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Propagation of elastic energy in a general anisotropic medium

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Abstract

Propagation of plane harmonic waves is studied in anisotropic elastic medium. Anisotropy is of general type, i.e., no symmetry enforced and no rotation of elastic tensor. The propagation is not restricted to a fixed plane but along a general direction in three-dimensional space. A new procedure is presented to study the reflection in anisotropic media. Phase direction of incident wave is calculated from its ray direction. For this incident phase direction, the Snell's law is used to calculate the phase direction of each of the homogeneous reflected waves, This identifies a critical angle of incidence for the reflected wave such that, for incidence beyond this angle, this reflected wave becomes inhomogeneous. Group (energy) velocities and ray directions of the homogeneous quasi-waves reflected at the free surface are calculated analytically and without using energy flux. An energy matrix is defined to explain the energy share of different reflected waves and interaction energy. The incidence of the quasi-waves is considered along a given (arbitrary) ray direction. The numerical results compute the group velocities and ray directions of reflected waves for the numerical model of Dolomite crystalline rock. The partition of incident energy among the homogeneous reflected waves is also calculated. The energies reflected as different homogeneous waves vary with the ray direction of the incident wave. These variations are plotted and discussed for the numerical model.

1. Introduction

Elastic anisotropy is a widespread observation in the areas of economic and scientific interest [1]. The preferential alignments in the Earth ranging from mineral orientations, grains, or microcracks to regional fractures result in the seismic anisotropy. Crampin [2] reviewed the various studies related to the observation of shear-wave splitting and confirmed that the anisotropy was present in almost all the rocks in the uppermost half of the crust. The mechanical behaviour of composite materials is represented by anisotropic elasticity [3,4]. In the last two decades, the applications of acoustic microscopy and fibre-reinforced composites have initiated the interest in the wave propagation in layered anisotropic media [3]. Physics of granular media [5–7] represents an active area of current research activity. Sound speeds in such media depend upon the stress-induced anisotropy existing there. The propagation in anisotropic media has enormous applications in the non-destructive evaluation of materials [8,9].

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The work of Synge [10] is one of the early detailed study of elastic waves in anisotropic media. Musgrave [11] presented a concise, elegant treatment of anisotropy with symmetry properties, based on algebraic solutions. Since then, there have been a large number of studies discussing wave propagation in anisotropic media. Almost all of the analytical studies, contain a few lines to convince the researchers that symmetry of one kind or the other is prevalent in the sediments/crystals/composites. This helped to restrict the anisotropic propagation to a fixed (symmetry or arbitrary) plane and hence to solve a 2-D (plane) problem. Symmetry planes are a special case of general anisotropy. It is usually impossible to extrapolate from such a special case of anisotropy to the general one.

The papers of Fryer and Frazer [12,13] are among the important analytical studies on general anisotropy. They derived the analytical expressions of the eigensolutions for the anisotropic propagation with monoclinic symmetry. Zilmer et al. [14,15] used perturbation theory of Frazer and Fryer [16] to approximate the reflection coefficients for weak anisotropic media. The approximations made were not valid near critical angles, near shear-wave singularities and when the anisotropy was too strong. Sharma [17] solved the Christoffel equation to obtain analytical expressions for the phase velocities of all the quasi-waves in a general anisotropic medium. This formulation also provided the analytical expressions for the directional derivatives of phase velocity. The work presented use this formulation to study the true reflection in a general anisotropic elastic medium.

2. Definition of the problem

The problem is to study the wave propagation in a general anisotropic solid half-space and is explained as follows. For the incidence of a quasi-wave, in a general anisotropic medium, with a given ray direction (arbitrary in 3-D space), at the free surface,

(i) find the phase direction and hence phase velocity of the incident wave;
(ii) then, use Snell's law to find the phase directions and phase velocities of all the three reflected (homogeneous) quasi-waves;
(iii) using the phase directions and phase velocities of all the quasi-waves to calculate
   (a) the group velocity and ray direction for each reflected wave;
   (b) the energy partition among reflected waves at the surface.

3. Field equations for anisotropic propagation

The governing equations for an elastic media, in the absence of body forces, are

$$\sigma_{ij,j} = \rho \ddot{u}_i. \tag{1}$$

The u_i are the components of the average displacements for the solid particles. The dot notation is used to denote time (partial) derivative. Summation convention is valid for repeated indices that can assume the values 1, 2 and 3. The comma (,) before an index represents partial space differentiation. ρ is the density of the medium. In an anisotropic elastic material, the constitutive equations for stresses (i.e., σ_{ij}) are

$$\sigma_{ij} = c_{ijkl} u_{k,l}. \tag{2}$$

The coefficients c_{ijkl} $(= c_{klij} = c_{jikl})$ are the 21 independent material constants for a most general anisotropic material. To seek the harmonic solution of Eq. (1), for the propagation of plane waves, write

$$u_j = S_j \exp\left\{\imath\omega\left(\frac{1}{v} n_k x_k - t\right)\right\} \quad (j = 1, 2, 3), \tag{3}$$

where ω is frequency. v is the phase velocity of wave, along the phase direction (n_1, n_2, n_3), denoted by unit vector **N**. Following Keith and Crampin [18], the Christoffel equation for an anisotropic medium is a system of three homogeneous equations, given by

$$F_{ij} S_j = 0 \quad (i = 1, 2, 3), \tag{4}$$

where $F_{ij} = c_{ijkl}p_k p_l - \rho\delta_{ij}$. δ_{ij} is Kronecker delta. Non-trivial solution of this system explains the propagation of three quasi-waves ($qP, qS1, qS2$) in an anisotropic elastic medium. Analogous to the P and S waves in elastic medium, the names qP and qS are chosen. These waves are identified with their velocities in decreasing order, i.e., fastest among them is qP. $qS1$ and $qS2$ represent the splitting of shear waves in anisotropic medium, where $qS1$ is the faster of the two split shear-waves [19]. These waves are called the quasi-waves because their polarisations may not be along the dynamical axes (with propagation direction as one of the coordinate axis). The analytical expressions for phase velocities ($v_j; j = 1, 2, 3$) of these quasi-waves are derived in Appendix A. The phase velocities depend upon the direction of phase propagation.

The eigenvectors of system (4), i.e., (S_1, S_2, S_3), define the general polarisations of quasi-waves in the anisotropic medium. These are expressed as follows:

$$\frac{S_1}{\Gamma_1} = \frac{S_2}{\Gamma_2} = \frac{S_3}{\Gamma_3} \tag{5}$$

for three sets of ($\Gamma_1, \Gamma_2, \Gamma_3$), given by

$$\begin{aligned}
&\text{(i) } \Gamma_1 = F_{22}F_{33} - F_{23}F_{32};\; \Gamma_2 = F_{23}F_{31} - F_{12}F_{33};\; \Gamma_3 = F_{21}F_{32} - F_{13}F_{22},\\
&\text{(ii) } \Gamma_1 = F_{13}F_{32} - F_{12}F_{33};\; \Gamma_2 = F_{11}F_{33} - F_{13}F_{31};\; \Gamma_3 = F_{12}F_{31} - F_{11}F_{32},\\
&\text{(iii) } \Gamma_1 = F_{12}F_{23} - F_{13}F_{22};\; \Gamma_2 = F_{13}F_{21} - F_{11}F_{23};\; \Gamma_3 = F_{11}F_{22} - F_{12}F_{21}.
\end{aligned} \tag{6}$$

Polarisations can be obtained from any of these three sets. Normalisation removes the extra degree of freedom. These sets of expressions are not independent but all three are required to find the polarisations in some situations, as explained in Fryer and Frazer [13].

4. Reflection

The analytical methods available in literature (e.g., Refs. [13,18]) to study anisotropic propagation, solve the eigensystem of Christoffel equation for vertical slowness in a fixed plane. For general anisotropy, this method requires to solve a polynomial of degree 6 to find the slowness values for all the waves propagating in an anisotropic elastic solid. Out of these six values, the suitable three are selected to calculate the energy-fluxes of the three quasi-waves. Such a selection may not be able to differentiate among $qS1$- and $qS2$-waves when their velocities are nearly same (i.e., around shear-wave singularities). The energy-fluxes are used in finding the ray directions of these waves. Fryer and Frazer [13] used this method to study the anisotropies with horizontal plane of symmetry where the equation of order 6 is reduced to a cubic equation.

The present study proposes a new method, which is more exploring and transparent one. Only cubic equation is required to be solved to calculate the velocities of three quasi-waves as functions of their propagation direction. After this, the waves will be identified with their velocity functions and, hence, the process of studying the reflected waves will be wave-specific. For each of the reflected waves, the group velocity and ray direction are calculated without involving their energy flux. This method is explained as follows.

(i) Christoffel equation is solved analytically to find the phase velocities of all the quasi-waves in an anisotropic elastic medium. The analytical expressions for phase velocity, derived in Sharma [17], enable to find the directional derivatives of phase velocity and hence the group velocity and ray direction without using any numerical method.
(ii) Phase velocity of a quasi-wave depends upon its phase direction. The phase direction of a reflected quasi-wave is obtained from Snell's law, which involves its phase velocity. This is a tricky situation. Here, the analytical expression for phase velocity of the quasi-wave manages the way out. The Snell's law and velocity function of a reflected wave, together, provide a nonlinear function in polar angle (θ_j, in Fig. 1) of the phase direction of the reflected wave. Bisection method (or Newton's method) is used to find the real root of this function. Existence of such a root defines the phase direction of the corresponding reflected wave for the given phase direction of the incident wave. Such a (real) phase direction represents the homogeneous reflected wave.
(iii) Phase direction from step (ii) is used to calculate the phase velocity of the reflected quasi-wave.

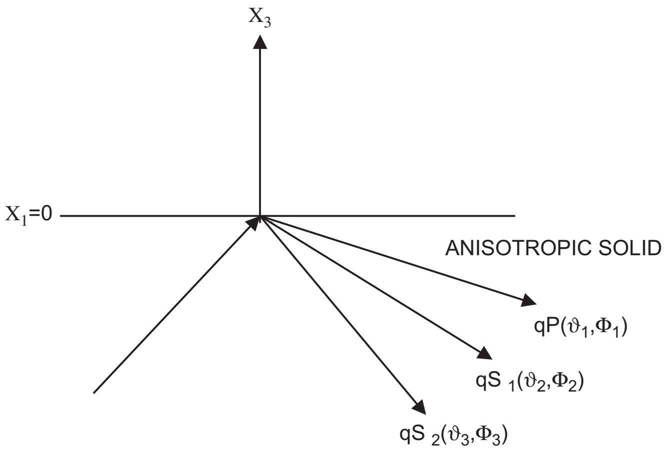

Fig. 1. Geometry of the medium (rays represent phase directions).

(iv) Phase direction from step (ii), phase velocity from step (iii) and analytical derivatives of phase velocity are used to calculate the ray (energy) direction and group velocity of the quasi-wave.

4.1. Geometry of the medium

Consider a rectangular coordinate system (x_1, x_2, x_3) to represent a three-dimensional space. Let (θ, ϕ) denotes a general direction in the space, where θ is inclination to the polar axis and ϕ is azimuth to the x_1-axis. The half-space $x_3 \leqslant 0$ is occupied by a general anisotropic elastic solid. The plane $x_3 = 0$ is the free surface of this medium with its outward normal along x_3-axis, as shown in Fig. 1. A quasi-wave travels through this medium and become incident at a point on the surface. In a spherical coordinate system centred at this point, let $(\theta_g \phi_g)$ is the ray direction of this incident wave. Corresponding to this ray direction, let (θ_I, ϕ_I) is phase direction of this quasi-wave, as shown in Fig. 1. The incident wave results in three quasi-waves $(qP, qS1, qS2)$ reflected back into the anisotropic medium. Rays showing the reflected waves, in Fig. 1, represent the phase directions (θ_j, ϕ_j), $(j = 1, 2, 3)$, of the three quasi-waves.

4.2. Displacements

The displacement components in the anisotropic elastic medium are expressed as

$$u_j = S_j^{(I)} \exp\left\{ \iota\omega \left(\frac{1}{v_I} n_k^{(I)} x_k - t \right) \right\} + \sum_{m=1}^{3} a(m) S_j^{(m)} \exp\left\{ \iota\omega \left(\frac{1}{v_m} n_k^{(m)} x_k - t \right) \right\}, \tag{7}$$

where $(n_1^{(m)}, n_2^{(m)}, n_3^{(m)}) = (\sin\theta_m \cos\phi_m, \sin\theta_m \sin\phi_m, \cos\theta_m)$ represents the phase direction of quasi-wave m. The $a(m)$ are relative excitation factors. The $I = 1, 2$ or 3 for the incidence of qP, $qS1$ or $qS2$ wave, respectively.

4.3. Boundary conditions

The boundary conditions represents the vanishing of stresses at the free surface. In this problem, the three boundary conditions, required to be satisfied at the plane $x_3 = 0$, are

$$\sigma_{k3} = 0 \quad (k = 1, 2, 3). \tag{8}$$

Eqs. (2) relate σ's to displacement components u_i. Satisfying the above boundary conditions yields a system of three linear inhomogeneous equations in $a(1), a(2)$ and $a(3)$. These equations are given by

$$\sum_{m=1}^{3} J_i^{(m)} \frac{1}{v_m} a(m) = -J_i^{(I)} \frac{1}{v_I} \quad (i = 1, 2, 3), \tag{9}$$

where, for c_{ij} defining the anisotropic elastic constants in two-suffix notation,

$$J_i^{(m)} = \sum_{k=1}^{3} (c_{lk} S_k^{(m)} n_k^{(m)}) + c_{l4}(S_2^{(m)} n_3^{(m)} + S_3^{(m)} n_2^{(m)}) + c_{l5}(S_1^{(m)} n_3^{(m)} + S_3^{(m)} n_1^{(m)}) + c_{l6}(S_1^{(m)} n_2^{(m)} + S_2^{(m)} n_1^{(m)}), \quad l = 6 - i \quad (i = 1, 2, 3). \tag{10}$$

4.4. Snell's law

In order to solve the system of Eqs. (8) for $a(m)$, $(m = 1, 2, 3)$, the values of $n_j^{(m)}, v_m$ and (v_I) are required for a given phase direction $(n_1^{(I)}, n_2^{(I)}, n_3^{(I)})$ of the incident wave. The continuity in boundary conditions requires the identical phase of all the waves at the surface $x_3 = 0$. The Snell's law in three dimensions is, then, explained by

$$\frac{n_i^{(I)}}{v_I} = \frac{n_i^{(m)}}{v_m} \quad (m = 1, 2, 3), \ (i = 1, 2). \tag{11}$$

This form of Snell's law helps to deduce the following points:

(i) $n_2/n_1 = n_2^{(m)}/n_1^{(m)}$, $(m = 1, 2, 3)$, imply that $\phi_m = \phi_I$. This means that phase directions of all the reflected waves lie in the same vertical plane, which contains the phase direction of incident wave. Hence, in the absence of azimuthal anisotropy, the velocity and direction of energy propagation will also confine to this plane only. So, in an anisotropic medium with anisotropy up to azimuthal isotropy, the study of wave propagation in a plane is sufficient to explain the reflection/refraction phenomenon. But, the presence of azimuthal anisotropy demands that the wave propagation needs to be studied in three dimensions.

(ii) The phase velocity v_m of the reflected quasi-wave m in anisotropic medium depends upon its phase direction (θ_m, ϕ_m). Using Snell's law and the condition $n_k^{(m)} n_k^{(m)} = 1$, an equation,

$$v_m^2 \sin^2 \theta_I - v_I^2 \sin^2 \theta_m = 0 \tag{12}$$

is obtained which relates θ_m and v_m^2. An expression of v_m^2 as a function of θ_m (Appendix A) enables to find the value of θ_m, for any given value of θ_I. A numerical method (bisection method or Newton's method) to find the real root of a nonlinear equation is used. This real value of θ_m is, further, used to calculate the (real) phase velocity v_m and hence, velocity, direction and partition of energy.

(iii) It may be noted that polar angles θ_m, $(m = 1, 2, 3)$ of quasi-waves are derived from the polar angle of incident wave. A real angle of reflection defines the real phase direction **N** for the reflected wave. In anisotropic medium, a wave with real phase direction will be a homogeneous wave. As the incident wave reaches the critical angle for any of the reflected waves, the reflected wave propagates along the surface with velocity $v_c = v_m(\pi/2, \phi_I)$. Such a critical angle is determined from the nonlinear equation

$$v_c \sin \theta_I - v_I(\theta_I, \phi_I) = 0. \tag{13}$$

This equation is solved for θ_I, with a given value of ϕ_I, numerically, for each of the reflected waves. A valid root of this equation, say θ_c, divides the each incidence plane $\phi = \phi_I$ into two parts. For incidence beyond this angle the corresponding reflected waves becomes an inhomogeneous wave.

(iv) In (ii), there may be some angles of the incident wave for which the real θ_m does not exist for a reflected wave. This implies that the concerned reflected wave is propagating as an inhomogeneous wave. The vector representing its phase direction (i.e., **N**) will be a complex vector. The velocity (v) of the reflected wave, corresponding to this complex vector, will also be complex. The propagation direction and attenuation direction of this wave is then obtained from the specification [20] of its complex slowness vector ($\mathbf{N}/v$). The energy-flux of the inhomogeneous waves is horizontal and, hence, travels along the surface/interface [10]. Hence, the suggested procedure calculates the group velocity and ray direction of the reflected wave until it is a homogeneous wave. So, the reflection process may be studied for the incidence that results all the reflected waves as homogeneous.

4.5. Energy ratios

Distribution of energy between different reflected homogeneous waves is considered across a surface element of unit area at the plane $x_3 = 0$. Following Achenbach [21], the scalar product of surface traction and particle velocity per unit area, denoted by P^*, represents the rate at which the energy is communicated per unit area of the surface. The time average of P^* over a period, denoted by $\langle P^* \rangle$, represents the average energy transmission per unit surface area per unit time. At incidence beyond the critical angle for a refracted wave, the waves become inhomogeneous and hence involves the concept of interaction energy. Borcherdt [22] explained the existence of interaction energy for the reflection and refraction of SH waves. A matrix of energy ratios, defined by $E_{jk} = \langle P^*_{jk} \rangle / \langle P^*_{II} \rangle$; $(j, k = I, 1, 2, 3)$, may be able to calculate the interaction energy among the quasi-waves present in the anisotropic medium. x_3-axis being the outer normal to the surface, the $\langle P^*_{jk} \rangle$ representing the average energy fluxes, are given by

$$\langle P^*_{jk} \rangle = -0.5\omega^2 \frac{1}{v_j} \mathrm{Re}\left\{ \sum_{i=1}^{3} J_i^{(j)} a(j) \bar{S}_i^{(k)} \bar{a}(k) \right\}. \tag{14}$$

The sum of all the non-diagonal entries of this energy matrix gives the interaction energy ratio for the reflection. At incidence yielding only the reflected homogeneous waves, this energy matrix is a skew symmetric one. Hence, the interaction energy vanishes. The diagonal entries of the matrix represent the energy share of reflected quasi-waves. The conservation of energy is given by the relation $\sum_{j=1}^{3}(\sum_{k=1}^{3} E_{jk} + E_{jI}) = -E_{II} = -1$.

5. Group velocity

Energy propagation in anisotropic media is, in fact, a three-dimensional phenomenon. Study of propagation in one plane (particularly a symmetry plane) may give no indication of its behaviour in neighbouring directions. In an anisotropic medium, energy associated with a quasi-wave travels with the group velocity along a ray at an angle to its direction of phase propagation. In a spherical coordinate system, let $v(\theta, \phi)$ define the phase velocity of a quasi-wave in the vertical plane $(\phi = \phi_I)$ along the phase direction which is making an angle θ with the polar axis. Following Ben-Menahem and Sena [23], the components of group velocity i.e., w_j, $(j = 1, 2, 3)$, are expressed as follows:

$$w_1/v = \cos\phi \sin\theta + \cos\phi \cos\theta T_\theta - \frac{\sin\phi}{\sin\theta} T_\phi,$$

$$w_2/v = \sin\phi \sin\theta + \sin\phi \cos\theta T_\theta + \frac{\cos\phi}{\sin\theta} T_\phi,$$

$$w_3/v = \cos\theta - \sin\theta T_\theta. \tag{15}$$

The magnitude of the group velocity is

$$w = v\sqrt{1 + T_\theta^2 + \frac{1}{\sin^2\theta} T_\phi^2} \tag{16}$$

and the ray direction, (θ_g, ϕ_g), is obtained from its components. T_θ and T_ϕ in Eqs. (15)–(16), are defined by

$$T_k = \frac{1}{v}(v)_{,k} = \frac{1}{2h}(h)_{,k} \quad (k = \theta, \phi). \tag{17}$$

The partial derivatives of $h(= v^2)$ are derived, analytically, in Appendix A.

6. Numerical computation and discussion

The purpose of numerical computation is to study the characteristics of the reflected homogeneous waves resulting from the incidence of each of the three quasi-waves at the free surface of the medium. This includes the computation of the group velocity and ray direction of the reflected waves along with the share of three reflected waves in the energy reflected back into the medium. For this purpose, numerical model of a real crystal rock may be more suitable. Therefore, Dolomite reservoir rock is chosen for the anisotropic medium. The elastic constants for this solid are obtained from Rasolofosaon and Zinszner [24]. These are (in GPa) as follows:

$$c_{11} = 65.53 \quad c_{12} = 9.77 \quad c_{13} = 12.19 \quad c_{14} = 0.18 \quad c_{15} = -0.81 \quad c_{16} = 2.94,$$

$$c_{22} = 50.77 \quad c_{23} = 11.61 \quad c_{24} = -0.09 \quad c_{25} = -0.50 \quad c_{26} = -0.19,$$

$$c_{33} = 60.11 \quad c_{34} = -1.61 \quad c_{35} = 1.78 \quad c_{36} = 0.84 \quad c_{44} = 23.51,$$

$$c_{45} = 1.49 \quad c_{46} = -1.17 \quad c_{55} = 24.57 \quad c_{56} = 0.26 \quad c_{66} = 20.21.$$

The density (ρ) of the rock is 2423 kg/m^3. The ray direction, (θ_g, ϕ_g), of the incident wave varies from $(0, 0)$ to $(90^\circ, 90^\circ)$. Using these numerical values, the variations in the (group) velocities and (ray) directions of the reflected waves with the ray direction of the incident wave, are calculated. The incidence of each of the qP, $qS1$ and $qS2$ waves is considered. The variations of energy ratios of reflected waves are, also, calculated. These variations are plotted in Figs. 2–8. Details are as follows.

The variations in the group velocities and ray directions of reflected waves from the incidence of qP wave are plotted in Fig. 2. Group velocities vary both with the polar angle and azimuth of the ray direction of incident wave. Variations are nearly uniform, in general. For the incidence of qP wave, all the reflected waves

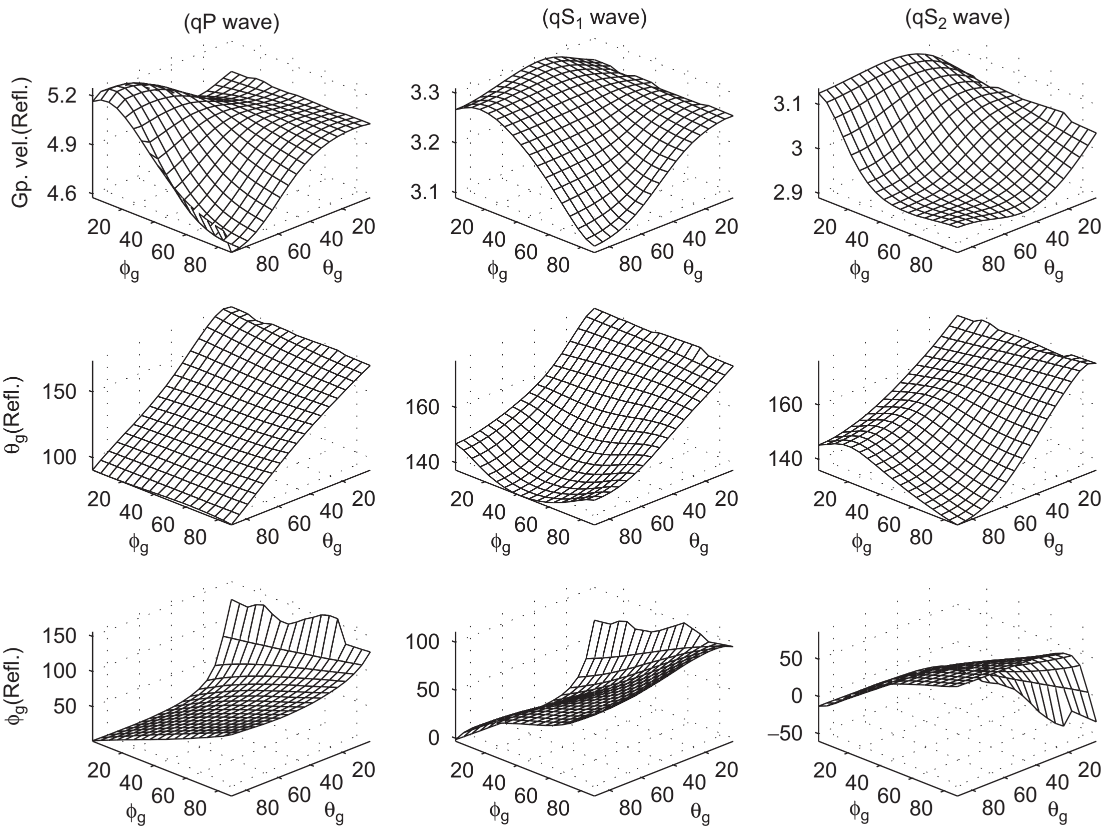

Fig. 2. Variations of group velocities (km/s) and ray directions of reflected homogeneous waves with the ray direction (θ_g, ϕ_g) of incident qP wave (all angles in degrees).

are homogeneous. This implies that the incidence of qP waves does not reflect inhomogeneous waves. The velocities of reflected waves decrease with the increase of θ_g and ϕ_g, both, of the incident wave. These velocity variations of three reflected waves with the ray directions of incident wave are up to 10%. Variations are larger with azimuth in comparison to with polar angle. The polar angle of reflected qP wave is nearly follows the reflection rule of the isotropic medium, i.e., θ_g of reflected qP wave $= \pi - \theta_g$ of incident qP wave. However, the azimuth of the reflected qP wave is quite different from the incident qP wave, except, near grazing incidence. This implies that the reflected energy is not confined to the vertical plane of incident wave. The ray directions of the reflected $qS1$ and $qS2$ waves are also different from the incident direction, as expected. The polar angles of reflected $qS1$ and $qS2$ waves may be up to $\pi/4$ away from reflected qP wave.

Fig. 3 contains the variations in the group velocities and ray directions of reflected waves for the incidence of $qS1$ wave. The reflected qP wave become inhomogeneous for incidence beyond, a critical value of θ_g, which varies with the value of ϕ_g. For incidence beyond these angles, the group velocities and ray direction of other two reflected waves changes rapidly with the directions of incident wave. Otherwise, these variations are nearly uniform. The ray directions of the three reflected waves are quite near to each other.

The variations of group velocities and ray directions of reflected waves from the incidence of $qS2$ wave are exhibited in Fig. 4. The variation patterns are quite similar to Fig. 2, except that the critical directions appear a bit early in this case. The similarity in the behaviour in Figs. 2 and 3 may imply that the propagation characteristics of $qS1$ and $qS2$ waves in the medium chosen are, very nearly, same to each other. The other difference is the second critical directions, incidence beyond which changes the reflected $qS1$ wave into an inhomogeneous wave.

In the above two figures, it is observed that the numerical values of critical directions after which the reflected $(qP, qS1)$ waves become inhomogeneous are important. Fig. 5, exhibits these critical directions for

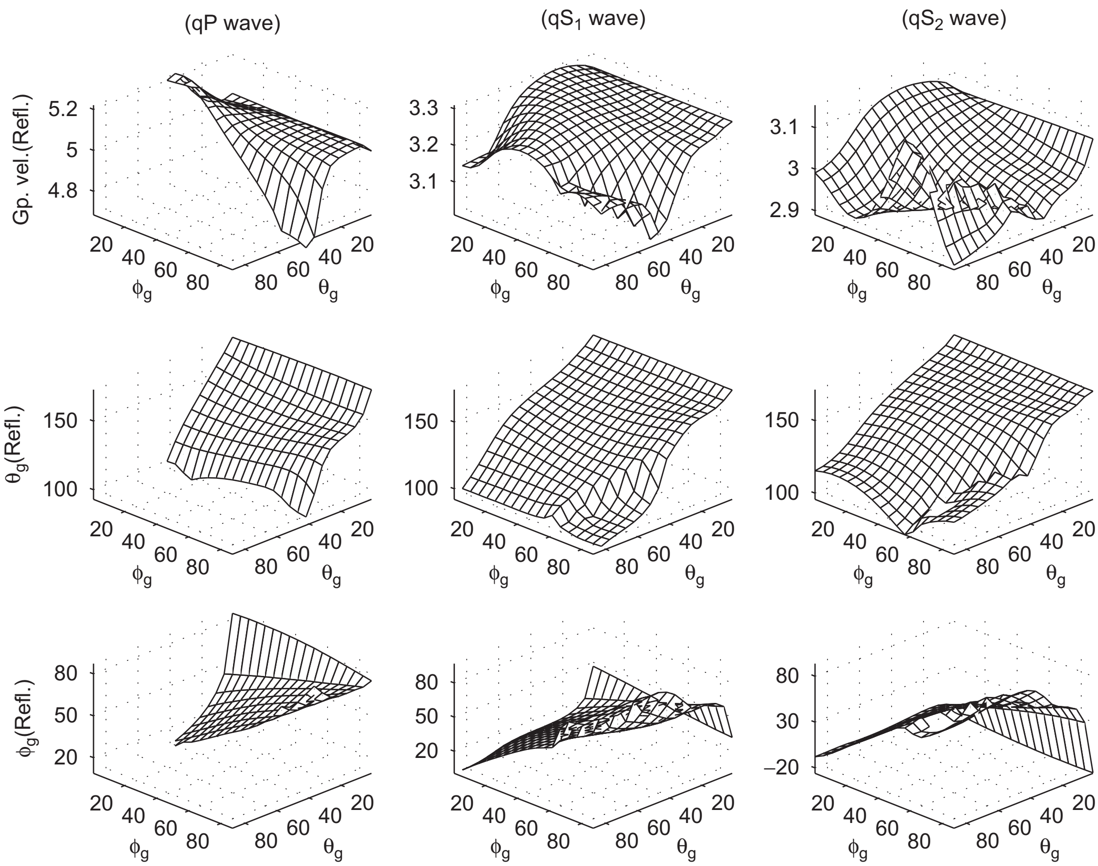

Fig. 3. Same as Fig. 2, but for incident $qS1$ wave.

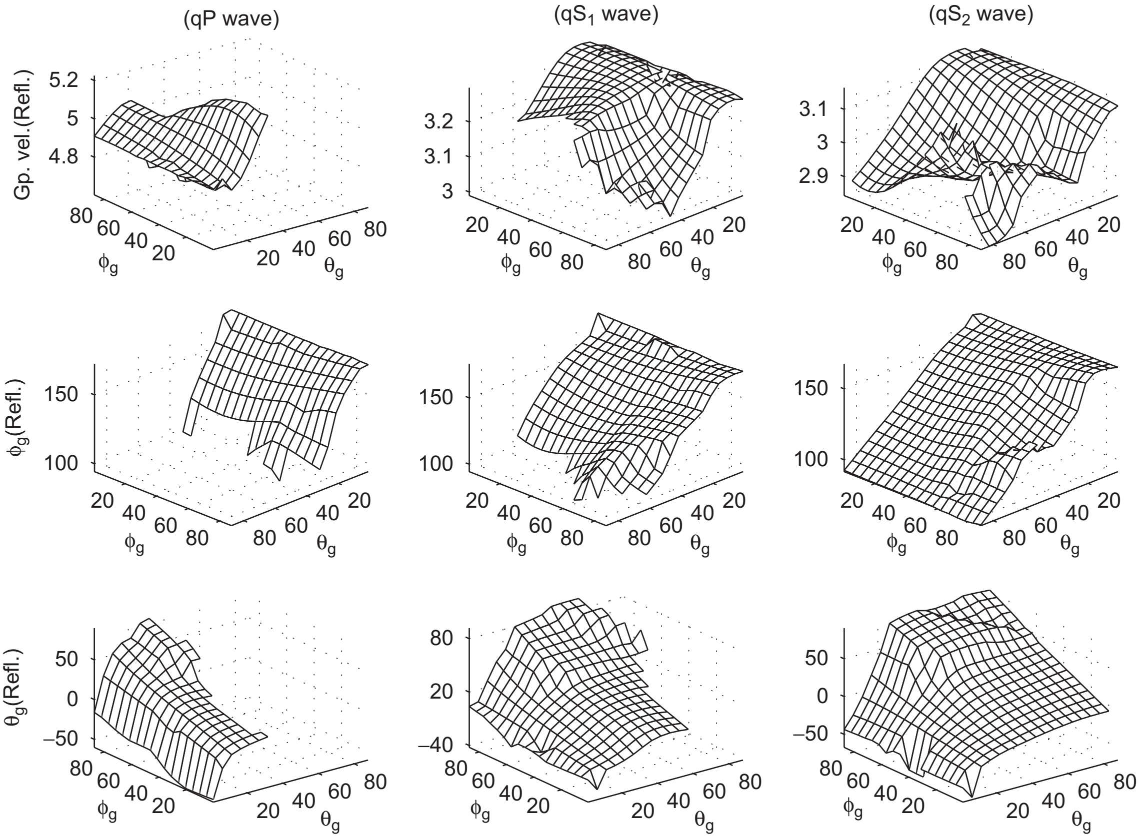

Fig. 4. Same as Fig. 2, but for incident $qS2$ wave.

the incidence of $qS1$ and $qS2$ waves. The plots in the figure at top shows the variations of the phase critical angle (θ_c) with the phase propagation plane ($\phi = \phi_I$) of the incident wave. The figure below shows the variations of ray critical angle (θ_c^g) with the ray propagation plane ($\phi_g = \phi_I^g$) of incident wave. Ray critical angles are little higher than phase critical angles. Triplication in one of the curve is the feature of wave surface of reflected $qS1$ wave [19]. For the corresponding reflected wave, these (directions) curves divide the planes (quadrants) of incidence into two parts, one for homogeneous reflected wave and other (near to the free plane surface) for inhomogeneous reflected wave.

Fig. 6 exhibits the variations of energy ratios of the reflected qP, $qS1$ and $qS2$ waves, with the ray direction of incident qP wave. All reflected waves are homogeneous, hence, no interaction energy for the incidence of qP wave. Change of the (vertical) plane of incidence ($\theta_g =$ constant) has little effect on the reflected qP wave but have significant effect on the reflected $qS1$ and $qS2$ waves. In general, the variations in the partition of reflected energy with the ray direction of incident qP wave are uniform. While going across normal to grazing incidence, the energy of reflected qP wave decreases whereas the energies of reflected $qS1$ and $qs2$ waves increase.

Fig. 7 shows the variations in partition of reflected energy with the ray direction of the incident $qS1$ wave. The direction of incidence resulting in all homogeneous reflected waves are considered. Share of $qS2$ wave in reflected energy is smaller as compared to two other waves. The abrupt edges of the surface plots (near critical directions) are due to the step-size of θ_I, chosen for numerical computation. The corresponding variations, for the incident $qS2$ waves, are presented in Fig. 8. The homogeneous reflected waves are obtained for a smaller incident region as compared to incident $qS1$ wave. For incidence in this region, the energy partition changes more steadily as compared to the incidence of other waves. In the reflected energy, the reflected $qS1$ wave has a smaller share as compared to qP and $qS2$ waves.

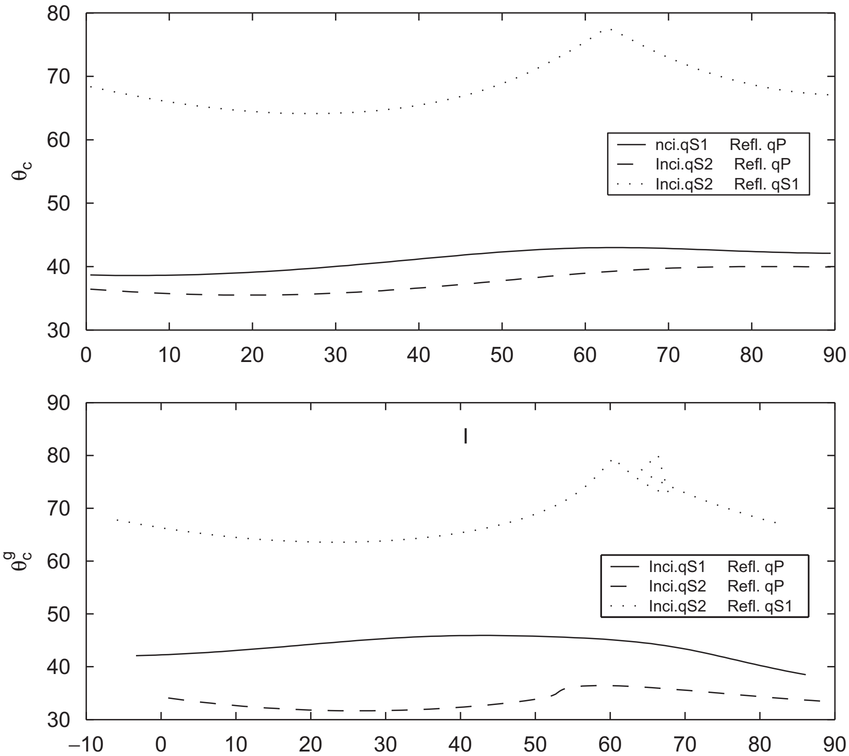

Fig. 5. Critical curves dividing incidence region among homogeneous and inhomogeneous reflected waves; phase curve: (θ_c, ϕ_I) and ray curve: (θ_c^g, ϕ_I^g) (all angles in degrees).

7. Conclusions

The numerical results, discussed above, are obtained for the numerical model of a particular rock. Any conclusion drawn from the discussion of these results may not qualify for generalisation but computation success does certify the applicability of the technique studied in this work. Few of the results drawn from the study may be expressed as follows:

(i) However, the phase directions of all the reflected waves lies in the incident plane but the reflected energies are not confined to a single plane. This demands that, in an anisotropic medium, the wave propagation should not be explained in two dimensions.
(ii) The velocities and directions of reflected waves varies considerably with the direction of incident plane. This implies that the study of reflection in any fixed plane may not be able to explain the propagation behaviour in the nearby plane.
(iii) The critical angle varies with the azimuth of incident wave. This implies that for incidence at a particular angle, near critical angle, a reflected wave may be homogenous or inhomogeneous depending upon the incident plane chosen.

This work presented studies the anisotropic reflection process in a true sense. The procedure introduced is a generalised analogy of that used for propagation in isotropic elastic medium. The transparent analytical expressions creates space for further research in the topic that includes the waves reflected as inhomogeneous waves. The procedures introduced may be used to study the problems of anisotropic scattering in an elastic medium. The researchers in this field would prefer to use the analytical expressions derived in this work. The variations of critical angles with the orientation of sagittal plane may be used as a diagnostic tool to recognise

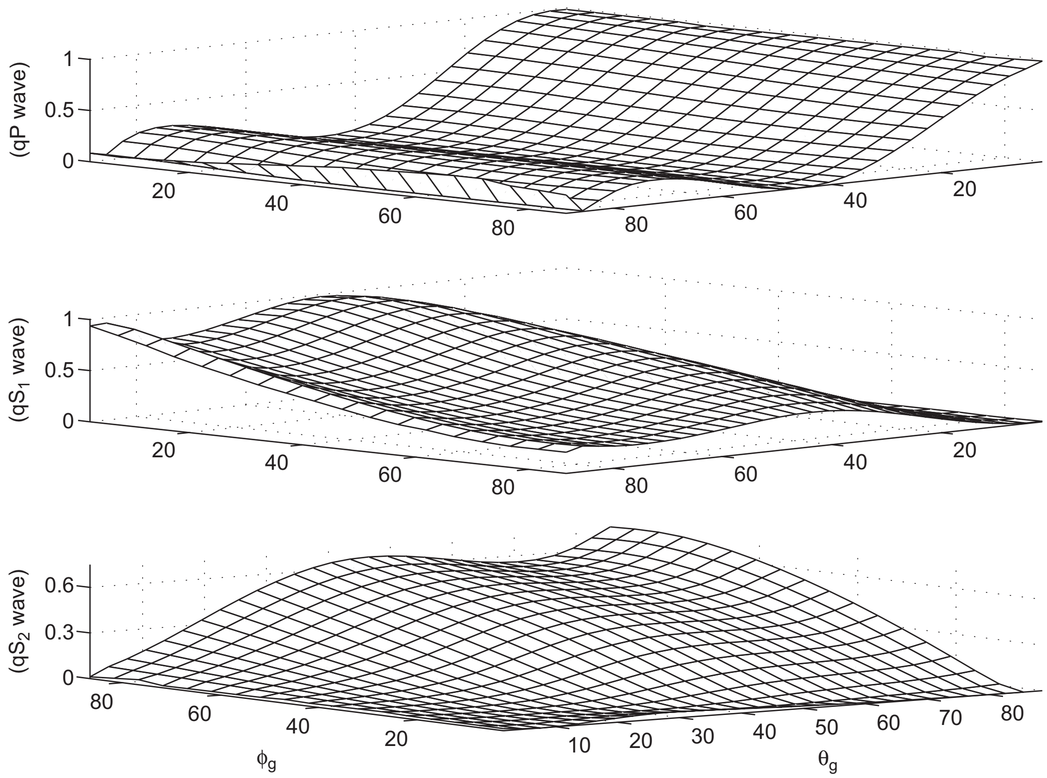

Fig. 6. Variations of energy ratios of reflected homogeneous waves with the ray direction (θ_g, ϕ_g) of incident qP wave (all angles in degrees).

and estimate anisotropy. It may help in improving the models used by exploration and NDE people in the interpretation of their complex data.

Appendix A

A.1. Phase velocities

Consider a general anisotropic medium represented by the elastic constants in two-suffix notation, c_{ij}. Define a row matrix $N = (n_x, n_y, n_z)$, where n_j denotes the components of a unit vector normal to wave surface and, hence, it represents the direction of phase propagation. Following Sharma [17], define:

$$\alpha = NAN', \quad \beta = NBN', \quad \gamma = NCN',$$
$$\delta = NDN', \quad \eta = NEN', \quad \zeta = NFN', \tag{18}$$

where N' is the transpose of N. A, B, C, D, E and F, the square matrices of order 3, are defined as follows:

$$A = \{a_{11}, a_{16}, a_{15}; a_{16}, a_{66}, a_{56}; a_{15}, a_{56}, a_{55}\}, \quad B = \{a_{66}, a_{26}, a_{46}; a_{26}, a_{22}, a_{24}; a_{46}, a_{24}, a_{44}\},$$

$$C = \{a_{55}, a_{45}, a_{35}; a_{45}, a_{44}, a_{34}; a_{35}, a_{34}, a_{33}\}, \quad D = \{a_{16}, a_{12}, a_{14}; a_{66}, a_{26}, a_{46}; a_{56}, a_{25}, a_{45}\},$$

$$E = \{a_{15}, a_{14}, a_{13}; a_{56}, a_{46}, a_{36}; a_{55}, a_{45}, a_{35}\}, \quad F = \{a_{56}, a_{46}, a_{36}; a_{25}, a_{24}, a_{23}; a_{45}, a_{44}, a_{34}\},$$

where $a_{ij} = c_{ij}/\rho$. ρ is the density of the medium. The eigenvalue problem for the medium is represented by

$$v^6 + 3av^4 + 3bv^2 + c = 0, \tag{19}$$

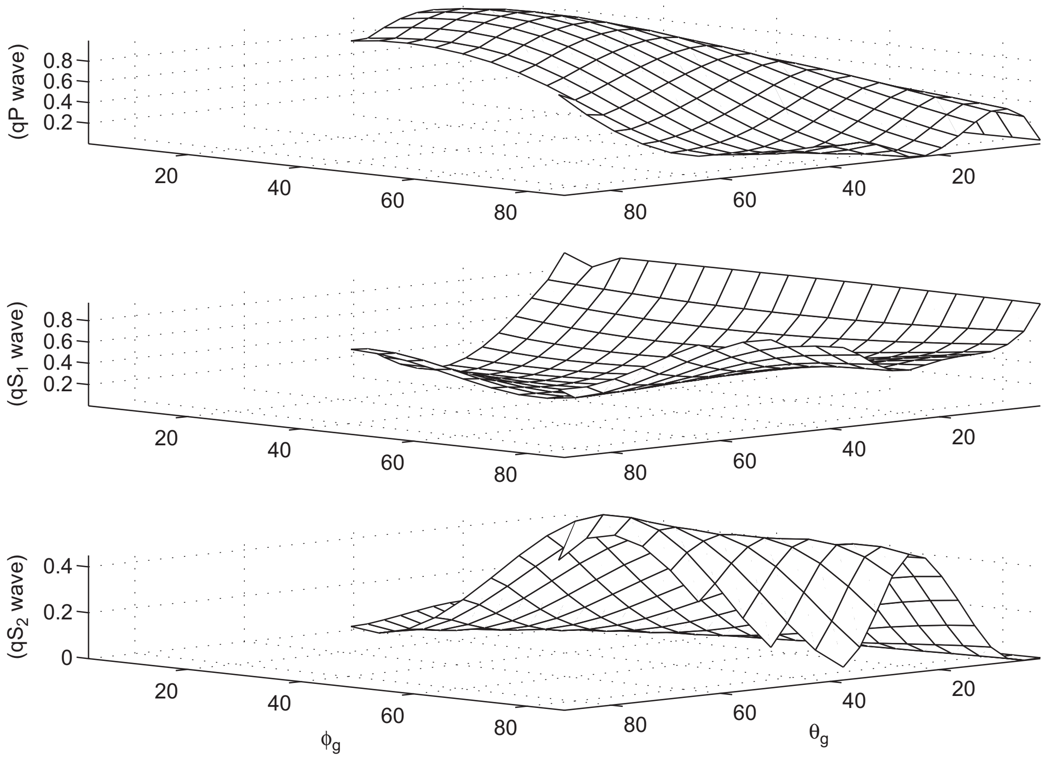

Fig. 7. Same as Fig. 6, but for incident $qS1$ wave.

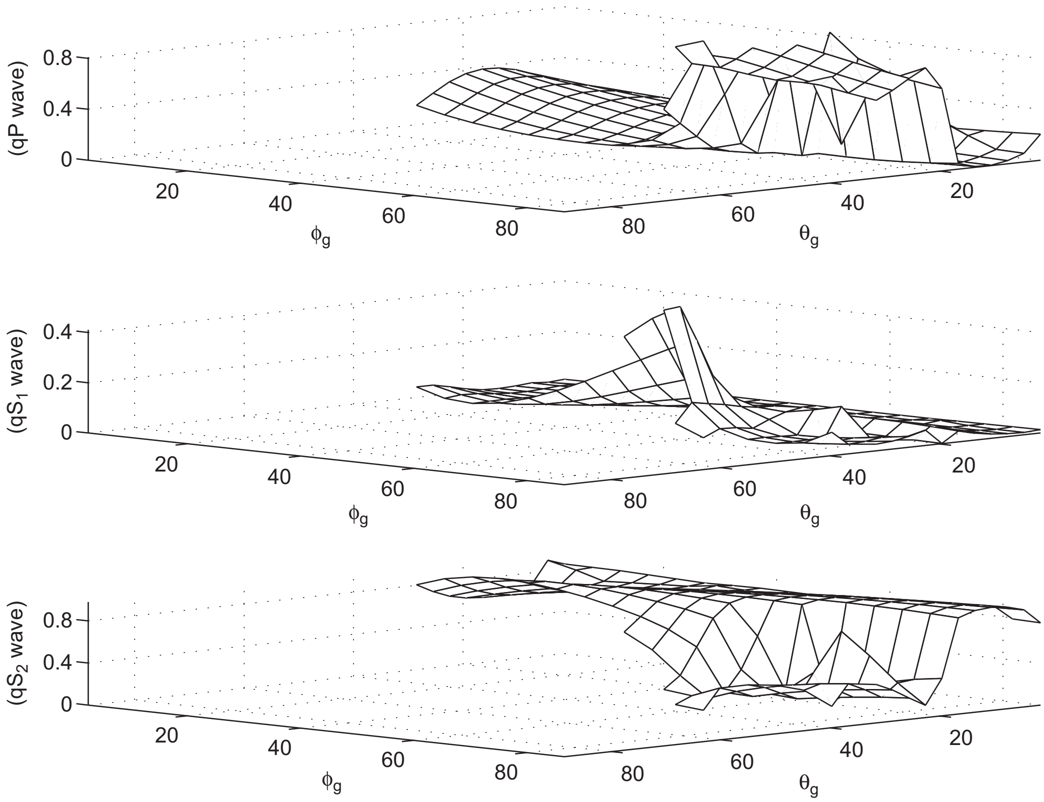

Fig. 8. Same as Fig. 6, but for incident $qS2$ wave.

where

$$a = -\tfrac{1}{3}(\alpha + \beta + \gamma),$$

$$b = \tfrac{1}{3}(\alpha\beta + \alpha\gamma + \beta\gamma - \delta^2 - \eta^2 - \zeta^2),$$

$$c = \alpha\zeta^2 + \gamma\delta^2 + \beta\eta^2 - \alpha\beta\gamma - 2\eta\delta\zeta. \tag{20}$$

The real roots of Eq. (18), which is cubic in V^2, are written as

$$v_m^2 = 2\sqrt{-H}\cos\left\{\frac{\psi - 2\pi(m-1)}{3}\right\} - a \quad (m = 1, 2, 3), \tag{21}$$

where $\psi = \tan^{-1}(\varDelta/G)$, $\varDelta = \sqrt{-(G^2 + H^3)}$; $H = b - a^2$; and $G = (3ab - c - 2a^3)/2$. v_m, $(m = 1, 2, 3)$, define the magnitudes of phase velocities of three quasi-waves in the direction of unit vector N. These waves, represented by $m = 1, 2$ and 3, are called the qP, $qS1$ and $qS2$ waves, respectively. $qS1$ is the faster of the two split shear-waves $qS1$ and $qS2$ [19].

A.2. Phase direction from ray direction

The purpose is to find phase direction, i.e., (θ, ϕ), for a given ray direction (θ_g, ϕ_g). This is possible only after reducing the relations (14) to a system of two nonlinear simultaneous equations. Such a system of equations is given by

$$\sin\phi_g f_1 - \cos\phi_g f_2 = 0$$

and

$$\sin\theta_g f_3 - \cos\theta_g \sqrt{f_1^2 + f_2^2} = 0, \tag{22}$$

where $f_1 = \cos\phi\sin\theta + \cos\phi\cos\theta T_\theta - \frac{\sin\phi}{\sin\theta}T_\phi$, $f_2 = \sin\phi\sin\theta + \sin\phi\cos\theta T_\theta + \frac{\cos\phi}{\sin\theta}T_\phi$; and $f_3 = \cos\theta - \sin\theta T_\theta$.

Newton's method for two variables is applied to solve this system numerically for θ and ϕ. The partial derivatives of functions f_j, for $j = 1, 2, 3$, with respect to θ and ϕ can be evaluated using the following relations:

$$h = v^2;\ T_j = \frac{h_{,j}}{2h} \quad (j = \theta, \phi), \quad T_{j,k} = \frac{h_{,jk}}{2h} - 2T_jT_k \quad (j, k = \theta, \phi),$$

$$h_{,j} = -(3a_{,j}h^2 + 3b_{,j}h + c_{,j})/(3h^2 + 6ah + 3b),$$

$$h,jk = -\frac{6(h+a)h_{,j}h_{,k} + 3(2a_{,k}h + b_{,k})h_{,j} + 3(2a_{,j}h + b_{,j})h_{,k} + 3a_{,jk}h^2 + 3b_{,jk}h + c_{,jk}}{3h^2 + 6ah + 3b}. \tag{23}$$

The derivatives of a, b and c can be derived from the relations (19). The phase direction can be calculated from a given ray direction but with following exceptions:

(i) Singularities are the phase directions along which the phase velocities of $qS1$ wave and $qS2$ wave approach each other. Near a singularity the directional derivatives of phase velocity change rapidly and the direction of local extremum represents the singularity. The existence of local extremum at the singularity in the root search region fails the Newton's method. This may be avoided by the slight shift of initial root.
(ii) For an assumed ray direction, there may be more than one phase direction possible and these may be quite near to each other. This defines the triplications or cusps in the wave fronts of $qS1$ and $qS2$-waves [25].

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