever plate is considered so the previous stated procedure is followed for three edges: in one direction the clamped-free beam is considered and in the perpendicular direction a free–free beam is considered, as illustrated in Fig. 1.

When dealing with an anisotropic rectangular plate finding coordinate functions which satisfy the free edge situation is exceedingly complicated (one must recall at this point that even in the case of a simply supported, rectangular, anisotropic plate, St Venant's method valid for an isotropic plate and also an orthotropic plate does not yield an exact solution since the sinusoidal solution does not satisfy exactly the governing boundary conditions), and it is necessary to make use of an approximate method to solve the problem, be it of a static or a dynamic nature.

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The presence of rigidly attached concentrated mass is taken into account in this study since this mechanical system is of interest in civil, naval and aeronautical engineering. Among the pertinent works on the subject one must quote the contributions by Chiba and Sugimoto [3] who carefully analyzed a cantilevered isotropic plate with a spring-mass system attached to it, Gorman who obtained an elegant, analytical solution for a cantilevered orthotropic plate [4] and the work of Rossi and Laura [5] about the effect of the Poisson's ratio and a concentrated mass.

In the general treatment of anisotropic vibrating rectangular plates, Laura and Grossi [6] considered elastic restraints in the edges, Laura et al. [7] studied forced vibrations of the simply supported case, Avalos et al. [8] considered doubly connected domains also for the simply supported case, and it is worthy of note the contribution of Grossi and Nallim [9] who studied the use of orthogonal polynomials in the application of Ritz method in the analysis of anisotropic plates.

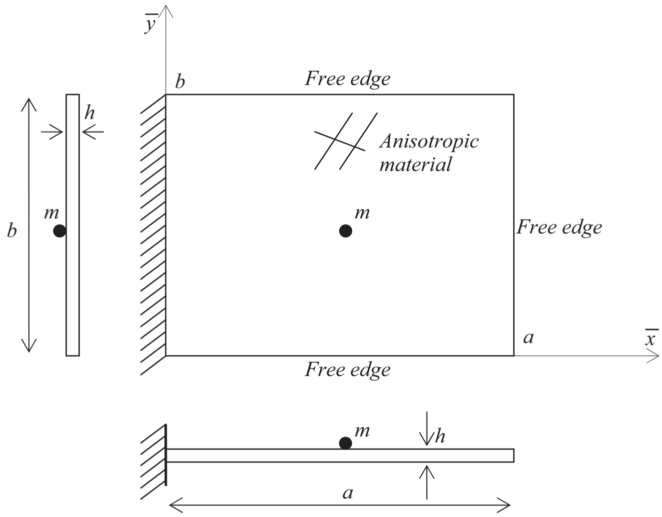

Fig. 1. Structural system under study.

Table 1
First six natural frequency coefficients $\Omega_i = \omega_i a^2 \sqrt{\rho h / D_{11}}$ for the cantilever plate of boron-epoxy composite components (Eq. (9))

λ	M	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
1/2	0	1.1222	3.6803	6.5669	10.2993	13.4473	19.7860
	0.1	1.1222	3.5622	6.4089	9.9454	13.1223	18.6502
	0.3	1.1221	3.3315	6.1284	9.3599	12.6589	16.7492
	0.5	1.12204	3.11902	5.91465	8.99269	12.4164	16.1201
	1	1.12188	2.69081	5.59914	8.56922	12.1706	15.6330
1	0	2.8454	10.0269	16.7404	21.7283	27.0418	35.5371
	0.1	2.7695	10.0221	16.7151	18.9435	25.2615	35.3644
	0.3	2.6329	10.0110	15.2794	16.8174	24.6652	35.2634
	0.5	2.5136	9.9971	13.5453	16.7840	24.5221	35.2276
	1	2.2719	9.9430	11.5758	16.7731	24.4113	35.1942
2	0	4.3249	15.0428	34.5083	44.0590	58.8009	71.2228
	0.1	4.0124	14.5226	34.5083	42.0799	56.5735	69.6054
	0.3	3.5534	13.9291	34.5082	39.6252	54.6370	68.3928
	0.5	3.2267	13.6038	34.5081	38.3308	53.8906	67.9534
	1	2.6967	13.2051	34.5079	36.8605	53.2024	67.5532

The quantity and variability of the parameters involved in the description of the mechanical behavior of these kinds of materials make of little interest the construction of tables of dynamical magnitudes, as it is usual for isotropic structural elements. Consequently, just a few representative cases will be considered to demonstrate the convenience of the procedure.

2. Approximate analytical solution

According to the classical thin anisotropic plate theory [10] the energy functional corresponding to the vibrating system shown in Fig. 1 is given by

$$J(W) = \frac{1}{2}\iint_A \left[D_{11}\left(\frac{\partial^2 W}{\partial \bar{x}}\right)^2 + 2D_{12}\frac{\partial^2 W}{\partial \bar{x}^2}\frac{\partial^2 W}{\partial \bar{y}^2} + D_{22}\left(\frac{\partial^2 w}{\partial \bar{y}^2}\right)^2 + 4D_{66}\left(\frac{\partial^2 W}{\partial \bar{x}\,\partial \bar{y}}\right)^2 + 4(D_{16})\frac{\partial^2 W}{\partial \bar{x}^2} + D_{26}\frac{\partial^2 W}{\partial \bar{y}^2}\right] \mathrm{d}\bar{x}\,\mathrm{d}\bar{y} - \frac{1}{2}\rho h\omega^2 \iint_A W^2\,\mathrm{d}\bar{x}\,\mathrm{d}\bar{y} + \frac{1}{2}m\omega^2[W(\bar{x}_m, \bar{y}_m)]^2, \quad (1)$$

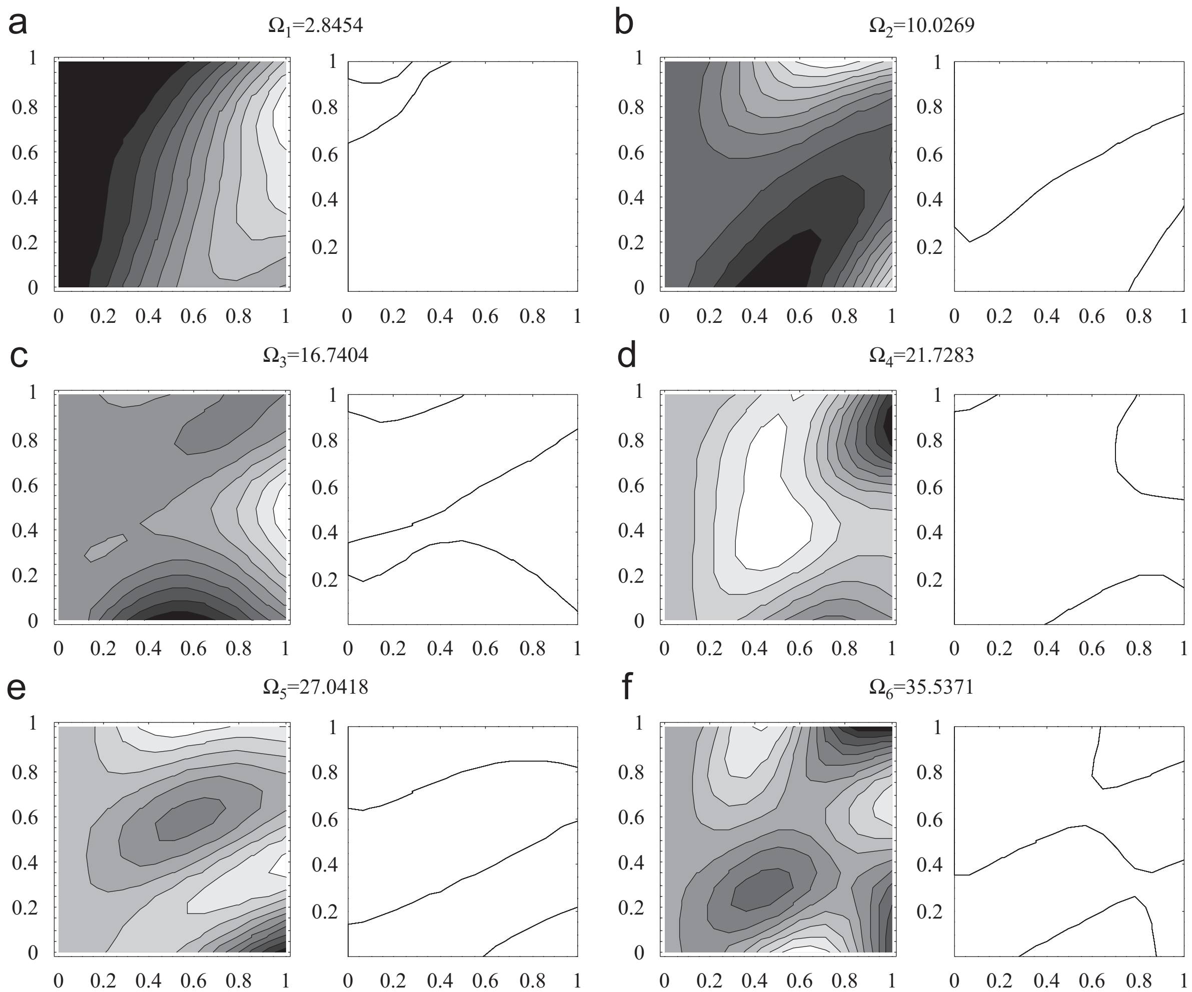

Fig. 2. Modal shapes and nodal lines in the vibration of the cantilever boron epoxy—Eq. (9)—plate of $\lambda = 1$ and without mass: (a) first mode, (b) second mode, (c) third mode, (d) fourth mode, (e) fifth mode, and (f) sixth mode.

where $W = W(\bar{x}, \bar{y})$ is the deflection amplitude of the middle plane of the plate, the D_{ij} are the well-known flexural rigidities of an anisotropic plate, A is the area of the plate plan form, ρ and h are, respectively, the density and the thickness of the plate, m is the magnitude of the concentrated mass, $W(\bar{x}_m, \bar{y}_m)$ is the plate displacement amplitude at the mass position $(\bar{x}_m, \bar{y}_m)$ and ω is the natural circular frequency of the system. The rotatory inertia of the concentrated mass is not taken into account in the present analysis. As the length of the sides of the rectangular plate are a and b in the $\bar{x}$ and $\bar{y}$ directions, respectively, the coordinates can be written in dimensionless form as

$$\begin{aligned} x &= \bar{x}/a, \qquad y = \bar{y}/b, \\ x_m &= \bar{x}_m/a, \quad y_m = \bar{y}_m/b, \end{aligned} \tag{2}$$

and the aspect ratio of the plate is denoted by

$$\lambda = \frac{a}{b}.$$

The expression of the deflection of the plate is approximated in the form of a truncated series of beam functions $X_m(x)$ and $Y_n(y)$:

$$W(x, y) \cong \sum_{m=1}^{M} \sum_{n=1}^{N} A_{mn} X_m(x) Y_n(y), \tag{3}$$

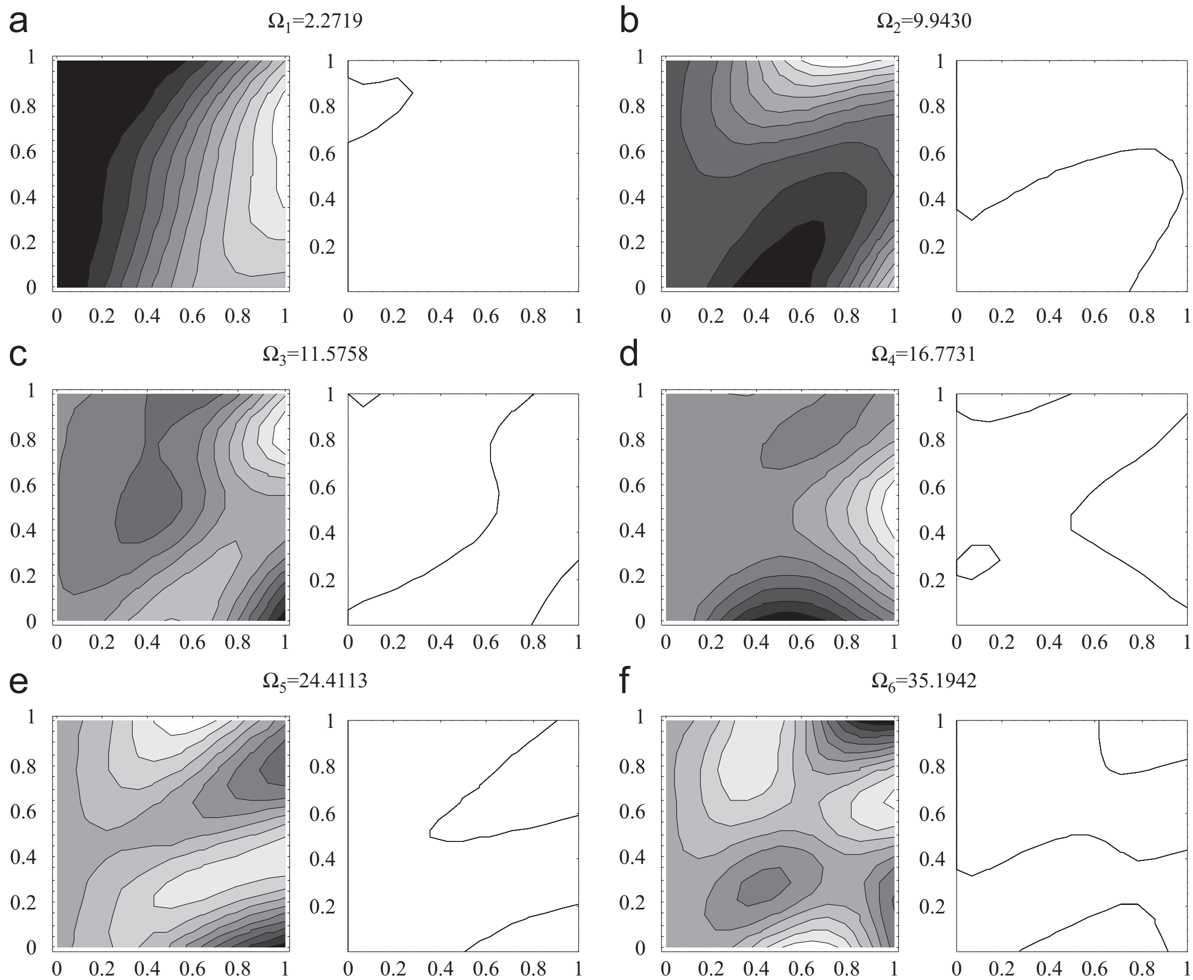

Fig. 3. Modal shapes and nodal lines in the vibration of the cantilever boron epoxy—Eq. (9)—plate of $\lambda = 1$ and with a centered mass $M = 1$: (a) first mode, (b) second mode, (c) third mode, (d) fourth mode, (e) fifth mode, and (f) sixth mode.

where $X_m(x)$ and $Y_m(y)$are the characteristic functions for the normal modes of vibration of beams with end conditions similar to the simplified expressions used for the plate at the free edges and, obviously, the same in the case of the clamped edge.

In the x-coordinate direction the corresponding beam function is

$$X_m(x) = \cosh(k_m x) - \cos(k_m x) - \alpha_m[\sinh(k_m x) - \sin(k_m x)], \tag{4}$$

where

$$\alpha_m = \frac{\cosh(k_m) + \cos(k_m)}{\sinh(k_m) + \sin(k_m)},$$

and k_m are the roots of

$$\cos(k_m)\cosh(k_m) = -1, \tag{5}$$

$k_m = 1.87504, 4.694091, 7.85457, \ldots$ while

$$\begin{aligned} &Y_1(y) = 1, \quad Y_2(y) = 2y - 1, \\ &Y_n(y) = \cosh(k_n y) + \cos(k_n y) - \alpha_n[\sinh(k_n y) + \sin(k_n y)], \end{aligned} \tag{6}$$

where

$$\alpha_n = \frac{\cosh(k_n) - \cos(k_n)}{\sinh(k_n) - \sin(k_n)},$$

and $k_n (n \geqslant 3)$ are the roots of

$$\cosh(k_n)\cos(k_n) = 1, \tag{7}$$

$k_n = 4.730040, 7.853204, 10.995607, \ldots$

Obviously Eqs. (4) and (6) do not satisfy the natural boundary conditions at the free ends, as previously stated but this is legitimate when using the Ritz method [9].

Substituting Eqs. (4) and (6) into Eq. (3) and Eq. (3) into Eq. (1) and, requiring that $J(W)$ be a minimum with respect to the $A_{mn's}$ coefficients:

$$\frac{\partial J[W]}{\partial A_{mn}} = 0, \quad m = 1, 2, \ldots. M; \quad n = 1, 2, \ldots. N, \tag{8}$$

one obtains a homogeneous linear system of equation in terms of the A_{mn} parameters.

Table 2
First six natural frequency coefficients $\Omega_i = \omega_i a^2 \sqrt{\rho h / D_{11}}$ for the cantilever plate of a generic anisotropic material (Eq. (10))

λ	M	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
1/2	0	3.1814	4.5016	9.4651	14.8854	19.8561	22.0135
	0.1	3.1339	4.4535	9.0704	14.3820	19.5741	19.9462
	0.3	3.0339	4.3661	8.3489	13.4933	17.8342	19.8764
	0.5	2.9306	4.2916	7.8027	12.9720	17.2977	19.9735
	1	2.6792	4.1579	7.0228	12.4284	16.8981	19.8719
1	0	2.8285	5.5269	18.9016	20.0922	27.5157	40.076
	0.1	2.7969	5.4113	17.5429	19.9259	26.0913	37.8911
	0.3	2.7331	5.2054	15.2921	19.8619	24.9061	36.3225
	0.5	2.6691	5.0316	13.9045	19.8441	24.4706	35.7685
	1	2.5129	4.7107	12.1729	19.8294	24.0831	35.2717
2	0	2.0317	6.5626	19.7769	27.1362	58.6434	66.4141
	0.1	2.0181	6.2841	18.6560	26.9542	58.3900	66.4141
	0.3	1.9907	5.8201	17.2197	26.7784	57.3895	62.4533
	0.5	1.9633	5.4541	16.3695	26.6940	55.8712	60.6937
	1	1.8954	4.8180	15.2821	26.6001	53.3167	59.8764

From the non-triviality condition, one can get natural frequency coefficients: $\Omega_i = \omega_i a^2 \sqrt{\rho h / D_{11}}$, as eigenvalues, and vibration modes as eigenvectors of the secular determinant.

The present study is concerned with the determination of the first six natural frequency coefficients Ω_1 to Ω_6 for the case of an anisotropic cantilever rectangular plate carrying a concentrated centered mass, and their respective modal shapes.

3. Numerical results

The numerical determinations have been performed for two kinds of materials:

(a) a boron-epoxy component characterized by the following parameters [9]:

$$\frac{D_{22}}{D_{11}} = 0.2130195, \quad \frac{D_{12}}{D_{11}} = 0.3245569, \quad \frac{D_{66}}{D_{11}} = 0.3387559, \quad \frac{D_{16}}{D_{11}} = 0.5120546 \quad \text{and} \quad \frac{D_{26}}{D_{11}} = 0.1694905, \tag{9}$$

and

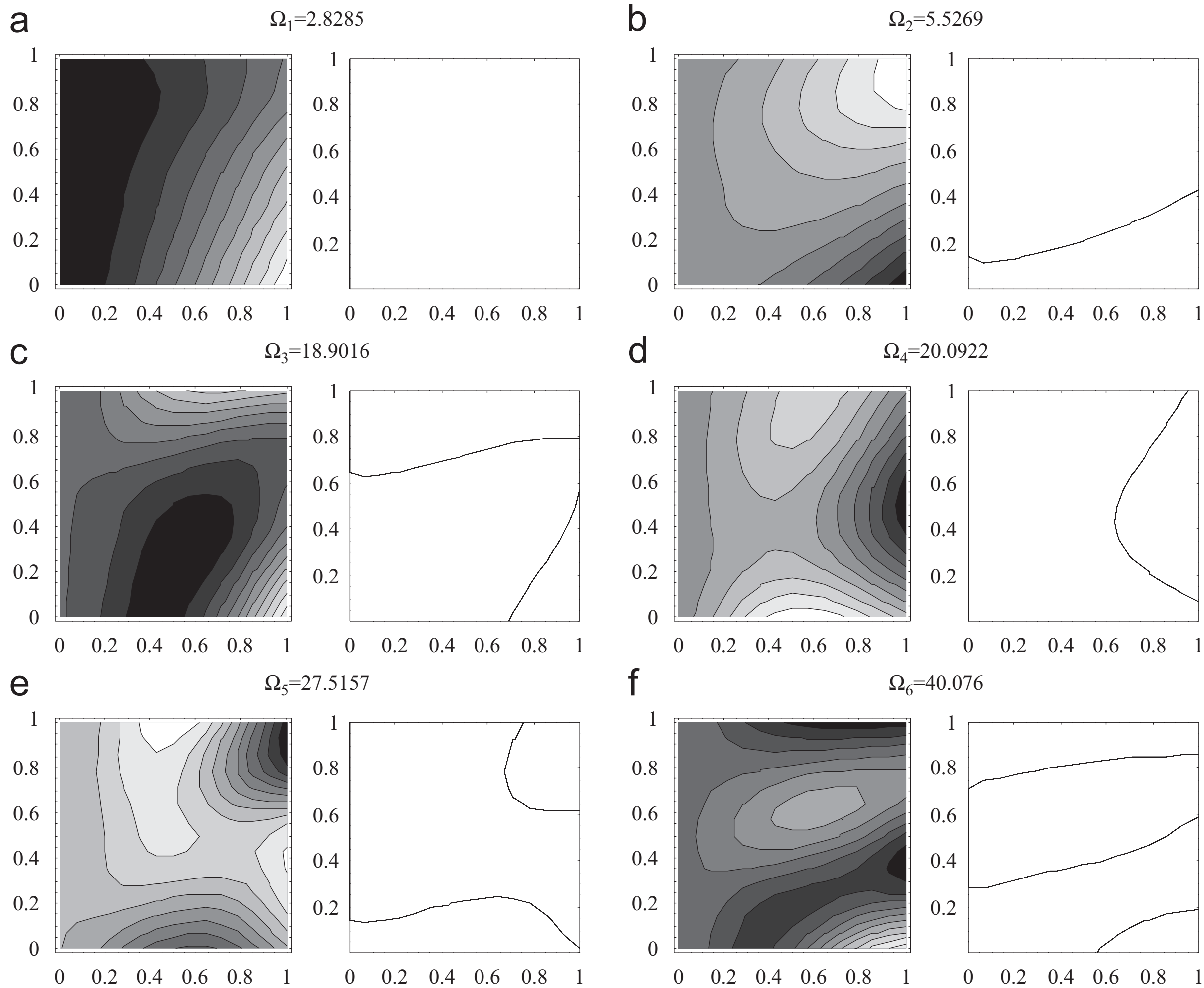

Fig. 4. Modal shapes and nodal lines in the vibration of the cantilever plate of a generic anisotropic material—Eq. (10)—of $\lambda = 1$ and without mass: (a) first mode, (b) second mode, (c) third mode, (d) fourth mode, (e) fifth mode, and (f) sixth mode.

(b) a generic anisotropic material for which:

$$\frac{D_{22}}{D_{11}} = \frac{D_{12}}{D_{11}} = \frac{D_{66}}{D_{11}} = \frac{1}{2}, \quad \frac{D_{16}}{D_{11}} = \frac{D_{26}}{D_{11}} = \frac{1}{3}. \tag{10}$$

The aspect ratio $\lambda = a/b$ of the plate varies from $\frac{1}{2}$ to 2, and the relation between the magnitude of the concentrated attached mass and the mass of the plate: $M = m/m_p$ is chosen to be 0, 0.1, 0.3, 0.5 and 1.

Eq. (1) was verified for the particular isotropic case by simply making

$$D_{11} = D_{22} = \frac{Eh^3}{12(1-\upsilon^2)}, \quad D_{12} = \nu D_{11}, \quad D_{66} = \frac{1-\nu}{2} D_{11}, \quad D_{16} = D_{26} = 0.$$

For this particular case, the obtained results are in excellent agreement with those obtained by Chiba and Sugimoto [3].

In Table 1 are shown the natural frequency coefficients for a cantilever boron-epoxy plate.

In Figs. 2 and 3 the modal shapes of vibration are shown along with the corresponding nodal lines of the boron-epoxy plate free of mass and with a mass ratio $M = 1$ attached, respectively, for the case $\lambda = 1$. In spite

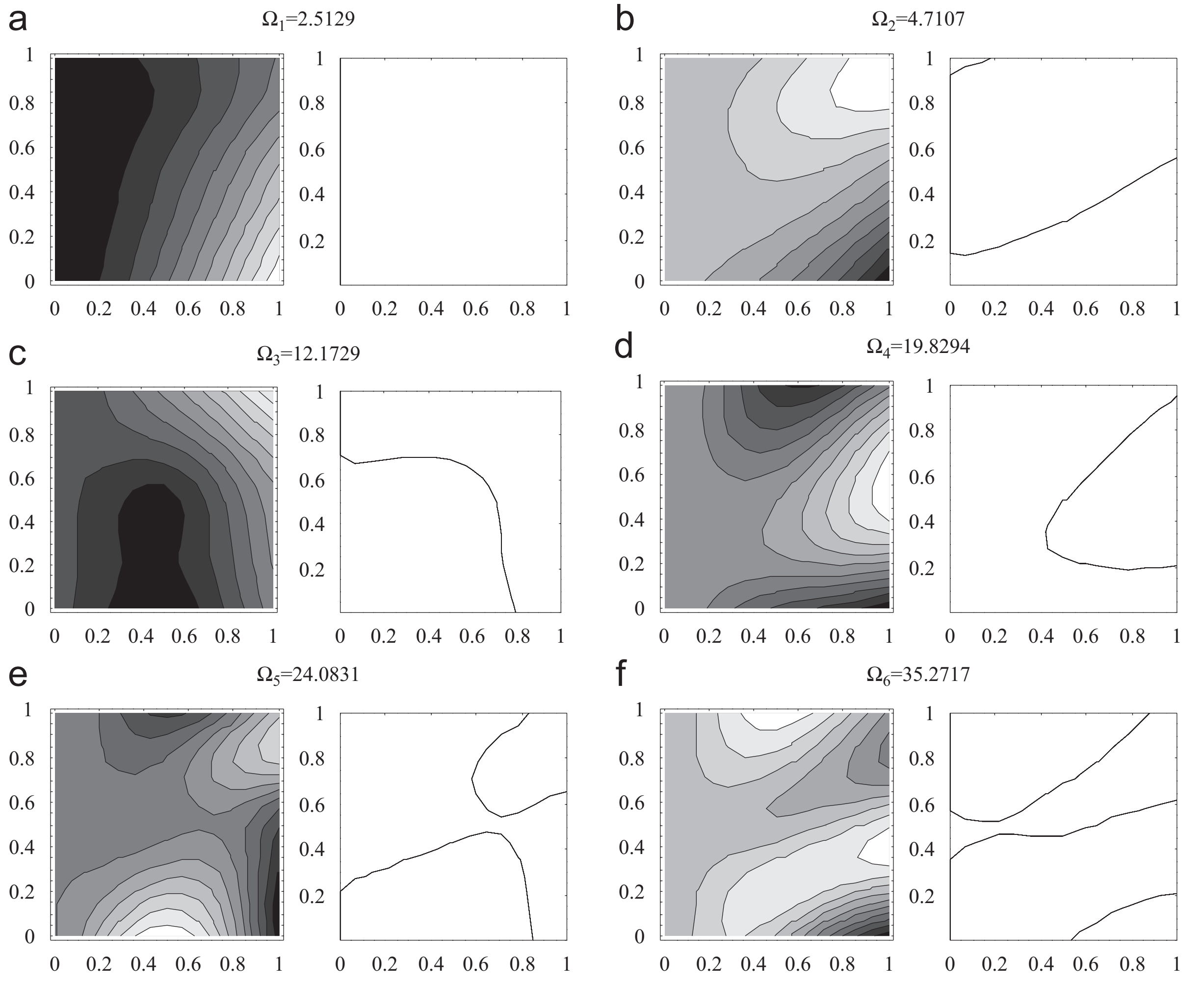

Fig. 5. Modal shapes and nodal lines in the vibration of the cantilever plate of a generic anisotropic material—Eq. (10)—of $\lambda = 1$ and with a centered mass $M = 1$: (a) first mode, (b) second mode, (c) third mode, (d) fourth mode, (e) fifth mode, and (f) sixth mode.

of the central location of the concentrated mass and the boundary conditions, in general, no symmetric behavior can be expected because of the anisotropy of the material.

The coefficients of Table 1 make evident the influence of the concentrated mass showing a decrease in almost all the frequency values. The influence on the fundamental frequency is in general higher, as in the isotropic case, except for the case where $\lambda = \frac{1}{2}$, where this natural frequency remains almost unchanged. In some cases, e.g., the second and sixth frequencies for $\lambda = 1$ and the third frequency for $\lambda = 2$, the values of the coefficients remain almost unchanged. In those cases a nodal line is situated near the location of the mass (see Figs. 2 and 3).

In Table 2 and Figs. 4 and 5 the frequency coefficients and modal shapes are shown for a cantilever plate made of a generic anisotropic material (Eq. (10)). They were obtained in order to show the influence of the material properties over the dynamical behavior of the plate. A surprising coincidence is that Ω_3 exhibits the highest variation with the increase of the mass, for $\lambda = 1$, in the present study for the boron epoxy plate as well as for a generic, arbitrary material.

4. Concluding remarks

The classical, variational method of Ritz has been successfully used in the present study to obtain an approximate, yet quite accurate, solution to a difficult elastodynamics problem. Natural frequencies and mode shapes are obtained for a meaningful combination of the governing parameters: material properties and ratios of the applied concentrated mass referred to the total plate mass, and different aspect radii of the plate.

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